Problem 1.

Show that for all \( x \geq 0 \), the following inequality holds
\[ x - \ln(1 + x) \geq 0. \]

Solution: Consider the function
\[ f(x) = x - \ln(1 + x). \]

We have
\[ f'(x) = 1 - \frac{1}{1 + x} = \frac{x}{x + 1} \geq 0 \]
for \( x \geq 0 \). Thus, \( f \) is increasing over \( [0, \infty) \) hence for \( x \geq 0 \) we have
\[ f(x) \geq f(0). \]

But \( f(0) = 0 \) so \( f(x) \geq 0 \).
Problem 2.

(i) Calculate the following limit or show that it does not exist
\[
\lim_{(x,y,z) \to (0,0,0)} \frac{(x^2 + y^2 + z^2)^2}{2x^2 + 3y^2 + 4z^2}.
\]

(ii) Show that the following limit does not exist
\[
\lim_{(x,y) \to (0,0)} \frac{x^3y^5}{x^8 + y^8}
\]

(iii) Using greek letters, rigorously prove that the sequence
\[
x_n = \frac{3n^2 - 1}{n^2 + 1}
\]
converges and find its limit.

Solution:

(i) We claim that the limit is 0. Indeed,
\[
\frac{(x^2 + y^2 + z^2)^2}{2x^2 + 3y^2 + 4z^2} = \frac{(x^2 + y^2 + z^2)}{2x^2 + 3y^2 + 4z^2} \cdot (x^2 + y^2 + z^2).
\]
This is a product of a fraction which is bounded by \(\frac{1}{2}\) and a function \(x^2 + y^2 + z^2\) which goes to 0. By a theorem in class, the limit is 0. To see that the fraction involved is bounded by \(1/2\) we need to prove
\[
\frac{x^2 + y^2 + z^2}{2x^2 + 3y^2 + 4z^2} \leq \frac{1}{2} \iff 2(x^2 + y^2 + z^2) \leq 2x^2 + 3y^2 + 4z^2
\]
which is clearly true.

(ii) We evaluate the limit along the line \(x = my\), for a fixed slope \(m\), as \(y \to 0\) (so that \(x = my \to 0\)).

The fraction
\[
\frac{x^3y^5}{x^8 + y^8} = \frac{(my)^3 \cdot y^5}{(my)^8 + y^8} = \frac{m^3y^8}{(m^8 + 1)y^8} = \frac{m^3}{m^8 + 1}.
\]
This fraction depends on \(m\) so the limit does not exist.

(iii) We show \(\lim_{n \to \infty} x_n = 3\). Fix \(\epsilon > 0\). We need to exhibit \(N\) such that if \(n \geq N\) then
\[
|x_n - 3| < \epsilon
\]
for \(n \geq N\).

Now,
\[
|x_n - 3| = \left| \frac{3n^2 - 1}{n^2 + 1} - 3 \right| = \frac{4}{n^2 + 1} < \epsilon \iff n^2 + 1 > \frac{4}{\epsilon} \iff n^2 > \frac{4}{\epsilon} - 1.
\]
If \(4/\epsilon - 1 < 0\), any \(N > 0\) works. Otherwise, if \(4/\epsilon - 1 > 0\), pick any
\[
N > \sqrt{\frac{4}{\epsilon} - 1}.
\]
For \(n \geq N\), the above inequalities are satisfied, proving convergence.
**Problem 3.**

Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by \( f(x, y) = e^{(x-1)^2+y^2} \).

(i) Draw the level diagram of the function \( f \), showing at least three different levels.
(ii) Draw the graph of the function \( f \).
(iii) Calculate the partial derivatives of \( f \) at the point \((0, 1)\). Write down the total derivative of \( f \) at \((2, 1)\).
(iv) Find the equation of the tangent graph of \( f \) at the point \((0, 1, e^2)\).
(v) Find the directional derivative of \( f \) at the point \((0, 1)\) in the direction \( u = 2i + 3j \).
(vi) For which unit vector \( u \) is the directional derivative of \( D_u f \) at \((0, 1)\) is maximal?

**Solution:**

(i) The level curve with level \( c \) is
\[
e^{(x-1)^2+y^2} = c \iff (x-1)^2 + y^2 = \ln c
\]
which is a circle of center \((1, 0)\) and radius \( \sqrt{\ln(c)} \) for \( c \geq 1 \) (the level curve is empty if \( c < 1 \) since the square root doesn’t make sense, as \( \ln(c) < 0 \)). Some possible level curves are obtained for \( c = e, e^4, e^9 \) which correspond to circles of center \((1, 0)\) and radii \(1, 2, 3\).

(ii) The graph has as level curves circles centered at \((1, 0)\) of radii \( \sqrt{\ln(c)} \) and placed at various heights \( c \). The graph resembles in shape the paraboloid except that the cross sections are exponentials as opposed to parabolas.

(iii) We have
\[
D_1 f = 2(x-1)e^{(x-1)^2+y^2} \Rightarrow D_1 f(0,1) = -2e^2,
\]
\[
D_2 f = 2ye^{(x-1)^2+y^2} \Rightarrow D_2 f(0,1) = 2e^2.
\]
The total derivative of \( f \) at \((1, 0)\) exists since the derivatives are continuous. It equals
\[
Df(0,1) = [-2e^2 \, 2e^2].
\]

(iv) The equation of the tangent plane is
\[
z - e^2 = -2e^2x + 2e^2(y - 1) \iff z = -2e^2x + 2e^2y - e^2.
\]

(v) The directional derivative is \( D_u f(0,1) = Df \cdot u = (-2e^2, 2e^2) \cdot (2, 3) = 2e^2 \).

(vi) The gradient is the direction of steepest increase so
\[
u = \frac{Df}{\|Df\|}(0,1) = \frac{1}{\sqrt{2}}(-1,1).
\]
Problem 4.

(i) Let $U$ be the set of $3 \times 3$ matrices $A$ such that $A^3 + A - I$ is invertible. Show that $U$ is open.

Solution: Consider the function

$$f : \text{Mat}(3,3) \to \mathbb{R}, \quad f(A) = \det(A^3 + A - I).$$

Clearly, $f$ is continuous being the composition of the determinant function which is continuous and of the function $A \to A^3 + A - I$ which is continuous as well (since it is given by polynomials in the entries of $A$). The set $U$ is the preimage $U = f^{-1}((-\infty, 0) \cup (0, \infty))$ of an open set under a continuous map. Thus $U$ must be open.

(ii) Indicate which of the following sets of $3 \times 3$ matrices are open, closed, compact or neither.

- The subset $S$ of matrices $A$ with $\|A - I\| = 1$.

Solution: Closed and compact. Indeed, the set is just the closed ball of radius 1 around the identity matrix, hence it is compact. The set is not open.

- The subset $S$ of matrices with $\frac{1}{2} < \|A - I\| < 1$.

Solution: Open. This is just an open annulus of radii $\frac{1}{2}$ and 1 around the identity matrix. The set is not closed (the inequalities are strict hence the boundary is not in the set), and therefore also not compact.

- The subset $S$ of matrices with trace 0.

Solution: Closed. The set $S$ is the preimage of 0, which is closed in $\mathbb{R}$, under the continuous trace map $A \to \text{Tr}(A)$. The set is not bounded, hence not compact. The set is also not open.

(ii) Indicate which of the following subsets are open, closed, compact. No justification is necessary.

- The subset $S \subset \mathbb{R}^2$ equal to $[0, 1] \times (0, 1)$.

Solution: Not open, nor closed, nor compact.

- The subset $S \subset \mathbb{R}^2$ of pairs $(x, y) \in \mathbb{R}^2$ such that $xy + x^3 \sin(x) \leq 1$.

Solution: Closed. The set $S$ is the preimage of the closed set $(-\infty, 1]$ under the continuous map $(x, y) \to xy + x^3 \sin(x)$. The set is not bounded from below hence it cannot be compact. It is also not open.

- The set $S = \{2 - \frac{1}{n} : n \text{ natural number}\} \cup \{1 + \frac{1}{n} : n \text{ natural number}\}$, viewed as a subset of $\mathbb{R}$.

Solution: Closed and compact. The set is clearly bounded. It is also closed since it contains its limit points. Indeed, the limit of $2 - \frac{1}{n}$ is 2 which belongs to the set (make $n = 1$ in $1 + 1/n$).
Likewise, the limit of $1 + \frac{1}{n}$ is 1 which belongs to the set (make $n = 1$ in $2 - \frac{1}{n}$). The set is not open.

- The set $S = \{2 + \frac{1}{n} : n \text{ natural number} \} \cup \{1 - \frac{1}{n} : n \text{ natural number} \}$, viewed as a subset of $\mathbb{R}$.

  **Solution:** Not closed, not open, not compact. Indeed, the set is not closed since the limit point of $2 + \frac{1}{n}$ is 2 and it does not belong to the set. Thus the set is also not compact. It cannot be open since no ball is contained in $S$.

- The subset $S \subset \mathbb{R}^2$ of pairs $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2$ is an integer.

  **Solution:** Closed. The complement of $S$ is the union $n < x^2 + y^2 < n + 1$ for all possible natural numbers $n$. Each of these sets is open, so the complement of $S$ is open as well. Thus $S$ is closed. Clearly, $S$ is unbounded so it cannot be compact. It is also not open.
Problem 5.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x^2 \cos(\pi x) + x^3 - \sin(\pi x)$.

(i) Calculate $f'$ and show $f'$ is continuous.

(ii) Explain why $f'$ is bounded over the interval $[0, 1]$.

(iii) Show that $f$ is uniformly continuous over the interval $[0, 1]$.

(iv) Show that there exists $c \in (0, 1)$ such that $f'(c) = 2$.

Solution:

(i) We have $f'(x) = -2x \cos(\pi x) + \pi x^2 \sin(\pi x) + 3x^2 - \pi \cos(\pi x)$. This is a sum of products of continuous functions hence it is continuous.

(ii) Since $f'$ is continuous, it is bounded over any compact interval such as $[0, 1]$.

(iii) Since $f'$ is bounded, it follows that $f$ is Lipschitz by a homework problem, hence uniformly continuous, as proved in class.

(iv) We have $f(1) = 2$, $f(0) = 0$. By the mean value theorem, there exists $c \in (0, 1)$ such that $f'(c) = \frac{f(1) - f(0)}{1 - 0} = 2$. 