The combinatorics of Lehn’s conjecture

A. Marian, D. Oprea, R. Pandharipande

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Lehn’s conjecture. The number of \((n-2)\)-subspaces in \(\mathbb{P}^{2n-2}\) which are \(n\)-secant to a smooth curve,

\[ C \subset \mathbb{P}^{2n-2}, \]

of genus \(g\) and degree \(d\) is a classical enumerative calculation [ACGH]. The answer can be expressed in terms of Segre integrals over the symmetric\(^1\) product \(C^{[n]}\) of \(C\). Let the line bundle \(H \to C\) be the degree \(d\) restriction of \(O_{\mathbb{P}^{2n-2}}(1)\). The \(n\)-secant problem is solved by the Segre integral, and the answer can be written in closed form [LeB], [C],

\[
\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} s_n(H^{[n]}) = \frac{(1-w)^{d+2\chi(O_C)}}{(1-2w)^{\chi(O_C)}},
\]

under the change of variables \(z = w(1-w)\).

Going further, consider a pair \((S,H)\) consisting of a smooth projective surface and a line bundle \(H \to S\). The Segre integrals

\[
\int_{S^{[n]}} s_{2n}(H^{[n]})
\]

over the Hilbert scheme of points \(S^{[n]}\) count the \(n\)-secants of dimension \(n-2\) to the image of the surface

\[ S \to \mathbb{P}^{3n-2}, \quad H = O_{\mathbb{P}^{3n-2}}(1)|_S. \]

The following conjecture was made by Lehn [L]:

\[
\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} s_{2n}(H^{[n]}) = \frac{(1-w)^a(1-2w)^b}{(1-6w+6w^2)^c},
\]

for constants

\[ a = H \cdot K_S - 2K_S^2, \quad b = (H - K_S)^2 + 3\chi(O_S), \quad c = \frac{1}{2}H(H - K_S) + \chi(O_S). \]

A more complicated change of variables is needed here,

\[ z = \frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3}. \]

For \(K\)-trivial surfaces, Lehn’s conjecture was established in [MOP] via a study of the virtual geometry of a suitable Quot scheme. The vanishing results of Theorem 3 in [V] for \(K3\)-blowups

\[ s_{2n}(H^{[n]}) = 0, \]

obtained via classical geometry, provide the missing geometric pieces needed to establish Lehn’s conjecture in full generality. It would be interesting to see if the latter vanishings can be obtained also by the methods of [MOP].

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\(^1\)The \(n^{th}\) symmetric product of \(C\) is the Hilbert scheme of points \(C^{[n]}\). For curves \(C\) and surfaces \(S\), we use the standard notation for the tautological bundles \(H^{[n]}\) of rank \(n\) and \(H_n\) of rank 1 on the Hilbert schemes \(C^{[n]}\) and \(S^{[n]}\) associated to a line bundle \(H\), see [EGL].
\textbf{Theorem 1.} \textit{Lehn’s conjecture holds for all surfaces.}

\textit{Proof.} Corollary 6 in [V] reduces Lehn’s conjecture to a simple combinatorial statement.\footnote{András Szenes and Michèle Vergne have also completed the proof of Lehn’s conjecture immediately after reading Voisin’s paper [V] by the same residue calculations explained here. The main last step in the proof of Lehn’s conjecture is Voisin’s paper.} Let

\begin{equation}
(3) \quad z = \frac{w(1 - w)(1 - 2w)^4}{(1 - 6w + 6w^2)^3}
\end{equation}

be Lehn’s change of variables,

\[ z = w + 9w^2 + 68w^3 + \ldots \iff w = z - 9z^2 + 94z^3 + \ldots. \]

In order to prove Lehn’s conjecture, we must show the following two properties hold after the change of variables (3):

(i) the coefficient of $z^k$ in the expression

\[ A(z) = \frac{(1 - w)^{k+1}(1 - 2w)^{5k}}{(1 - 6w + 6w^2)^{3k-1}} \]

vanishes,

(ii) the coefficient of $z^k$ in the expression

\[ B(z) = \frac{(1 - w)^{k+2}(1 - 2w)^{5k-1}}{(1 - 6w + 6w^2)^{3k-1}} \]

vanishes.

Both follow from residue calculations.

To prove property (i), we write

\[ f(w) = \frac{(1 - w)(1 - 2w)^5}{(1 - 6w + 6w^2)^3}, \quad g(w) = (1 - w)(1 - 6w + 6w^2) \]

so that

\[ A(z) = f(w)^k g(w). \]

We have

\begin{align*}
  f(w) &= 1 + 7z - 13z^2 + 88z^3 + \cdots, \\
  g(w) &= 1 - 7z + 75z^2 - 880z^3 + \cdots.
\end{align*}

For the coefficient in (i), we must prove

\[ \text{Res}_{z=0} \frac{A(z)}{z^{k+1}} \, dz = 0 \]

which is equivalent to

\begin{equation}
(4) \quad \text{Res}_{z=0} \left( \frac{f(w)}{z} \right)^k \cdot \frac{g(w)}{z} \, dz = 0.
\end{equation}

We compute

\[ \frac{f(w)}{z} = \frac{1 - 2w}{w}, \quad \frac{g(w)}{z} = \frac{(1 - 6w + 6w^2)^4}{w(1 - 2w)^4}. \]

Lehn’s change of variables is a nonsingular coordinate change near $w = 0$,

\[ dz = \frac{(1 - 2w)^3}{(1 - 6w + 6w^2)^4} \, dw. \]
The residue appearing on the left side of (4) equals the residue
\[ \text{Res}_{w=0} \left( \frac{1 - 2w}{w} \right)^k \cdot \frac{(1 - 6w + 6w^2)^4}{w(1 - 2w)^4} \cdot \frac{(1 - 2w)^3}{(1 - 6w + 6w^2)^4} \, dw \]
which in turn may be written as
\[ \text{Res}_{w=0} \left( \frac{1 - 2w}{w} \right)^{k-1} \, dw. \]

The latter residue obviously vanishes since the numerator has degree \( k - 1 \). Thus equation (4) holds and property (i) is established.

The proof of property (ii) is identical. The claim is equivalent to vanishing residue
\[ \text{Res}_{w=0} \left( \frac{1 - 2w}{w} \right)^{k-2} \cdot \frac{1 - w}{w^{k+1}} \, dw = 0. \]
\( \square \)

**Remark.** While the Segre vanishings of Theorem 3 in [V] are sufficient to complete the proof of Lehn’s conjecture, there are further vanishings predicted by the formula. The same residue calculations which appear above show that
\[ s_{2k}(H^k) = 0 \]
provided that the pair
\( (d, \pi) = (H^2, H \cdot K_S) \)
falls in a favorable range. For instance,

(i) if \( S \) is a blowup of K3 surface at one point, vanishing holds in the range
\[ \pi \geq k - 1, \quad d - 2\pi \geq 4k - 4, \quad d - \pi \leq 6k - 6. \]

The numerics considered by [V] satisfy these requirements
\( (d, \pi) = (7(k - 1), k - 1), \quad (7(k - 1) + 1, k). \)

(ii) if \( S \) is a blowup of an abelian, Enriques, bielliptic surface at one point, vanishing holds in the range
\[ \pi \geq k - 1, \quad d - 2\pi \geq 4k + 2, \quad d - \pi \leq 6k - 4. \]

It would be interesting to interpret these vanishings geometrically.

**Exponential form.** Following [EGL], it is customary to rewrite the above formulas in exponential notation.

- For curves, two power series are needed,

\[ \sum_{n=0}^{\infty} z^n \int_{C \cap n} s_n(H^{[n]}) = \exp \left( A_1(z) \cdot d + A_2(z) \cdot \chi(\mathcal{O}_C) \right). \]

By (1), the expressions for \( A_1, A_2 \) become particularly simple under the change of variables
\[ z = -t(1 + t). \]

We have
\[ A_1(z) = \log(1 + t), \quad A_2(z) = 2\log(1 + t) - \log(1 + 2t) \]
for formula (5).

- For surfaces, four power series are needed,

\[ \sum_{n=0}^{\infty} z^n \int_{S \cap n} s_{2n}(H^{[n]}) = \exp \left( A_1(z) \cdot H^2 + A_2(z) \cdot \chi(\mathcal{O}_S) + A_3(z) \cdot (H \cdot K_S) + A_4(z) \cdot K_S^2 \right). \]
After the change of variables
\[ z = \frac{1}{2}t(1 + t)^2, \]
a straightforward calculation using (2) for surfaces yields:
\[
\begin{align*}
A_1(z) &= \frac{1}{2} \log(1 + t), \\
A_2(z) &= \frac{3}{2} \log(1 + t) - \frac{1}{2} \log(1 + 3t), \\
A_3(z) &= -\log 2 - \log(1 + t) + \log (\sqrt{1 + t} + \sqrt{1 + 3t}), \\
A_4(z) &= \log 4 + \frac{1}{2} \log(1 + t) + \frac{1}{2} \log(1 + 3t) - 2 \log (\sqrt{1 + t} + \sqrt{1 + 3t}).
\end{align*}
\]

**Higher rank.** The exponential formulas above have higher rank analogues. For a pair \((C, V)\) consisting of a smooth projective curve \(C\) and a rank \(r\) vector bundle \(V\) of degree \(d\), we have
\[
\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} s_n(V^{[n]}) = \exp \left( d \cdot A_1(z) + \chi(C) \cdot A_2(z) \right)
\]
for power series \(A_1(z)\) and \(A_2(z)\) depending upon \(r\). The series \(A_1\) was conjectured in [W], though not in closed form, while the expression for \(A_2\) was left open. Here, we prove the following result.

**Theorem 2.** For formula (6) in rank \(r\), we have
\[
A_1(-t(1 + t)^r) = \log(1 + t), \quad A_2(-t(1 + t)^r) = (r + 1) \log(1 + t) - \log(1 + t(r + 1)).
\]

**Proof of Theorem 2.** To find the series \(A_1\) and \(A_2\), we need only consider the projective line \(C \cong \mathbb{P}^1\) with the vector bundle
\[
V = \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}^{r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(d).
\]
We obtain
\[
V^{[k]} = \mathcal{O}^{[k]} \otimes \mathcal{O}^{r-1} \oplus (\mathcal{O}(d))^{[k]}.
\]
The Hilbert scheme of points is simply \((\mathbb{P}^1)^{[k]} \cong \mathbb{P}^k\), and the universal subscheme \(Z \hookrightarrow \mathbb{P}^k \times \mathbb{P}^1\) is given by
\[
\mathcal{O}(-Z) = \mathcal{O}_{\mathbb{P}^k}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-k).
\]
It follows that
\[
\text{ch } \mathcal{O}(d)^{[k]} = \text{ch } \mathbb{R}pr_* (\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}^1}(d))
\]
\[
= \text{ch } \mathbb{R}pr_* ((\mathcal{O} - \mathcal{O}(-Z)) \otimes \mathcal{O}_{\mathbb{P}^1}(d))
\]
\[
= \text{ch } (H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \otimes \mathcal{O}_{\mathbb{P}^k} - H^0(\mathcal{O}_{\mathbb{P}^1}(d - k)) \otimes \mathcal{O}_{\mathbb{P}^k}(-1))
\]
\[
= (d + 1) - (d - k + 1) \cdot \exp(-h)
\]
Here, we write \(h\) for the hyperplane class on \(\mathbb{P}^k\). We can then find the Chern roots of \((\mathcal{O}(d))^{[k]}\) yielding the following expression for the Segre class
\[
s(\mathcal{O}(d)^{[k]}) = (1 - h)^{d-k+1}.
\]
Consequently
\[
s(V^{[k]}) = (1 - h)^{d-rk+r} \Rightarrow \int_{\mathbb{P}^k} s_k(V^{[k]}) = (-1)^k \binom{d-rk+r}{k}.
\]
We conclude that
\[
\sum (-1)^k \binom{d-rk+r}{k} \cdot z^k = \exp(d \cdot A_1(z) + A_2(z)).
\]
To finish the proof, we invoke the following result which was first proved in [MOP] for $r = 2$. We follow the same argument here.

**Lemma 3.** After the change of variables

$$z = t(1 + t)^r,$$

we have

$$\sum_{k=0}^{\infty} \binom{d - rk + r}{k} \cdot z^k = \frac{(1 + t)^{d+r+1}}{1 + t(r + 1)}.$$

**Proof.** First, we already know from (7) that the left hand side takes the form

$$F_1 \cdot F_2$$

for power series $F_1 = \exp(A_1)$, $F_2 = \exp(A_2)$. In fact, we claim that

$$F_1(z) = 1 + t, \quad F_2(z) = \frac{(1 + t)^{r+1}}{1 + t(r + 1)}.$$

To confirm the formulas for $F_1$ and $F_2$ above, it suffices to verify the Lemma for two different values of $d$. We use $d = -2r$ and $d = -r$.

First, when $d = -2r$, we establish

$$\sum_{k=0}^{\infty} \binom{-rk - r}{k} \cdot z^k = \frac{1}{(1 + t)^{r-1}(1 + t(r + 1))}.$$ 

This is contained in Lemma 5 of [MOP] for $r = 2$. There, it is shown that the solution to the equation

$$z = t(1 + t)^r$$

has the Taylor expansion

$$t = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} \binom{-rk - r}{k}.$$ 

Differentiating, we find identity (8)

$$\sum_{k=0}^{\infty} z^k \binom{-rk - r}{k} = \frac{dt}{dz} = (\frac{dz}{dt})^{-1} = ((1 + t)^r(1 + (r + 1)t))^{-1}.$$

The case $d = -r$ uses the identity

$$\frac{1 + t}{1 + (r + 1)t} = (1 + t) - \frac{1}{(1 + t)^{r-1}(1 + (r + 1)t)} \cdot (r + 1) \cdot t(1 + t)^r.$$

For $1 + t$ we substitute the expression (9), while for the fraction that follows it we use (8). We obtain

$$\frac{1 + t}{1 + (r + 1)t} = 1 + \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} \binom{-rk - r}{k} - (r + 1) \sum_{k=0}^{\infty} z^{k+1} \binom{-rk - r}{k} \quad \sum_{k=0}^{\infty} z^k \binom{-rk}{k}$$

which verifies the Lemma in this case. □

**Surfaces.** For surfaces, a complete higher rank analogue of Lehn’s conjecture is an open question. In this direction, several conjectures were recently formulated by Drew Johnson in [J]. Johnson’s formulation of the conjecture was inspired by counts of points of 0-dimensional Quot schemes
and strange duality, much like the strategy used to prove strange duality for curves in [MO]. We sharpen these conjectures, by providing closed formulas for some of the series involved.

Specifically, consider a pair \((S, V)\) where \(V\) is a rank \(s\) vector bundle on a smooth projective surface \(S\). The associated vector bundle \(V^{[n]}\) over the Hilbert scheme has rank \(sn\). By passing to resolutions, \(V^{[n]}\) makes sense for all \(K\)-theory classes \(V\).

It is remarked in [J] that the following integrals of \(V^{[n]}\) depend on 5 different power series

\[
\sum_{n=0}^{\infty} z^n \int_{\tilde{S}^{[n]}} c_{2n}(V^{[n]}) = A_1(z) c_1(V)^s \cdot A_2(z) \chi(c_1(V)) \cdot A_3(z) \frac{1}{2} \chi(O_S) \cdot A_4(z) K_S \cdot A_5(z) K_S^2 .
\]

After changing \(V\) into \(-V\) in \(K\)-theory, the above expressions turn into Segre integrals of higher rank vector bundles. Hence, equation (10) generalizes Lehn’s formula.

To say a bit more about the above series, we recall a result of [EGL] regarding the holomorphic Euler characteristics of tautological line bundles twisted by powers of the exceptional divisor \(E\):

\[
\sum_{n=0}^{\infty} z^n \chi(S^{[n]}, H_n \otimes E^r) = f_r(z) \chi(O_S) \cdot g_r(z) \chi(H) \cdot a_r(z) \chi(E) \cdot b_r(z) \chi(k).
\]

By [J] and [EGL], the two series corresponding to \(K\)-trivial surfaces are determined in closed form

\[
f_r(z) = \frac{(1+t)^r}{1+r^2 t}, \quad g_r(z) = 1 + t
\]

under the change of variables

\[
z = t(1+t)^{r-1}.
\]

As is usually the case, the series \(a_r, b_r\) are unknown.

Refining the conjectures in [J], we provide closed expressions for the series in (10) corresponding to \(K\)-trivial surfaces. The last two series are quite surprisingly connected in a very precise fashion to the unknown series \(a_r, b_r\) of (11).

**Conjecture 1.** Write \(r = s - 1\), where \(s\) is the rank of \(V\). Under the change of variables

\[
z = -\frac{1}{r} t(1+t)^{-r}, \quad w = \frac{t(-r+(-r+1)t)^{r-1}}{(-r(1+t))^{r^2}},
\]

we have

\[
A_1(z) = (-r)^{-r-1} \cdot (1+t)^{-r} \cdot (-r+(-r+1)t)^{r+1},
\]

\[
A_2(z) = (-r)^r \cdot (1+t)^{-r} \cdot (-r+(-r+1)t)^{-r},
\]

\[
A_3(z) = (-r)^2 \cdot (1+t-rt)^{-1} \cdot (1+t)^{(r-1)^2} \cdot (-r+t(-r+1))^{-r},
\]

\[
A_4(z) = a_r(w),
\]

\[
A_5(z) = b_r(w).
\]

Furthermore, using the solution of Lehn’s conjecture, we are able to predict the first nontrivial examples of the unknown series \(a_r, b_r\) corresponding to \(r = \pm 2\).

**Conjecture 2.** Under the change of variables

\[
w = \frac{t(2+3t)^3}{16(1+t)^4},
\]

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3The series \(A_1, \ldots, A_5\) up to order 6 in \(z\) were calculated in [J]. The numerical data in [J] played an important role in our formulation of Conjecture 1.

4We have \(a_0 = a_\pm 1 = b_0 = b_\pm 1 = 1\).
we have
\[
    a_{-2}(w) = a_2(w) = \frac{2 + 3t}{\sqrt{1 + t} \cdot \sqrt{1 + t + \sqrt{1 + 3t}}},
\]
\[
    b_{-2}(w) = b_2(w) = 4\sqrt{2 + 3t} \cdot \frac{(1 + t)^{1/4} \cdot \sqrt{1 + 3t}}{(\sqrt{1 + t} + \sqrt{1 + 3t})^{5/2}}.
\]
These expressions are connected to the series appearing in Lehn's rank 1 formula. We have checked the term by term expansions pertaining to both \(a_{-2}\) and \(b_{-2}\) to high order.

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References


