# SHEAVES ON ABELIAN SURFACES AND STRANGE DUALITY

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ABSTRACT. We formulate three versions of a *strange duality* conjecture for sections of the Theta bundles on the moduli spaces of sheaves on abelian surfaces. As supporting evidence, we check the equality of dimensions on dual moduli spaces, answering a question raised by Göttsche-Nakajima-Yoshioka [GNY].

## 1. INTRODUCTION

Let $(A, H)$ be a polarized abelian surface. In this paper, we consider the moduli spaces of Gieseker $H$-semistable sheaves on $A$, and sections of the Theta line bundles defined over them.

It will be convenient to bookkeep coherent sheaves $E$ on $A$ by their Mukai vectors, setting

$$v(E) = r + c_1(E) + \chi(E)\,\omega \in H^{2\star}(A),$$

where $\omega$ stands for the class of a point. As customary, we will equip the even cohomology $H^{2\star}(A)$ with the Mukai pairing. For any two vectors $x = (x_0, x_2, x_4) \in H^{2\star}(A)$ and $y = (y_0, y_2, y_4) \in H^{2\star}(A)$, we set

$$\langle x, y \rangle = -\int_A x^\vee \cup y = \int_A (x_2 y_2 - x_0 y_4 - x_4 y_0)\,.$$

It follows that for any two sheaves $E$ and $F$, we have

$$\chi(E, F) = \sum_{i=0}^{2} (-1)^i \ \mathrm{Ext}^i(E, F) = -\langle v(E), v(F) \rangle.$$

For an arbitrary $v \in H^{2\star}(A)$, let us write $\mathfrak{M}_v$ for the moduli space of Gieseker $H$-semistable sheaves $E$ on $A$, with Mukai vector $v$. To keep things simple, we will make the following assumption throughout:

**Assumption 1.** (i) *The polarization $H$ is generic i.e., it belongs to the complement of a locally finite union of hyperplane walls in the ample cone of $A$;*

(ii) *The vector $v$ is a primitive element of the lattice $H^{2\star}(A, \mathbb{Z})$;*

(iii) *The vector $v$ is positive i.e., one of the following is true:*
- *rank $(v) > 0$;*

– *rank* $(v) = 0$ *and* $c_1(v)$ *is effective,* $\chi(v) \neq 0$, *and* $\langle v, v\rangle \neq 0, 4$.

The choice of generic polarization will play only a minor role in what follows, and as such, we will supress it from the notation. Note that (i) and (ii) together imply that $\mathfrak{M}_v$ consists of stable sheaves only. Therefore, by [Muk3], $\mathfrak{M}_v$ is a smooth manifold of dimension $2d_v + 2$, with

$$d_v = \frac{1}{2}\langle v, v\rangle. \tag{1}$$

The main characters of our story will be a collection of naturally defined Theta line bundles on $\mathfrak{M}_v$. Consider the subgroup $v^\perp$ inside the holomorphic $K$-theory of $A$ generated by the sheaves $F$ whose Mukai vectors $w$ are orthogonal to $v$:

$$\chi(v \otimes w) = -\langle v^\vee, w\rangle = 0. \tag{2}$$

There is a morphism

$$\Theta : v^\perp \to \mathrm{Pic}(\mathfrak{M}_v), \quad [F] \to \Theta_F,$$

constructed and studied by Li and Le Potier [Li] [LP] in the case of surfaces, and also by Drézet-Narasimhan in the case of curves [DN]. The construction is easiest to explain assuming that $\mathfrak{M}_v$ is a fine moduli space, such that $\mathcal{E}$ is the universal sheaf on $\mathfrak{M}_v \times A$. In this case, for a sheaf $F$ with Mukai vector $w$, we set

$$\Theta_F = \det \mathbf{R}p_!(\mathcal{E} \otimes q^\star F)^{-1}, \tag{3}$$

where $p$ and $q$ are the two projections from $\mathfrak{M}_v \times A$. The orthogonality condition (2) is used to obtain a well defined line bundle $\Theta_F$, even in the absence of universal structures, by descent from the Quot scheme.

Note that even though the line bundle $\Theta_F$ depends on the $K$-theory class of $F$, the Chern class $c_1(\Theta_F)$ depends only on the Mukai vector $w$. For simplicity of notation, in all *cohomological* computations below, we will write $\Theta_w$ for any one of the line bundles $\Theta_F$ as above. The finer dependence of the line bundles $\Theta_F$ on the sheaf $F$ will be discussed in more detail in Section 2.

Our goal in this paper is to compute the Euler characteristics of the line bundles $\Theta_w$, which can be interpreted as the $K$-theoretic Donaldson invariants of the abelian surface $A$. We will provide a simple expression for these Euler characteristics, valid in any rank. We will thus answer a question raised in [GNY], as part of a general study of the rank two $K$-theoretic Donaldson invariants of surfaces.

To explain the results, let us first fix a reference line bundle $\Lambda$ on $A$ with $c_1(\Lambda) = c_1(v)$. Then, we have a well-defined determinant morphism

$$\alpha_\Lambda^+ : \mathfrak{M}_v \to \widehat{A} = \mathrm{Pic}^0(A), \;\; E \to \det E \otimes \Lambda^{-1}. \tag{4}$$

Its fiber over the origin is the moduli space $\mathsf{M}_v^+(\Lambda)$ of sheaves with fixed determinant $\Lambda$. The choice of the determinant is unimportant for our arguments, and therefore we will omit it from our notation when no confusion is likely to arise. We will show:

**Theorem 1.** *For any vectors $v$ and $w$ satisfying Assumption 1, and such that $\chi(v \otimes w) = 0$, we have*

$$\chi(\mathsf{M}_v^+, \Theta_w) = \chi(\mathsf{M}_w^+, \Theta_v) = \frac{1}{2}\frac{c_1(v\otimes w)^2}{d_v + d_w}\binom{d_v + d_w}{d_v}. \tag{5}$$

There is yet another moduli space of interest to us, which is Fourier-Mukai 'dual' to the one considered above. Letting $\mathcal{P}$ denote the normalized Poincaré bundle on $A \times \widehat{A}$, the Fourier-Mukai transform is defined by

$$\mathbf{R}\mathcal{S}(E) = \mathbf{R}p_!(\mathcal{P} \otimes q^\star E) \in \mathbf{D}(\widehat{A}), \tag{6}$$

where $p, q$ are the two projections. With this understood, let us set

$$\alpha_\Lambda^- : \mathfrak{M}_v \to A, \;\; E \to \det \mathbf{R}\mathcal{S}(E) \otimes \det \mathbf{R}\mathcal{S}(\Lambda)^{-1}.$$

The fiber of the morphism $\alpha_\Lambda^-$ over the origin is denoted by $\mathsf{M}_v^-$, and parametrizes sheaves $E$ with fixed determinant of the Fourier-Mukai transform. We will prove:

**Theorem 2.** *For any vectors $v$ and $w$ as in Theorem 1, we have*

$$\chi(\mathsf{M}_v^-, \Theta_w) = \chi(\mathsf{M}_w^-, \Theta_v) = \frac{1}{2}\frac{c_1(\hat{v}\otimes \hat{w})^2}{d_v + d_w}\binom{d_v + d_w}{d_v}. \tag{7}$$

*Here $\hat{v}$ and $\hat{w}$ denote the Fourier-Mukai transforms of the two vectors $v$ and $w$.*

Finally, we may consider the morphism

$$\mathsf{a}_v = (\alpha_\Lambda^+, \alpha_\Lambda^-) : \mathfrak{M}_v \to \widehat{A} \times A.$$

This is the Albanese map of the moduli space $\mathfrak{M}_v$, cf. [Y1]. Its fiber over the origin will henceforth be denoted by $K_v$. We will show:

**Theorem 3.** *Assume that the Néron-Severi group of $A$ has rank 1. With the same hypotheses as in Theorem 1, we have*

$$\chi(K_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v}. \tag{8}$$

The manifest symmetry of the formulas in Theorems 1, 2 and 3 matches first of all that of their counterpart for the case of sheaves on a $K3$ surface. Indeed, the theta Euler characteristics for the moduli space of sheaves on a $K3$ were shown to be [GNY][OG]

$$\chi(\mathfrak{M}_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \binom{d_v + d_w + 2}{d_v + 1}. \tag{9}$$

This symmetry further suggests a general *strange duality* for surfaces, reminiscent of the case of moduli spaces of bundles on curves. There, the analogous invariance of the Verlinde formula reflects a geometric isomorphism between generalized theta functions with dual ranks and levels [B][MO1]. In the case of sheaves on an abelian or $K3$ surface, it is tempting to assert, similarly, that whenever defined and nonzero, the morphisms

$$\mathsf{SD}^{\pm} : H^0(\mathsf{M}_v^{\pm}, \Theta_w)^{\vee} \to H^0(\mathsf{M}_w^{\pm}, \Theta_v)$$

are isomorphisms. The same considerations should apply to the companion morphism

$$\mathsf{SD} : H^0(K_v, \Theta_w)^{\vee} \to H^0(\mathfrak{M}_w, \Theta_v).$$

In the above, for each of the three pairs of moduli spaces *i.e.*, $(\mathsf{M}_v^{\pm}, \mathsf{M}_w^{\pm})$ and $(K_v, \mathfrak{M}_w)$, the line bundle $\Theta_v$ stands for any one of the $\Theta_E$'s, for $E$ in the corresponding moduli space of sheaves with Mukai vector $v$; similarly, $\Theta_w$ is any one of the line bundles $\Theta_F$, for $F$ in the dual moduli space of sheaves with Mukai vector $w$. We will review the definition of the three *strange duality* morphisms in Section 2 below, assuming that

**Assumption 2.** *For any two (semi)-stable sheaves $E$ and $F$ with Mukai vectors $v$ and $w$, we have*

$$H^2(E \otimes F) = 0.$$

*This is automatic if $c_1(v \otimes w) \cdot H > 0$, by Serre duality and stability.*

In order to use the numerics provided by Theorems 1, 2 and 3, one has to assume in addition that the line bundle $\Theta_w$ has no higher cohomology on the various moduli spaces considered. The vanishing of higher cohomology is a delicate question, which can be answered satisfactorily only in few cases. For smooth moduli spaces, or for moduli spaces with mild singularities - *e.g.* rational - one may invoke standard vanishing theorems. These require the understanding of the positivity properties of $\Theta_w$ *i.e.*, determining whether $\Theta_w$ is big and nef. In the case under study, smoothness is assumed, bigness is easy to detect, and nefness is hoped for. This last point of nefness is a subtle issue, even though the presence of the holomorphic symplectic structure on the moduli spaces considered here makes the question more tractable. A study for the Hilbert scheme of two points on $K3$ surfaces and their deformations can be found for instance in [HT].

Nevertheless, Le Potier [LP] and Li [Li] proved the following results, which can be viewed as a higher dimensional generalization of the ampleness of the determinant line bundle on the moduli space of bundles over a curve.

**Fact 1.** (i) *If $w$ has positive rank, and $c_1(w)$ is a high multiple of the polarization $H$, then $\Theta_w$ is relatively ample on the fibers of the determinant map $\alpha^+$.*

(ii) *If $w$ has rank $0$, and $c_1(w)$ is a positive multiple of the polarization, then $\Theta_w$ is big and nef on the fibers of $\alpha^+$.*

Similar results should hold for the morphism $\alpha^-$. When the Picard rank of $A$ is 1, this is obtained for free in many cases, using the remarks following Conjecture 2(ii).

One may then speculate

**Conjecture 1.** *When Assumptions 1 and 2 are satisfied, the three morphisms $\mathsf{SD}^+, \mathsf{SD}^-$ and $\mathsf{SD}$ are either isomorphisms or zero.*

This has the immediate

**Corollary 1.** *As $E$ varies in $K_v$, the Theta sections $\Theta_E$ on the dual moduli space $\mathfrak{M}_w$ span the linear series $|\Theta_v|$. Same statements apply to the moduli spaces $\mathsf{M}_w^+$ and $\mathsf{M}_w^-$, letting $E$ vary in $\mathsf{M}_v^+$ and $\mathsf{M}_v^-$ respectively.*

The conjecture was demonstrated in a number of cases, in this and other geometric setups. An overview of the already existing arguments, as well as proofs of new cases, can be found in [MO2].

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## 2. Preliminaries on Theta divisors

We review here the definition of the *strange duality* morphisms. We will fix $\Lambda$ an arbitrary determinant, and we let $\widehat{\Lambda} = \det \mathbf{R}\mathcal{S}(\Lambda)$. We recall the notation of the Introduction, letting $\mathsf{M}_v^+$ and $\mathsf{M}_v^-$ denote the moduli spaces of sheaves with determinant $\Lambda$ and determinant of the Fourier-Mukai transform equal to $\widehat{\Lambda}$ respectively; $K_v$ consists of sheaves satisfying both requirements.

We explained in the Introduction that the line bundle $\Theta_F$ only depends on the $K$-theory class of the reference sheaf $F$. We establish here the following more precise result:

**Lemma 1.** *Consider a sheaf $F$ of Mukai vector $w$, and consider the line bundle $\Theta_F$ on the moduli space $\mathfrak{M}_v$. Then,*

(i) *for $F \in \mathsf{M}_w^+$, the restriction of $\Theta_F$ to $\mathsf{M}_v^+$ is independent of the choice of $F$;*
(ii) *for $F \in \mathsf{M}_w^-$, the restriction of $\Theta_F$ to $\mathsf{M}_v^-$ is independent of the choice of $F$;*
(iii) *for $F \in \mathfrak{M}_w$, the restriction of $\Theta_F$ to $K_v$ is independent of the choice of $F$.*

*Proof.* To prove (i), pick two sheaves $F_1$ and $F_2$ with Mukai vector $w$ and the same determinant. Considering the virtual element in $K$-theory

$$\mathsf{f} = F_1 - F_2,$$

we need to show that

$$\Theta_{F_1} \otimes \Theta_{F_2}^{-1} = \Theta_{\mathsf{f}}$$

is trivial. We will show that in $K$-theory,

$$\mathsf{f} = \mathcal{O}_Z - \mathcal{O}_W \tag{10}$$

for two zero-dimensional schemes $Z$ and $W$, which necessarily have to be of the same length. This follows by induction on the rank of the $F$'s. The rank 0 case is obvious. When the rank is 1, then

$$F_1 = L \otimes I_Z, F_2 = L \otimes I_W$$

with $L = \det F_1 = \det F_2$, and the result is immediate. For the inductive step, note that it suffices to replace $F_1$ and $F_2$ by the twists $F_1(D)$ and $F_2(D)$, for some ample divisor $D$. In this case, we reduce the rank by constructing exact sequences

$$0 \to \mathcal{O}_A \to F_i(D) \to F_i' \to 0$$

with $F_1'$, and $F_2'$ of the same lower rank and the same determinant. The claim then follows by the induction hypothesis applied to $F_1' - F_2'$. Once (10) is understood, it suffices to assume that $Z$ and $W$ are supported on single points *i.e.,* that $\mathsf{f}$ is a formal sum

$$\mathsf{f} = \sum_{z,w} \left(\mathcal{O}_z - \mathcal{O}_w\right).$$

In this case, we will check $\Theta_{\mathsf{f}}$ is trivial by testing against any $S$-family $\mathcal{E} \to S \times A$ of sheaves with fixed determinant $\Lambda$. In particular, the latter requirement implies that

$$\det \mathcal{E} \cong M \boxtimes \Lambda$$

for some line bundle $M$ on $S$. Then, the pullback of $\Theta_{\mathsf{f}}$ under the classifying morphism $S \to \mathsf{M}_v^+$ is

$$\det p_!(\mathcal{E} \otimes q^\star \mathsf{f})^{-1} = \left(\det \mathcal{E}\big|_{S\times\{z\}}\right)^{-1} \otimes \det \mathcal{E}\big|_{S\times\{w\}} \cong M^{-1} \otimes M \cong \mathcal{O}_S,$$

completing the proof of (i).

For (ii), the same reasoning applies to

$$\mathbf{R}\mathcal{S}(\mathsf{f}) = \mathbf{R}\mathcal{S}(F_1) - \mathbf{R}\mathcal{S}(F_2),$$

which can be assumed to be the difference of the structure sheaves of $z$ and $w$ on the dual abelian variety $\widehat{A}$. Therefore,

$$\mathsf{f} = \mathcal{P}_z - \mathcal{P}_w,$$

where $\mathcal{P}_z$ and $\mathcal{P}_w$ are the line bundles on $A$ represented by $z$ and $w$. We need to check that

$$\det p_!(\mathcal{E} \otimes q^\star \mathcal{P}_z) \cong \det p_!(\mathcal{E} \otimes q^\star P_w).$$

Consider the relative Fourier-Mukai sheaf on $S \times \widehat{A}$

$$\det p_{13!}(p_{12}^\star \mathcal{E} \otimes p_{23}^\star \mathcal{P}) \cong M \boxtimes \widehat{\Lambda},$$

for some line bundle $M$ on $S$. Here the pushforward and pullbacks are taken along the projections from $S \times A \times \widehat{A}$. The conclusion follows by looking at the isomorphic pullbacks of this sheaf over $S \times \{z\}$ and $S \times \{w\}$.

Finally, the third statement is obvious since $K_v$ is simply connected [Y1]. Indeed, it suffices to check that $c_1(\Theta_F)$ is independent of $F$. This is a Grothendieck-Riemann-Roch computation, using the defining formula (3), with $\mathcal{E}$ replaced by a quasi universal sheaf, if needed.

**Remark 1.** The Lemma above is sufficient for our purposes. It would be useful to have a more detailed understanding of how the Theta line bundles $\Theta_F$ vary with $F$. For the case of curves, the requisite formulas were established by Drézet-Narasimhan [DN]. We speculate that the following holds:

**Conjecture 2.** *Consider two sheaves $F_1$ and $F_2$ with the same Mukai vector orthogonal to $v$.*

(i) *On $\mathsf{M}_v^+$, we have*

$$\Theta_{F_1} = \Theta_{F_2} \otimes \left(\alpha^-\right)^\star \left(\det F_1 \otimes \det F_2^{-1}\right).$$

(ii) *On $\mathsf{M}_v^-$, we have*

$$\Theta_{F_1} = \Theta_{F_2} \otimes \left((-1)\circ \alpha^+\right)^\star \left(\det \mathbf{R}\mathcal{S}(F_1) \otimes \det \mathbf{R}\mathcal{S}(F_2)^{-1}\right).$$

(iii) *If $c_1(v)=0$, then on $\mathfrak{M}_v$ we have*

$$\Theta_{F_1}=\Theta_{F_2}\otimes\left((-1)\circ\alpha^+\right)^{\star}\left(\det\mathbf{R}\mathcal{S}(F_1)\otimes\det\mathbf{R}\mathcal{S}(F_2)^{-1}\right)\otimes\left(\alpha^-\right)^{\star}\left(\det F_1\otimes\det F_2^{-1}\right).$$

Formula (i) is easily checked on the Hilbert scheme of points using that

$$\Theta_F=(\det F)_{(n)}\otimes E^{\,\mathrm{rank}F},$$

where $E$ is the exceptional divisor and $(\det F)_{(n)}$ is the pullback of the symmetrization of $\det F$ on the symmetric product [EGL]. Assuming (i), evidence for (ii) is provided by the change of the Theta line bundles under Fourier-Mukai transform. Indeed, when the Picard number of $A$ is 1, Yoshioka [Y1] exhibited very general examples of birational isomorphisms between the moduli spaces $\mathsf{M}_v^{\pm}$ and $\mathsf{M}_{\hat{v}}^{\mp}$ on $A$ and $\widehat{A}$, interchanging the maps $\alpha^+$ and $\alpha^-$; arguments of Maciocia [Ma] can be used to show that under this isomorphism the line bundle $\Theta_F$ corresponds to $\Theta_{(-1)^{\star}\widehat{F}}$, at least for generic $F$ satisfying $WIT$. Finally item (iii) is consistent with (i) and (ii), and with Grothendieck-Riemann-Roch. It may be possible to prove all three formulas using suitable degeneration arguments.

Assuming Lemma 1, it is now standard to define the three *strange duality* morphisms. The construction is contained in [D] and [OG], but we will review it briefly here for the sake of completeness.

Recall that for any pair of sheaves $(E,F)\in\mathfrak{M}_v\times\mathfrak{M}_w$ we have

$$\chi(E\otimes F)=0.$$

We will assume furthermore that

**Assumption 2.** *(modified version)*

(a) *either $H^2(E\otimes F)=0$; by stability this happens if $c_1(E\otimes F).H>0$;*
(b) *or $H^0(E\otimes F)=0$; by stability this happens if $c_1(E\otimes F).H<0$.*

To treat all cases at once, let us denote by $\mathcal{M}_v$ and $\mathcal{M}_w$ any one of the three pairs $(\mathsf{M}_v^+,\mathsf{M}_w^+)$, $(\mathsf{M}_v^-,\mathsf{M}_w^-)$ and $(K_v,\mathfrak{M}_w)$. We set

$$\mathcal{L}_w=\begin{cases}\Theta_F, \text{ for } F\in\mathcal{M}_w, \text{ if } \textit{Assumption 2 (a)} \text{ holds,}\\ \Theta_F^{-1}, \text{ for } F\in\mathcal{M}_w, \text{ if } \textit{Assumption 2 (b)} \text{ holds.}\end{cases}\tag{11}$$

By Lemma 1, this is a well-defined line bundle on $\mathcal{M}_v$. We similarly define the line bundle $\mathcal{L}_v$ on $\mathcal{M}_w$.

Descent arguments, presented in detail in Dănilă's paper [D], show the existence of a natural divisor

$$\Delta_{v,w}\hookrightarrow\mathcal{M}_v\times\mathcal{M}_w$$

which is supported set-theoretically on the locus

$$\Delta_{v,w} = \left\{(E,F) \in \mathcal{M}_v \times \mathcal{M}_w, \text{such that } h^1(E \otimes F) \neq 0\right\}.$$

This divisor is obtained as the vanishing locus of a section of a naturally defined line bundle $\Theta_{v,w}$ on $\mathcal{M}_v \times \mathcal{M}_w$. The splitting

$$\Theta_{v,w} = \mathcal{L}_w \boxtimes \mathcal{L}_v$$

follows from Lemma 1 by the see-saw theorem. Therefore, $\Delta_{v,w}$ becomes an element of

$$H^0(\mathcal{M}_v, \mathcal{L}_w) \otimes H^0(\mathcal{M}_w, \mathcal{L}_v)$$

inducing the duality morphism

$$H^0(\mathcal{M}_v, \mathcal{L}_w)^\vee \to H^0(\mathcal{M}_w, \mathcal{L}_v).$$

Note that when *Assumption 2 (a)* holds, the construction gives rise to the three duality morphisms of the Introduction:

$$\mathsf{SD}^\pm : H^0(\mathsf{M}_v^\pm, \Theta_w)^\vee \to H^0(\mathsf{M}_w^\pm, \Theta_v), \text{ and} \tag{12}$$

$$\mathsf{SD} : H^0(K_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v). \tag{13}$$

## 3. Euler characteristics on Albanese fibers

3.1. **The Albanese map.** In this section, we will compute the Euler characteristics of line bundles on $K_v$. We begin by reviewing a few facts about the Albanese map of $\mathfrak{M}_v$. Recall from the Introduction the modified determinant morphism

$$\alpha_\Lambda^+ : \mathfrak{M}_v \to \widehat{A}, E \to \det E \otimes \Lambda^{-1},$$

and its Fourier-Mukai 'dual'

$$\alpha_\Lambda^- : \mathfrak{M}_v \to A, \ \ E \to \det \mathbf{R}\mathcal{S}(E) \otimes \widehat{\Lambda}^{-1}.$$

Putting these two morphisms together, we obtain the map

$$\mathsf{a}_v = (\alpha^+, \alpha^-) : \mathfrak{M}_v \to \widehat{A} \times A. \tag{14}$$

Yoshioka proved that $\mathsf{a}_v$ is the Albanese map of the moduli space $\mathfrak{M}_v$ [Y1].

The morphism (14) is easiest to understand for the vector $v = (1, 0, n)$. Then, the moduli space $\mathfrak{M}_v$ is isomorphic to the product $\widehat{A} \times A^{[n]}$ of the dual abelian variety and the Hilbert scheme $A^{[n]}$ of points on $A$. The morphism $\mathsf{a}_v$ can be identified with

$$1 \times s : \widehat{A} \times A^{[n]} \to \widehat{A} \times A$$

where the first map is the identity, while the second is induced by summation on the abelian surface. That is, for a zero-cycle $Z$ supported on points $z_i$ with length $n_i$, we let

$$s : A^{[n]} \to A, \;\; s([Z]) = \sum_i n_i z_i.$$

The fiber of $s$ over $(0,0)$ is the generalized Kummer variety $K_{n-1}$ of dimension $2(n-1)$.

In general, Yoshioka studied the fiber of the Albanese map $\mathsf{a}_v$ over the origin

$$K_v = \mathsf{a}_v^{-1}(0,0),$$

under the assumption that the vector $v$ is primitive and positive, in the sense discussed in the Introduction. In this situation, and when $\langle v, v\rangle \geq 6$, Yoshioka proved in [Y1] that

- $K_v$ is an irreducible holomorphic symplectic manifold, deformation equivalent to the generalized Kummer surface $K_{n-1}$, with $n = \langle v, v\rangle/2$.
- There is an isomorphism
$$H^2(K_v, \mathbb{Z}) \cong v^\perp.$$
Here $v^\perp$ is computed in the cohomology of the surface $A$. Under the above isomorphism, Mukai vectors $w \in v^\perp$ of sheaves orthogonal to $v$ correspond to the Chern class $c_1(\Theta_w) \in H^2(K_v, \mathbb{Z})$.
- Moreover, if one endows $H^2(K_v, \mathbb{Z})$ with the Beauville-Bogomolov form, and $v^\perp$ with the intersection form, the above isomorphism is an isometry.

3.2. **Generalized Kummer varieties.** We start the calculation of Euler characteristics by considering the case of line bundles on the generalized Kummer varieties. In other words, we assume that $v = (1, 0, n)$. It may be possible to derive the answers from the calculations of [BN], but for completeness we include a direct argument, which also expresses the numerics in a form more convenient to us.

For any divisor $D$ on the abelian surface $A$, we let $D_{(n)}$ be the divisor on $A^{[n]}$ consisting of zero-cycles which intersect $D$. This divisor is a pull-back under the support morphism

$$f : A^{[n]} \to A^{(n)},$$

from the symmetric power $A^{(n)}$ of $A$,

$$D_{(n)} = f^\star \left(D \boxtimes D \boxtimes \ldots \boxtimes D\right)^{S_n}.$$

Further, let $E$ be the exceptional divisor of $A^{[n]}$ consisting of schemes with two coincident points in their support. Any line bundle on the Hilbert scheme is of the form $D_{(n)} \otimes E^r$. We will denote by the same symbol the restriction of these bundles to the generalized

Kummer variety. We also do not distinguish notationally between line bundles and divisors.

**Lemma 2.**

$$\chi(K_{n-1}, D_{(n)} \otimes E^r) = n\binom{\chi(D) - (r^2-1)n - 1}{n-1}$$

*Proof.* The expression given by the Lemma is a consequence of the known formula

$$\chi(A^{[n]}, D_{(n)} \otimes E^r) = \frac{\chi(D)}{n}\binom{\chi(D) - (r^2-1)n - 1}{n-1}, \tag{15}$$

which was deduced in [EGL]. To relate the two, we use the cartesian diagram

$$\begin{tikzcd} K_{n-1} \times A \arrow[r, "\sigma"] \arrow[d, "p"] & A^{[n]} \arrow[d, "s"] \\ A \arrow[r, "n"] & A \end{tikzcd} .$$

The upper horizontal map is induced by the translation on $A$:

$$\sigma : K_{n-1} \times A \to A^{[n]},\ (Z, a) \mapsto t_a^\star Z.$$

The bottom morphism is the multiplication by $n$ in the abelian surface. It follows that $\sigma$ is an étale cover of degree $n^4$.

By the see-saw theorem, we find

$$\sigma^\star D_{(n)} = D_{(n)} \boxtimes D^{\otimes n},\ \text{while } \sigma^\star E = E \boxtimes \mathcal{O}_A.$$

Therefore

$$\chi(A^{[n]}, D_{(n)} \otimes E^r) = \frac{1}{n^4}\chi(K_{n-1}, D_{(n)} \otimes E^r)\,\chi(A, D^n) = \frac{\chi(D)}{n^2}\chi(K_{n-1}, D_{(n)} \otimes E^r). \tag{16}$$

Putting (15) and (16) together we obtain the Lemma.

3.3. **General Albanese fibers.** We can now consider the case of an arbitrary vector $v$. We claim

**Proposition 1.** *If $d_v \neq 0$, then*

$$\chi(K_v, \Theta_w) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

*Proof.* We prove here the statement when $d_v \neq 2$. The case $d_v = 2$ will be considered separately in the next subsection.

When $d_v = 1$ the Proposition follows immediately from Mukai and Yoshioka's results [Muk2], [Y2], as they proved that the Albanese map $\mathsf{a}_v : \mathfrak{M}_v \to \widehat{A} \times A$ is an isomorphism. Then $K_v$ is a point, and both sides of the equation in Proposition 1 equal 1.

When $d_v \geq 3$, we follow the same arguments as in the case of $K3$ surfaces. We will make use of the Beauville-Bogomolov form $B$. This quadratic form is defined on the second cohomology of any irreducible holomorphic symplectic manifold, and can be considered as a generalization of the intersection pairing on $K3$ surfaces. In the case of the generalized Kummer varieties $K_{n-1}$, the form $B$ gives an orthogonal decomposition

$$H^2(K_{n-1},\mathbb{Z}) = H^2(A,\mathbb{Z}) \oplus \mathbb{Z}[E]$$

such that

$$B(c_1(D_{(n)})) = D^2, \quad B(c_1(E)) = -2n.$$

In particular,

$$B(c_1(D_{(n)} \otimes E^r)) = 2\left(\chi(D) - r^2 n\right).$$

Therefore, the result of Lemma 2 can be restated as

$$\chi(K_{n-1}, L) = n\binom{\frac{B(c_1(L))}{2} + n - 1}{n-1},$$

for any line bundle $L = D_{(n)} \otimes E^r$ on the Kummer variety $K_{n-1}$.

To get the result of the Proposition, we will use the fact that the Euler characteristics $\chi(X, \mathcal{L})$ of any line bundle on an irreducible holomorphic symplectic manifold $X$ can be expressed as a universal polynomial in the Beauville-Bogomolov form $B(c_1(\mathcal{L}))$ [H] *i.e.,* a polynomial depending only on the underlying holomorphic symplectic manifold. This polynomial is an invariant of the deformation type. By [Y1], $K_v$ is deformation equivalent to the generalized Kummer variety $K_{n-1}$, for $n = \langle v, v\rangle/2$, and

$$B(c_1(\Theta_w)) = \langle w, w\rangle.$$

Therefore,

$$\chi(K_v, \Theta_w) = d_v\binom{\frac{B(c_1(\Theta_w))}{2} + d_v - 1}{d_v - 1} = d_v\binom{d_v + d_w - 1}{d_v - 1} = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v},$$

completing the proof of the Proposition when $d_v \neq 2$.

3.4. **Two-dimensional Albanese fibers.** In this subsection, we establish Proposition 1 when $d_v = 2$, by analyzing the special geometry of the situation. The case $r = 1$ is covered by Lemma 2, so we assume that $r \geq 2$. $K_v$ is now a $K3$ surface. It suffices to show that

$$\chi(K_v, \Theta_w) = \frac{c_1(\Theta_w)^2}{2} + 2 = 2d_w + 2$$

or equivalently,

$$c_1(\Theta_w)^2 = 2\langle w, w\rangle. \tag{17}$$

Yoshioka identifies the $K3$ surface $K_v$ as a Fourier-Mukai partner of the Kummer surface $X$ associated to $A$ [Y1]. We review his construction below. To fix the notation, consider the following diagram

$$\begin{tikzcd} \bigsqcup_{i=1}^{16} E_i \arrow[r, hook] & \tilde{A} \arrow[r, "p"] \arrow[d, "\pi"] & X & \bigsqcup_{i=1}^{16} C_i \arrow[l, hook'] \\ & A & & \end{tikzcd}$$

Here $\tilde{A}$ is the blowup of $A$ at the 16 two-torsion points, with exceptional divisors denoted by $E_i$. Let

$$j : \bigsqcup_{i=1}^{16} E_i \to \tilde{A}$$

denote the inclusion of all 16 exceptional di‘visors in $\tilde{A}$. Finally, $p$ is the morphism quotienting the $\mathbb{Z}/2\mathbb{Z}$ automorphisms, and $C_i$ are the $(-2)$ curves on $X$ which are the images of the exceptional divisors $E_i$ under $p$.

Yoshioka proves that for a suitable isotropic Mukai vector $\tau$ on $X$ *i.e.*, $\langle \tau, \tau \rangle = 0$, and a suitable polarization $L$, the following isomorphism holds

$$K_v \cong \mathfrak{M}_X(\tau). \tag{18}$$

Here $\mathfrak{M}_X(\tau)$ denotes the moduli space of $L$-semistable sheaves on $X$, with Mukai vector $\tau$. We will use Yoshioka's explicit isomorphism to identify the theta bundle $\Theta_w$ on the moduli space $\mathfrak{M}_X(\tau)$.

We will explain the argument when $r + c_1(v)$ is indivisible. The remaining case when $r + c_1(v)$ equals twice a primitive class is entirely similar. Let us assume first that $r \neq 2$, and that either $r$ and $\chi(v)$ are both even, or $r$ is odd. The isomorphism (18) associates to each $F \in \mathfrak{M}_X(\tau)$ a sheaf $E$ on $A$, via elementary modifications along the exceptional divisors on the blowup $\widetilde{A}$. Concretely, the sheaf $p^\star F|_{E_i}$ splits as a sum

$$p^\star F|_{E_i} \cong \mathcal{O}_{E_i}(-1)^{\oplus a_i} \oplus \mathcal{O}_{E_i}^{\oplus (r - a_i)}, \tag{19}$$

for suitable integers $a_i$. Then, $E$ is defined by the exact sequence

$$0 \to \pi^\star E \to p^\star F \to j_\star \left( \bigoplus_{i=1}^{16} \mathcal{O}_{E_i}(-1)^{\oplus a_i} \right) \to 0. \tag{20}$$

The assignment

$$\mathfrak{M}_X(\tau) \ni F \to E \in \mathfrak{M}_v,$$

establishes an isomorphism onto the image $K_v$.

In fact, Yoshioka's construction works in families, giving a natural transformation between the moduli functors

$$\underline{\mathfrak{M}}_X(\tau) \to \underline{K}_v.$$

The exact description of this transformation will be useful later. In what follows, let us agree that the base change of various morphisms to an arbitrary base $S$ will be decorated by overlines. Fix any flat $S$-family $\mathcal{F}$ of sheaves in $\underline{\mathfrak{M}}_X(\tau)(S)$. Define the sheaf $\mathcal{G}$ on the union $\bigsqcup_{i=1}^{16} E_i \times S$ via the exact sequence

$$0 \to \bar{\pi}^\star \bar{\pi}_\star \bar{j}^\star \bar{p}^\star \mathcal{F} \to \bar{j}^\star \bar{p}^\star \mathcal{F} \to \mathcal{G} \to 0. \tag{21}$$

The short exact sequence

$$0 \to \bar{\pi}^\star \mathcal{E} \to \bar{p}^\star \mathcal{F} \to \bar{j}_\star \mathcal{G} \to 0 \tag{22}$$

then defines a new $S$-family $\mathcal{E}$ in $\underline{K}_v(S)$.

Now let

$$\zeta = \mathrm{ch}\left(p_!\left(\pi^\star W\right)\right)(1+\omega) \in H^\star(X), \tag{23}$$

be the Mukai vector of the pushforward $p_!(\pi^\star W)$, for an arbitrary sheaf $W$ on $A$ with Mukai vector $w$. Using the exact sequence (20), and the fact that

$$\chi(j_\star \mathcal{O}_{E_i}(-1) \otimes \pi^\star w) = 0,$$

we find

$$2\,\chi(\tau \otimes \zeta) = \chi(p^\star \tau \otimes \pi^\star w) = \chi(\pi^\star(v \otimes w)) = \chi(v \otimes w) = 0.$$

The above computation shows that $\zeta$ defines a Theta line bundle $\Theta_\zeta$ on $\mathfrak{M}_X(\tau)$. We claim that under the isomorphism $K_v \cong \mathfrak{M}_X(\tau)$ we have an identification

$$\Theta_w \cong \Theta_\zeta. \tag{24}$$

As for any good quotient, the Picard group of the moduli scheme $\mathfrak{M}_X(\tau)$ injects into that of the moduli functor $\underline{\mathfrak{M}}_X(\tau)$. Therefore, it suffices to check equality of the two line bundles $\Theta_w$ and $\Theta_\zeta$ over arbitrary base schemes $S$, and for arbitrary $S$-families $\mathcal{F}$ of $\underline{\mathfrak{M}}_X(\tau)$.

Let $q : S \times \tilde{A} \to S$ and $\widetilde{q} : S \times X \to S$ be the two projections. The exact sequence (22) and the push-pull formula then give

$$\Theta_w = \det \mathbf{R}q_!\left(\bar{\pi}^\star \mathcal{E} \otimes \mathrm{pr}_A^\star w\right)^{-1} = \det \mathbf{R}q_!\left(\bar{p}^\star \mathcal{F} \otimes \mathrm{pr}_A^\star w\right)^{-1} = \det \mathbf{R}\widetilde{q}_!\left(\mathcal{F} \otimes \mathrm{pr}_X^\star \zeta\right)^{-1} = \Theta_\zeta.$$

Here, we used that the contribution of the last term of (22) vanishes. Indeed, since

$$\bar{j}^\star \mathrm{pr}_A^\star w = \mathrm{rank}\ (w) \cdot 1,$$

we have

$$\det \mathbf{R}q_! \left(\bar{j}_\star \mathcal{G} \otimes \mathrm{pr}_A^\star w\right) = \mathrm{rank}\ (w) \cdot \det \mathbf{R} \left(q\bar{j}\right)_! (\mathcal{G}) = 0.$$

The last equality follows from the fact that all direct images of $\mathcal{G}$ vanish. This is implied by the base change theorem, observing that the restriction of $\mathcal{G}$ to each fiber of the morphism $q\bar{j} : S \times E_i \to S$ splits as a sum of line bundles $\mathcal{O}_{E_i}(-1)$. In turn this latter fact is a consequence of the defining exact sequence (22), in conjunction with equation (19).

To complete the proof, recall that Mukai [Muk4] established an isometric isomorphism

$$H^2(\mathfrak{M}_X(\tau)) \cong \tau^\perp/\tau$$

where the left hand side is endowed with the intersection pairing, while the right hand side carries the Mukai form induced from the cohomology $H^\star(X)$. Then,

$$c_1(\Theta_w)^2 = c_1\left(\Theta_\zeta\right)^2 = \langle \zeta, \zeta \rangle = 2\langle w, w \rangle.$$

This proves (17).

The case $r \neq 2$ and $\chi(v)$ odd is entirely similar. In this case, the exact sequence (20) is replaced by

$$0 \to \pi^\star E \to p^\star F\left(E_1\right) \to j_\star \left( \bigoplus_{i=1}^{16} \mathcal{O}_{E_i}(-1)^{\oplus a_i} \right) \to 0.$$

The argument identifying the Theta bundles carries through, for the vector

$$\zeta = \mathrm{ch}(p_!(\pi^\star W(E_1))(1+\omega). \tag{25}$$

The case $r = 2$ requires a different discussion, since in this case, the description of the isomorphism (18) via the assignment $F \to E$ is valid only on the complement of four rational curves $R_i$, $1 \leq i \leq 4$. In fact, one cannot pick an isotropic vector $\tau$ such that for each of the 16 exceptional divisors, the splitting type (19) is independent of the choice of a point $[F] \in \mathfrak{M}_X(\tau)$. At best, for a suitable $\tau$, the rigid splitting

$$p^\star F|_{E_i} \cong \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(-1)$$

holds for 12 exceptional divisors $E_i$, $5 \leq i \leq 16$. For the remaining four divisors $E_i, 1 \leq i \leq 4$, the splitting type varies within the moduli space.

When $\chi(v)$ is even, for generic $F$ in $\mathfrak{M}_X(\tau)$, the splitting type is

$$p^\star F|_{E_i} = \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i},\ 1 \leq i \leq 4. \tag{26}$$

The loci of non-generic splitting give rational curves $R_i$ in $K_v$. Indeed, the nongeneric splitting is

$$p^\star F|_{E_i} = \mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(1). \tag{27}$$

These $F$'s are shown to sit in exact sequences

$$0 \to G_i \to F \to \mathcal{O}_{C_i}(-1) \to 0 \tag{28}$$

for certain rigid stable bundles $G_i$ on the $K3$ surface $X$, cf. Lemma 4.23 in [Y1]. The nontrivial extensions (28) are parametrized by a rational curve

$$R_i \cong \mathbb{P}(\mathrm{Ext}^1_X(O_{C_i}(-1), G_i)).$$

When $\chi(v)$ is odd, all these statements are true for the exceptional divisors $E_2, E_3, E_4$, but equations (26) and (27) fail for $E_1$. In fact, generically

$$p^\star F|_{E_1} = \mathcal{O}_{E_1}(1) \oplus \mathcal{O}_{E_1}(1), \tag{29}$$

while nongenerically

$$p^\star F|_{E_1} = \mathcal{O}_{E_1} \oplus \mathcal{O}_{E_1}(2). \tag{30}$$

The nongeneric splitting occurs along the rational curve

$$R_1 = \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{C_1}, G_1))$$

parametrizing extensions of the type

$$0 \to G_1 \to F \to \mathcal{O}_{C_1} \to 0, \tag{31}$$

for some rigid vector bundle $G_1$ on $X$.

We claim that the Theta bundles agree in this case as well *i.e.*, we check that the isomorphism (24)

$$\Theta_w \cong \Theta_\zeta$$

is satisfied. Let us first discuss the case when $\chi(v)$ is even, with $\zeta$ given by (23). To begin, $\Theta_w$ and $\Theta_\zeta$ agree on the complement of the four rational curves $R_i$, $1 \leq i \leq 4$, since the exact sequence (20) is valid outside these curves. We will check that the Theta bundles agree along the curves $R_i$ as well. Precisely, we claim that

$$c_1(\Theta_w) \cdot R_i = c_1(\Theta_\zeta) \cdot R_i = s, \;\; 1 \leq i \leq 4, \tag{32}$$

with

$$s = \mathrm{rank}\ w.$$

Moreover, the four curves $R_i, 1 \leq i \leq 4$, are disjoint. These facts will establish the isomorphism (24).

To calculate the first intersection in (32), we will use the explicit description of the rational curves $R_i$ in the moduli space $K_v$. Specifically, Yoshioka notes that the curve

$R_i$ corresponds to those sheaves $E$ on $A$ which fail to be locally free at a two-torsion point $x_i$. We can construct these sheaves as elementary modifications of a fixed $V$:

$$0 \to E \to V \to \mathcal{O}_{\{x_i\}} \to 0. \tag{33}$$

The middle sheaf $V$ has Mukai vector $v + \omega$, so it sits in a moduli space of dimension 2. We may assume that $V$ is locally free at $x_i$; this may be arranged by starting with an arbitray $V$ with Mukai vector $v + \omega$, and then pulling back by a translation on the abelian variety $A$, if necessary to obtain local-freeness at $x_i$. With this understood, it follows that $E$ is not locally free at $x_i$. The nonlocally free elementary modifications $E$ are parametrized by a $\mathbb{P}^1$, which should therefore be the rational curve $R_i$ above.

The universal structure on $R_i \times A$, associated to the elementary modifications (33), becomes

$$0 \to \mathcal{E} \to \mathrm{pr}_A^\star V \to \mathcal{O}_{R_i}(1) \boxtimes \mathcal{O}_{\{x_i\}} \to 0.$$

Therefore,

$$c_1(\Theta_w) \cdot R_i = -c_1(p_!(\mathcal{E} \otimes \mathrm{pr}_A^\star w)) = c_1\left(p_!\left(\mathcal{O}_{R_i}(1) \boxtimes \left(\mathcal{O}_{\{x_i\}} \otimes w\right)\right)\right) = c_1(\mathcal{O}_{R_i}(1)^{\oplus s}) = s.$$

To prove the second equality of (32), we will use the description of the rational curves $R_i$ provided by equation (28). The universal extension

$$0 \to \mathrm{pr}_X^\star G_i \to \mathcal{F} \to \mathcal{O}_{R_i}(-1) \boxtimes \mathcal{O}_{C_i}(-1) \to 0$$

on $R_i \times X$ restricts to (28) on the fibers of the projection $p : R_i \times X \to R_i$. Using this exact sequence, we compute

$$\begin{aligned} c_1(\Theta_\zeta) \cdot R_i &= -c_1(p_!(\mathcal{F} \otimes \mathrm{pr}_X^\star \zeta)) = -c_1\left(p_!\left(\mathcal{O}_{R_i}(-1) \boxtimes (\mathcal{O}_{C_i}(-1) \otimes \zeta)\right)\right) \\ &= -c_1(\mathcal{O}_{R_i}(-1))\, \chi(\mathcal{O}_{C_i}(-1) \otimes \zeta) = c_1(\zeta) \cdot C_i = s. \end{aligned}$$

The last evaluation follows from (23) via Riemann-Roch.

When $\chi(v)$ is odd, the numerics are slightly different, but (24) still holds for the vector $\zeta$ given by (25). In this case, we check that

$$c_1(\Theta_\zeta) \cdot R_1 = s,$$

using equation (31), instead of (28).

Finally, to see that the curves $R_i$'s are disjoint, assume for a contradiction that the intersection of $R_i$ and $R_j$ contains a sheaf $E$, for $i \neq j$. It follows that $E$ fails to be locally free at two different two-torsion points. Then the quotient $E^{\vee\vee}/E$ is supported at least at two points. Thus, $E^{\vee\vee}$ has Mukai vector $v + a\omega$, for $a \geq 2$. Its self-intersection is

$$\langle v(E^{\vee\vee}), v(E^{\vee\vee}) \rangle = \langle v, v \rangle - 4a = 4 - 4a \leq -4.$$

This contradicts the Bogomolov inequality for the vector bundle $E^{\vee\vee}$, proving our claim.

This completes our analysis of the two-dimensional Albanese fibers, establishing Proposition 1.

## 4. Sheaves with fixed determinant

In this section we will prove Theorem 1. We begin by fixing the notation. Specifically, let us write $r, \Lambda, \chi$ for the rank, determinant and Euler characteristic of the vector $v$. The notation $r', \Lambda', \chi'$ will be used for the vector $w$. The orthogonality of $v$ and $w$ translates into

$$r'\chi + c_1(\Lambda) \cdot c_1(\Lambda') + r\chi' = 0. \tag{34}$$

Let $\mathcal{P}$ be the normalized Poincaré bundle on $A \times \widehat{A}$. We make the convention that $x$ will stand for a point of $A$, while $y$ will be a point of $\widehat{A}$. We will write

$$\mathcal{P}_x = \mathcal{P}|_{\{x\}\times \widehat{A}}, \;\; \mathcal{P}_y = \mathcal{P}|_{A\times\{y\}} \cong y.$$

We denote by $t_x$ and $t_y$ the translations by $x$ and $y$ on the abelian varieties $A$ and $\widehat{A}$ respectively.

The following two facts about the Fourier-Mukai transform of an arbitrary $E \in \mathbf{D}(A)$, proved in [Muk2], will be used below:

$$\mathbf{R}\mathcal{S}(E \otimes \mathcal{P}_y) = t_y^\star \mathbf{R}\mathcal{S}(E), \tag{35}$$

$$\mathbf{R}\mathcal{S}(t_x^\star E) = \mathbf{R}\mathcal{S}(E) \otimes \mathcal{P}_{-x}. \tag{36}$$

It is moreover useful to recall that the two line bundles $\Lambda$ and $\widehat{\Lambda}$ standardly induce morphisms

$$\Phi_\Lambda : A \to \widehat{A}, \;\; x \mapsto t_x^\star \Lambda \otimes \Lambda^{-1}, \;\text{ and}$$

$$\Phi_{\widehat{\Lambda}} : \widehat{A} \to A, \;\; y \to t_y^\star \widehat{\Lambda} \otimes \widehat{\Lambda}^{-1},$$

satisfying [Y1]

$$\Phi_\Lambda \circ \Phi_{\widehat{\Lambda}} = -\chi(\Lambda)1, \;\; \Phi_{\widehat{\Lambda}} \circ \Phi_\Lambda = -\chi(\Lambda)1. \tag{37}$$

To start the proof of Theorem 1, consider the diagram

$$\begin{tikzcd} K_v \times A \arrow[r, "\Phi^+"] \arrow[d, "p"] & \mathsf{M}_v^+ \arrow[d, "\alpha^-"] \\ A \arrow[r, "\Psi^+"] & A \end{tikzcd} .$$

The upper horizontal map is given by

$$\Phi^+(E,x) = t_{rx}^\star E \otimes t_x^\star \Lambda^{-1} \otimes \Lambda.$$

This is well defined since

$$\det \Phi^+(E,x) = t_{rx}^\star \Lambda \otimes \left(t_x \Lambda^{-1} \otimes \Lambda\right)^r = \Lambda.$$

**Lemma 3.** *The morphism $\Psi^+$ is the multiplication by $d_v$ in the abelian variety.*

*Proof.* Using (35) and (36), we compute

$$\begin{aligned}
\alpha^- \circ \Phi^+(E,x) &= \det \mathbf{R}\mathcal{S}(t_{rx}^\star E \otimes t_x^\star \Lambda^{-1} \otimes \Lambda) \otimes \widehat{\Lambda}^{-1} = \det \mathbf{R}\mathcal{S}(t_{rx}^\star E \otimes \mathcal{P}_{-\Phi_\Lambda(x)}) \otimes \widehat{\Lambda}^{-1} \\
&= \det \left(t_{-\Phi_\Lambda(x)}^\star \mathbf{R}\mathcal{S}(t_{rx}^\star E)\right) \otimes \widehat{\Lambda}^{-1} = t_{\Phi_\Lambda(-x)}^\star \det \mathbf{R}\mathcal{S}(t_{rx}^\star E) \otimes \widehat{\Lambda}^{-1} \\
&= t_{\Phi_\Lambda(-x)}^\star \det\left(\mathbf{R}\mathcal{S}(E) \otimes \mathcal{P}_{-rx}\right) \otimes \widehat{\Lambda}^{-1} \\
&= t_{\Phi_\Lambda(-x)}^\star \left(\det \mathbf{R}\mathcal{S}(E) \otimes \mathcal{P}_{-rx}^\chi\right) \otimes \widehat{\Lambda}^{-1} \\
&= t_{\Phi_\Lambda(-x)}^\star \widehat{\Lambda} \otimes \mathcal{P}_{-r\chi x} \otimes \widehat{\Lambda}^{-1} = \Phi_{\widehat{\Lambda}}(\Phi_\Lambda(-x)) \otimes \mathcal{P}_{-r\chi x} \\
&= \chi(\Lambda)x \otimes \mathcal{P}_{-r\chi x} = (\chi(\Lambda) - r\chi)x = d_v x.
\end{aligned}$$

The first equality on the penultimate line follows from the fact that the Poincaré bundle $\mathcal{P}_x$ is invariant under translations [M]. Equation (37) was used in the last line.

**Lemma 4.** *When $d_v \neq 0$, the diagram above is cartesian. Therefore, the morphism $\Phi^+$ has degree $d_v^4$.*

*Proof.* This is almost immediate. Together, $\Phi^+$ and $p$ give rise to a morphism

$$i : K_v \times A \to \mathsf{M}_v^+ \times_{A,(\alpha^-,\Psi^+)} A.$$

We show that $i$ is an isomorphism. Since $\Psi^+$ is étale, the natural morphism

$$\mathsf{M}_v^+ \times_{A,(\alpha^-,\Psi^+)} A \to \mathsf{M}_v^+$$

is also étale, so the fibered product $\mathsf{M}_v^+ \times_{A,(\alpha^-,\Psi^+)} A$ is smooth. The fibered product is also connected, as it follows by looking at the connected fibers of the projection to $A$; note that the projection is surjective, as $\alpha^-$ has this property, according to the previous Lemma. Therefore, it suffices to check that $i$ is injective. If $i(E,x) = i(E',x')$ then, by composing $i$ with $\Phi^+$ and $p$, we see that

$$t_{rx}^\star E \otimes t_x^\star \Lambda^{-1} \otimes \Lambda = t_{rx'}^\star E' \otimes t_{x'}^\star \Lambda^{-1} \otimes \Lambda, \text{ and } x = x'.$$

This immediately implies $E = E'$ as well. The diagram is therefore cartesian.

**Proposition 2.** *We have*

$$\left(\Phi^{+}\right)^{\star}\Theta_w \cong \Theta_w \boxtimes \mathcal{L}^{+}$$

*where $\mathcal{L}^{+}$ is a line bundle on $A$ with*

$$c_1(\mathcal{L}^{+}) = d_v c_1(v \otimes w).$$

*Proof.* This follows by the see-saw theorem. Letting $\Phi_x = \Phi^{+}|_{K_v \times \{x\}}$, we claim that the pullback $\Phi_x^{\star}\Theta_w$ is independent of $x$, and therefore, by specializing to $x = 0$, it should coincide with $\Theta_w$. Since $K_v$ is simply connected, it suffices to check that the Chern class $c_1(\Phi_x^{\star}\Theta_w)$ is independent of $x$. This is clear when a universal sheaf $\mathcal{E}$ exists on $\mathsf{M}_v^{+} \times A$. Indeed, for $F$ a sheaf on $A$ with Mukai vector $w$,

$$\Phi_x^{\star}\Theta_w = \Phi_x^{\star}\left(\det \mathbf{R}p_!(\mathcal{E} \otimes q^{\star}F)\right)^{-1} = \left(\det \mathbf{R}p_!\left((1 \times t_{rx})^{\star}\mathcal{E} \otimes q^{\star}\left(t_x^{\star}\Lambda^{-1} \otimes \Lambda \otimes F\right)\right)\right)^{-1}.$$

The first Chern class can then be computed by Grothendieck-Riemann-Roch. The answer does not depend on the point $x \in A$ since the maps $(1 \times t_{rx})^{\star}$ and $t_x^{\star}$ act as the cohomological restriction associated with $K_v \times A \hookrightarrow \mathsf{M}_v^{+} \times A$, and as the identity on the cohomology of $A$, respectively. When a universal family does not exist, one can use a quasi-universal family instead.

The above argument shows that $\left(\Phi^{+}\right)^{\star}\Theta_w$ should be of the form $\Theta_w \boxtimes \mathcal{L}^{+}$ for some line bundle $\mathcal{L}^{+}$ coming from $A$. We can express this line bundle explicitly as follows. Write

$$m = m_1 : A \times A \to A, (a, b) \to a + b$$

for the addition map, and consider the morphism

$$m_r : A \times A \to A, (a, b) \to a + rb.$$

Then,

$$m_r = m \circ (1, r).$$

Letting $p_1, p_2$ be the two projections, we have

$$\mathcal{L}^{+} = \left(\det \mathbf{R}p_{2!}\left(m_r^{\star}E \otimes m^{\star}\Lambda^{-1} \otimes p_1^{\star}(\Lambda \otimes F)\right)\right)^{-1}.$$

Letting $\lambda = c_1(\Lambda)$, we get by Grothendieck-Riemann-Roch,

$$c_1(\mathcal{L}^{+}) = -p_{2!}\left[m_r^{\star}v \cdot m^{\star}e^{-\lambda} \cdot p_1^{\star}(e^{\lambda}w)\right]_{(3)}.$$

Expanding each of the terms, we obtain

$$c_1(\mathcal{L}^{+}) = -p_{2!}\left[(r + m_r^{\star}\lambda + \chi\, m_r^{\star}\omega) \cdot \left(1 - m^{\star}\lambda + \frac{\lambda^2}{2}m^{\star}\omega\right)\right.$$
$$\left.\cdot p_1^{\star}\left(r' + \left(r'\lambda + \lambda'\right) + \left(\chi' + \lambda\lambda' + r'\frac{\lambda^2}{2}\right)\omega\right)\right]_{(3)}.$$

The precise evaluation of this product relies on the following intersections

$$p_{2!}(m^\star\lambda\cdot p_1^\star\omega)=\lambda,\ \ p_{2!}(m_r^\star\lambda\cdot p_1^\star\omega)=r^2\lambda,$$

$$p_{2!}(m_r^\star\omega\cdot m^\star\lambda)=(r-1)^2\lambda,\ \ p_{2!}(m_r^\star\lambda\cdot m^\star\omega)=(r-1)^2\lambda,$$

$$p_{2!}(m^\star\omega\cdot p_1^\star\alpha)=\alpha,\ \ p_{2!}(m_r^\star\omega\cdot p_1^\star\alpha)=r^2\alpha,\ \text{for any } \alpha\in H^2(A).$$

The last pair of intersections is to be used for the class $\alpha = r'\lambda+\lambda'$. The formulas above are easily justified either by explicit computations in coordinates, or directly, by interpreting geometrically the intersections involved. For instance, the third pushforward $p_{2!}(m_r^\star\omega\cdot m^\star\lambda)$ is computed as the image under $p_2$ of the cycle

$$\{(a,b),\ a+rb=0,\ \ a+b\in\lambda\}\hookrightarrow A\times A.$$

This pushforward can be identified with $(r-1)^\star\lambda=(r-1)^2\lambda$.

The value of the Chern class is obtained immediately from the previous intersections and a last one calculated by the Lemma below. Equation (34) has to be used to bring the answer in the form claimed by Proposition 2.

**Lemma 5.** *For any $\lambda,\alpha\in H^2(A)$, we have*

$$p_{2!}(m_r^\star\lambda\cdot m^\star\lambda\cdot p_1^\star\alpha)=(r-1)^2\left(\int_A\alpha\lambda\right)\cdot\lambda+r\lambda^2\cdot\alpha.$$

*Proof.* First, note the isomorphism

$$m^\star\Lambda\cong p_1^\star\Lambda\otimes p_2^\star\Lambda\otimes(1\times\Phi_\Lambda)^\star\mathcal{P}.$$

This shows that

$$m^\star\lambda=p_1^\star\lambda+p_2^\star\lambda+(1\times\Phi_\Lambda)^\star c_1(\mathcal{P}),\ \text{and} \tag{38}$$

$$m_r^\star\lambda=(1\times r)^\star m^\star\lambda=p_1^\star\lambda+r^2p_2^\star\lambda+r\cdot(1\times\Phi_\Lambda)^\star c_1(\mathcal{P}).$$

It follows that

$$\begin{aligned}p_{2!}(m_r^\star\lambda\cdot m^\star\lambda\cdot p_1^\star\alpha)&=(r^2+1)\left(\int_A\alpha\lambda\right)\cdot\lambda+r\cdot p_{2!}\left(p_1^\star\alpha\cdot(1\times\Phi_\Lambda)^\star c_1(\mathcal{P})^2\right)\\&=(r^2+1)\left(\int_A\alpha\lambda\right)\cdot\lambda+2r\cdot\Phi_\Lambda^\star\left\{p_{2!}\left(p_1^\star\alpha\cdot\frac{c_1(\mathcal{P})^2}{2}\right)\right\}.\end{aligned}$$

We will prove

$$\Phi_\Lambda^\star\left\{p_{2!}\left(p_1^\star\alpha\cdot\frac{c_1(\mathcal{P})^2}{2}\right)\right\}=-\left(\int_A\alpha\lambda\right)\cdot\lambda+\frac{\lambda^2}{2}\cdot\alpha. \tag{39}$$

This follows by a computation in coordinates. Explicitly, let us write $A=V/\Gamma$. We regard $V$ as a four-dimensional real vector space. The dual abelian variety has as underlying *real* manifold the torus $V^\vee/\Gamma^\vee$, where $V^\vee$ stands for the *real* dual of $V$. Pick a basis $f_1, f_2, f_3, f_4$ for $V$, which is symplectic for $\Lambda$. This means that in the dual basis,

$$\lambda = c_1(\Lambda) = d \cdot f_1^\vee \wedge f_2^\vee + e \cdot f_3^\vee \wedge f_4^\vee \in \Lambda^2 V^\vee,$$

for some (integers) $d$ and $e$. Moreover, the Chern class of the Poincaré line bundle on $A \times \widehat{A}$ takes the form

$$c_1(\mathcal{P}) = f_1^\vee \wedge f_1 + f_2^\vee \wedge f_2 + f_3^\vee \wedge f_3 + f_4^\vee \wedge f_4. \tag{40}$$

Note that pullbacks of the 1-forms $f_i^\vee$ and $f_i$ from the two factors $A$ and $\widehat{A}$ to their product are understood in the above equation. Throughout the paper, it should be clear from the context when such notational simplifications are employed.

To prove (39), it suffices to assume that

$$\alpha = f_1^\vee \wedge f_2^\vee, \text{ or } \alpha = f_1^\vee \wedge f_3^\vee.$$

Let us consider only the first case, the second being similar. Then,

$$p_{2!}\left(p_1^\star \alpha \cdot \frac{c_1(\mathcal{P})^2}{2}\right) = -f_3 \wedge f_4.$$

The discussion in [LB], chapter 2, and in particular Lemma 4.5 therein, shows that the map

$$\Phi_\Lambda^\star : H^1(\widehat{A}, \mathbb{R}) \cong V \to H^1(A, \mathbb{R}) \cong V^\vee$$

is induced by the contraction of the first Chern class $c_1(\Lambda)$. It follows that

$$\Phi_\Lambda^\star \left\{p_{2!}\left(p_1^\star \alpha \cdot \frac{c_1(\mathcal{P})^2}{2}\right)\right\} = -\Phi_\Lambda^\star f_3 \wedge \Phi_\Lambda^\star f_4 = -e^2\, f_3^\vee \wedge f_4^\vee.$$

But this is also the result on the right hand side of (39):

$$-\left(\int_A \alpha\lambda\right) \cdot \lambda + \frac{\lambda^2}{2} \cdot \alpha = -e \cdot \lambda + de \cdot \alpha = -e^2\, f_3^\vee \wedge f_4^\vee.$$

*Proof of Theorem 1.* When $d_v \neq 0$, Theorem 1 follows immediately from Propositions 1 and 2, and Lemma 4. Indeed, we have

$$\chi(\mathsf{M}_v^+, \Theta_w) = \frac{1}{d_v^4}\chi((\Phi^+)^\star\, \Theta_w) = \frac{1}{d_v^4}\chi(K_v, \Theta_w)\, \chi(A, \mathcal{L}^+)$$

$$= \frac{1}{d_v^4} \cdot \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v} \cdot \frac{(d_v c_1(v \otimes w))^2}{2} = \frac{1}{2}\frac{c_1(v\otimes w)^2}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

When $d_v = 0$, the Theorem is equivalent to the equality

$$\chi(\mathsf{M}_v^+, \Theta_w) = r^2.$$

It suffices to explain that the moduli space $\mathsf{M}_v^+$ consists of $r^2$ smooth points. By work of Mukai, it is known that $\mathfrak{M}_v$ is an abelian surface. In fact, fixing $E \in \mathsf{M}_v^+$, we have an isogeny

$$A \to \mathfrak{M}_v, \;\; x \to t_x^\star E,$$

whose kernel is the group

$$K(E) = \{x \in A \text{ such that } t_x^\star E \cong E\}\,.$$

Note that we may need to replace $E$ by a twist $E \otimes H^{\otimes n}$ to ensure that $K(E)$ is finite. In this case, $K(E)$ has $\chi^2$ elements. This is a result of Mukai [Muk1]; to apply it, we need to observe that $E$ is a simple semi-homogeneous sheaf. Restricting to sheaves with determinant $\Lambda$, we see that

$$\mathsf{M}_v^+ \cong K(\Lambda)/K(E),$$

has length $\frac{\chi(\Lambda)^2}{\chi^2} = r^2$.

## 5. Sheaves with fixed determinant of the Fourier-Mukai transform.

This section is devoted to the proof of Theorem 2. It is possible to deduce this result from Theorem 1 when the Picard number of $A$ is 1, by explicitly studying how the relevant moduli spaces and Theta divisors change under the Fourier-Mukai transform [Y1][Ma]. However, the following proof is simpler, covers all cases, and it is in the spirit of this paper. Note that the cohomological computation below may be regarded as the Fourier-Mukai 'dual' of last section's calculations.

We will crucially make use of the diagram

$$\begin{tikzcd} K_v \times \widehat{A} \arrow[r, "\Phi^-"] \arrow[d, "p"] & \mathsf{M}_v^- \arrow[d, "\alpha^+"] \\ \widehat{A} \arrow[r, "\Psi^-"] & \widehat{A} \end{tikzcd} .$$

The upper horizontal morphism $\Phi^-$ is defined as

$$\Phi^-(E, y) = t^\star_{\Phi_{\widehat{\Lambda}}(y)} E \otimes y^\chi.$$

To check that $\Phi^-$ is well defined, we compute

$$\begin{aligned} \det \mathbf{R}\mathcal{S}\left(\Phi^-(E,y)\right) &= \det \mathbf{R}\mathcal{S}\left(t^\star_{\Phi_{\widehat{\Lambda}}(y)} E \otimes y^\chi\right) = \det\left(t^\star_{\chi y}\, \mathbf{R}\mathcal{S}(E) \otimes \Phi_{\widehat{\Lambda}}(y)^{-1}\right) \\ &= t^\star_{\chi y} \det \mathbf{R}\mathcal{S}(E) \otimes \Phi_{\widehat{\Lambda}}(y)^{-\chi} = t^\star_{\chi y} \widehat{\Lambda} \otimes \Phi_{\widehat{\Lambda}}(y)^{-\chi} \\ &= \widehat{\Lambda} \otimes \Phi_{\widehat{\Lambda}}(\chi y) \otimes \Phi_{\widehat{\Lambda}}(y)^{-\chi} = \widehat{\Lambda}. \end{aligned}$$

The next two results are the versions of Lemma 3 and Proposition 2 suitable to the present context.

**Lemma 6.** *The morphism $\Psi^-$ is the multiplication by $-d_v$ on the abelian variety $\widehat{A}$.*

*Proof.* Using (36) and (37), we compute

$$\alpha^+ \circ \Phi^-(E,y) = \det\left(t^\star_{\Phi_{\widehat{\Lambda}}(y)}E \otimes y^\chi\right) \otimes \Lambda^{-1} = t^\star_{\Phi_{\widehat{\Lambda}}(y)}\Lambda \otimes y^{r\chi} \otimes \Lambda^{-1}$$
$$= \Phi_\Lambda\left(\Phi_{\widehat{\Lambda}}(y)\right) \otimes y^{r\chi} = (-\chi(\Lambda) + r\chi)y = -d_v y.$$

**Proposition 3.** *We have*

$$\left(\Phi^-\right)^\star \Theta_w \cong \Theta_w \boxtimes \mathcal{L}^-,$$

*where*

$$c_1(\mathcal{L}^-) = -d_v c_1(\hat{v} \otimes \hat{w}).$$

*Proof.* The proof of this result parallels that of Proposition 2. It suffices to show that the line bundle $\mathcal{L}^-$ corresponding to the divisor

$$\left\{y \in \widehat{A}, \text{ with } H^0\left(t^\star_{\widehat{\Lambda}(y)}(E) \otimes y^\chi \otimes F\right) \neq 0\right\}$$

has the first Chern class given by the Proposition. Note that

$$\mathcal{L}^- = \left(\det p_{2!}\left(f^\star E \otimes p_1^\star F \otimes \mathcal{P}^\chi\right)\right)^{-1},$$

where

$$f : A \times \widehat{A} \to A \times A \to A$$

denotes the composition

$$f = m \circ (1 \times \Phi_{\widehat{\Lambda}}),\ (x,y) \to x + \Phi_{\widehat{\Lambda}}(y). \tag{41}$$

By Riemann-Roch, we compute

$$c_1(\mathcal{L}^-) = -p_{2!}\left[(r + f^\star\lambda + \chi f^\star\omega) \cdot \left(r' + p_1^\star\lambda' + \chi' p_1^\star\omega\right) \cdot \left(1 + \chi c_1(\mathcal{P}) + \chi^2 \frac{c_1(\mathcal{P})^2}{2}\right)\right]_{(3)}.$$

The following observations allow for the explicit evaluation of the expression above:

$$p_{2!}\left(\frac{c_1(\mathcal{P})^2}{2} \cdot p_1^\star\lambda'\right) = \widehat{\lambda}',\ p_{2!}\left(\frac{c_1(\mathcal{P})^2}{2} \cdot f^\star\lambda\right) = \widehat{\lambda},$$

$$p_{2!}(f^\star\omega \cdot p_1^\star\lambda') = \frac{\lambda^2}{2} \cdot \widehat{\lambda}' - (\lambda \cdot \lambda') \cdot \widehat{\lambda}, \tag{42}$$

$$p_{2!}\left(f^\star\lambda \cdot p_1^\star\omega\right) = -\frac{\lambda^2}{2} \cdot \widehat{\lambda}, \tag{43}$$

$$p_{2!}\left(f^\star\omega \cdot c_1(\mathcal{P})\right) = -2\widehat{\lambda}, \tag{44}$$

$$p_{2!}\left(f^{\star}\lambda\cdot p_1^{\star}\lambda'\cdot c_1(\mathcal{P})\right)=-\lambda^2\cdot\widehat{\lambda}'. \tag{45}$$

The Proposition follows by substitution, also making straightforward use of the orthogonality constraint

$$r\chi'+\lambda\cdot\lambda'+r'\chi=0.$$

It remains to explain the four numbered equations claimed above. Let us first consider (42). Interpreting the pushforward geometrically, and recalling the definiton of $f$ in (41), we find that

$$p_{2!}(f^{\star}\omega\cdot p_1^{\star}\lambda')=(-\Phi_{\widehat{\Lambda}})^{\star}\lambda'=\Phi_{\widehat{\Lambda}}^{\star}\lambda'=\frac{\lambda^2}{2}\cdot\widehat{\lambda}'-\left(\int_A\lambda\cdot\lambda'\right)\cdot\widehat{\lambda}.$$

The dual of the last equality was verified in (39). The case at hand is a corollary of what we have already shown there, using the fact that the Fourier-Mukai transform is an isometry. Equation (43) is very similar. To prove it, we observe that $f$ restricts to $\Phi_{\widehat{\Lambda}}$ on $\{0\}\times\widehat{A}$, hence

$$p_{2!}\left(f^{\star}\lambda\cdot p_1^{\star}\omega\right)=\Phi_{\widehat{\Lambda}}^{\star}\lambda=-\frac{\lambda^2}{2}\cdot\widehat{\lambda}.$$

In turn, (44) follows by a computation in local coordinates. First, pick a basis $f_1,f_2,f_3,f_4$ for $V$ such that

$$c_1(\mathcal{P})=f_1^{\vee}\wedge f_1+f_2^{\vee}\wedge f_2+f_3^{\vee}\wedge f_3+f_4^{\vee}\wedge f_4.$$

From the definition of $f$ in (41), we calculate

$$p_{2!}(f^{\star}\omega\cdot c_1(\mathcal{P}))=p_{2!}\left((1\times\Phi_{\widehat{\Lambda}})^{\star}m^{\star}\omega\cdot c_1(\mathcal{P})\right)=$$

$$=-p_{2!}\left((1\times\Phi_{\widehat{\Lambda}})^{\star}\left(\sum_{j=1}^{4}\mathrm{PD}(f_j^{\vee})\wedge f_j^{\vee}\right)\cdot\left(\sum_{j=1}^{4}f_j^{\vee}\wedge f_j\right)\right)=\sum_{j=1}^{4}\Phi_{\widehat{\Lambda}}^{\star}f_j^{\vee}\wedge f_j.$$

Taking

$$\lambda=d\cdot f_1^{\vee}\wedge f_2^{\vee}+e\cdot f_3^{\vee}\wedge f_4^{\vee},$$

this last expression is

$$2d\cdot f_3\wedge f_4+2e\cdot f_1\wedge f_2=-2\widehat{\lambda}, \tag{46}$$

confirming (44).

Finally, for (45), we observe that

$$\begin{aligned}
p_{2!}\left(f^{\star}\lambda\cdot p_1^{\star}\lambda'\cdot c_1(\mathcal{P})\right) &= p_{2!}\left((1\times\Phi_{\widehat{\Lambda}})^{\star}m^{\star}\omega\cdot p_1^{\star}\lambda'\cdot c_1(\mathcal{P})\right)\\
&= p_{2!}\left((1\times\Phi_{\widehat{\Lambda}})^{\star}(1\times\Phi_{\Lambda})^{\star}c_1(\mathcal{P})\cdot p_1^{\star}\lambda'\cdot c_1(\mathcal{P})\right)\\
&= p_{2!}\left((1\times(-\chi(\Lambda)))^{\star}\,c_1(\mathcal{P})\cdot p_1^{\star}\lambda'\cdot c_1(\mathcal{P})\right)\\
&= -\chi(\Lambda)\cdot p_{2!}(c_1(\mathcal{P})^2\cdot p_1^{\star}\lambda')=\lambda^2\cdot\widehat{\lambda}'.
\end{aligned}$$

The first line follows by the definition of $f$ in (41), the second uses (38), while the third uses (37).

*Proof of Theorem 2.* As before, when $d_v \neq 0$, the Theorem follows immediately from Propositions 1 and 3, and Lemma 6. Using the cartesian diagram, we compute

$$\chi(\mathsf{M}_v^-, \Theta_w) = \frac{1}{d_v^4}\chi\big(\big(\Phi^-\big)^\star \Theta_w\big) = \frac{1}{d_v^4}\chi(K_v, \Theta_w)\,\chi(A, \mathcal{L}^-)$$

$$= \frac{1}{d_v^4} \cdot \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v} \cdot \frac{\big(d_v c_1(\hat{v} \otimes \hat{w})\big)^2}{2} = \frac{1}{2}\frac{c_1(\hat{v} \otimes \hat{w})^2}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

When $d_v = 0$, we observe that $\mathsf{M}_v^-$ consists of $\chi^2$ smooth points. First, for any sheaf $E$ in the moduli space $\mathfrak{M}_v$, consider the isogeny

$$\widehat{A} \to \mathfrak{M}_v,\ y \mapsto E \otimes \mathcal{P}_y.$$

The kernel

$$\Sigma(E) = \Big\{ y \in \widehat{A} \text{ such that } E \otimes \mathcal{P}_y \cong E \Big\}$$

has length $r^2$, cf. [Muk1] (as in the proof of Theorem 1, twisting by powers of $H$ may be necessary). Note that the points in $\mathsf{M}_v^-$ have the property

$$\det \mathbf{R}\mathcal{S}\,(E \otimes \mathcal{P}_y) \otimes (\det \mathbf{R}\mathcal{S} E)^{-1} \cong t_y^\star \widehat{\Lambda} \otimes \widehat{\Lambda}^{-1} \cong \mathcal{O}.$$

Therefore,

$$\mathsf{M}_v^- = K(\widehat{\Lambda})/\Sigma(E)$$

has length $\chi(\widehat{\Lambda})^2/r^2 = \chi^2$, as claimed.

## 6. Sheaves with arbitrary determinant.

This last section contains the proof of Theorem 3. The Euler characteristic on $K_v$ was calculated in Proposition 1. To compute the one on $\mathfrak{M}_w$, we use the diagram

$$\begin{tikzcd} K_w \times A \times \widehat{A} \arrow[r, "\Phi"] \arrow[d, "p"] & \mathfrak{M}_w \arrow[d, "\mathsf{a}_w"] \\ A \times \widehat{A} \arrow[r, "\Psi"] & A \times \widehat{A} \end{tikzcd}.$$

Here, $\Phi : K_w \times A \times \widehat{A} \to \mathfrak{M}_w$ is defined as

$$\Phi(E, x, y) = t_x^\star E \otimes y.$$

Using (35) and (36), Yoshioka proved in detail that

$$\Psi(x, y) = (-\chi' x + \Phi_{\widehat{\Lambda}'}(y), \Phi_{\Lambda'}(x) + r' y),$$

which has degree $d_w^4$ [Y1].

**Proposition 4.** *We have*

$$\Phi^\star\Theta_v = \Theta_v \boxtimes \mathcal{L}$$

*where $\mathcal{L}$ is a line bundle on $A \times \widehat{A}$ with*

$$\chi(\mathcal{L}) = d_v^2 d_w^2.$$

*Proof.* It suffices to compute the Euler characteristic of the line bundle $\mathcal{L}$ corresponding to the divisor

$$\{(x, y) \in A \times \widehat{A}, \text{ such that } H^0(t_x^\star E \otimes y \otimes F) \neq 0\}.$$

In other words

$$\mathcal{L} = \left(\det p_{23!}\left(m_{12}^\star E \otimes p_{13}^\star \mathcal{P} \otimes p_1^\star F\right)\right)^{-1},$$

where the $p$'s denote the projections on the corresponding factors of $A \times A \times \widehat{A}$, while

$$m_{12} : A \times A \times \widehat{A} \to A$$

is the addition on the first two factors. Keeping the previous notations,

$$c_1(\mathcal{L}) = -p_{23!}\left[(r + m_{12}^\star\lambda + \chi m_{12}^\star\omega) \cdot \left(1 + p_{13}^\star c_1(\mathcal{P}) + \frac{p_{13}^\star c_1(\mathcal{P})^2}{2}\right) \cdot \left(r' + p_1^\star\lambda' + \chi' p_1^\star\omega\right)\right]_{(3)}.$$

Expanding, we easily obtain

$$-c_1(\mathcal{L}) = (\chi\lambda' + \chi'\lambda) + (r\widehat{\lambda}' + r'\widehat{\lambda}) - r\chi' c_1(\mathcal{P}) + p_{23!}\left(m_{12}^\star\lambda \cdot p_{13}^\star c_1(\mathcal{P}) \cdot p_1^\star\lambda'\right).$$

We claim that

$$\chi(\mathcal{L}) = \frac{c_1(\mathcal{L})^4}{4!} = d_v^2 d_w^2.$$

The computation makes use of the fact that the Picard number of $A$ is 1, so we may assume that either $\lambda' = 0$, or otherwise that

$$\lambda = a\lambda'$$

for some constant $a$. In the first case, we have

$$c_1(\mathcal{L}) = -\chi'\lambda - r'\widehat{\lambda} + r\chi' c_1(\mathcal{P}).$$

To prove the claim, we first note that

$$\lambda \cdot \widehat{\lambda} \cdot \frac{c_1(\mathcal{P})^2}{2} = \lambda^2 \tag{47}$$

This follows easily by a computation in local coordinates. Indeed, writing

$$\lambda = d \cdot f_1^\vee \wedge f_2^\vee + e \cdot f_3^\vee \wedge f_4^\vee, \tag{48}$$

and recalling that $\widehat{\lambda}$ and $c_1(\mathcal{P})$ have the form (46) and (40) respectively, we calculate

$$\lambda \cdot \widehat{\lambda} \cdot \frac{c_1(\mathcal{P})^2}{2} = 2de = \lambda^2.$$

With (47) understood, and using the fact that the Fourier-Mukai is an isometry, we obtain

$$\frac{c_1(\mathcal{L})^4}{4!} = \frac{1}{4!}(\chi'\lambda + r'\widehat{\lambda} - r\chi' c_1(\mathcal{P}))^4 = \frac{r'^2\chi'^2(\lambda^2)^2}{4} + r'\chi'(r\chi')^2\lambda^2 + (r\chi')^4$$
$$= \left(\frac{r'\chi'\lambda^2}{2} + (r\chi')^2\right)^2 = \left[r'\chi'\left(\frac{\lambda^2}{2} - r\chi\right)\right]^2 = d_v^2 d_w^2.$$

The penultimate equality made use of the fact that $r\chi' + r'\chi = 0$.

Finally, the more general second case $\lambda = a\lambda'$ is similar. Using (38), we get

$$p_{23!}\left(m_{12}^\star\lambda \cdot p_{13}^\star c_1(\mathcal{P}) \cdot p_1^\star\lambda'\right) = (\Phi_\Lambda \times 1)^\star q_{23!}\left(q_{12}^\star c_1(\mathcal{P}) \cdot q_{13}^\star c_1(\mathcal{P}) \cdot q_1^\star\lambda'\right),$$

with the $q$'s standing for the projections of the factors of $A \times \widehat{A} \times \widehat{A}$. In turn, we claim that

$$(\Phi_\Lambda \times 1)^\star q_{23!}\left(q_{12}^\star c_1(\mathcal{P}) \cdot q_{13}^\star c_1(\mathcal{P}) \cdot q_1^\star\lambda\right) = -\frac{\lambda^2}{2} c_1(\mathcal{P}). \tag{49}$$

Again, this is easiest to check in local coordinates. Assuming that (48) holds, we have

$$q_{23!}\left(q_{12}^\star c_1(\mathcal{P}) \cdot q_{13}^\star c_1(\mathcal{P}) \cdot q_1^\star\lambda\right) = -d \cdot f_3 \wedge f_4 + d \cdot f_4 \wedge f_3 - e \cdot f_1 \wedge f_2 + e \cdot f_2 \wedge f_1.$$

Note that here we follow our previous notational conventions; in particular, pullbacks of the 1-forms $f_i$ from the two factors of $\widehat{A} \times \widehat{A}$ are understood in the above wedge products. After pullback by $\Phi_\Lambda \times 1$, the left hand side of (49) becomes

$$-de\, c_1(\mathcal{P}) = -\frac{\lambda^2}{2} c_1(\mathcal{P}),$$

as claimed. Putting things together, we obtain

$$c_1(\mathcal{L}) = -(\chi' a + \chi)\lambda' - (r'a + r)\widehat{\lambda}' + \left(r\chi' + \frac{a\lambda'^2}{2}\right) c_1(\mathcal{P}).$$

The same type of calculation as the one done above yields the answer

$$\chi(\mathcal{L}) = \frac{c_1(\mathcal{L})^4}{4!} = \left[\frac{(\chi' a + \chi)(r'a + r)\lambda'^2}{2} + \left(r\chi' + \frac{a\lambda'^2}{2}\right)^2\right]^2.$$

To conclude the proof, it remains to observe that the expression in square brackets can be equated with

$$-\left(\frac{a^2\lambda'^2}{2} - r\chi\right)\left(\frac{\lambda'^2}{2} - r'\chi'\right) = -d_v d_w,$$

so that

$$\chi(\mathcal{L}) = (d_v d_w)^2.$$

The latter algebraic manipulation will be left to the reader, who may wish to use the fact that

$$a\lambda'^2 + r\chi' + r'\chi = 0.$$

It is very likely that the Lemma holds true for arbitrary abelian surfaces, without any restrictions on the Néron-Severi group, but the computation seems to be more involved.

*Proof of Theorem 3.* We compute

$$\chi(\mathfrak{M}_w, \Theta_v) = \frac{1}{d_w^4}\chi(K_w, \Theta_v)\chi(A \times \widehat{A}, \mathcal{L}) = \frac{1}{d_w^4} \cdot \frac{d_w^2}{d_v + d_w}\binom{d_v + d_w}{d_v} \cdot \left(d_v^2 d_w^2\right)$$
$$= \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v} = \chi(K_v, \Theta_w).$$

The case $d_w = 0$ requires, as usual, special care. We need to show

$$\chi(\mathfrak{M}_w, \Theta_v) = d_v.$$

Using the degree $\chi'^2$ isogeny:

$$\pi : A \to \mathfrak{M}_w, x \to t_x^\star F,$$

where $F$ is a semi-homogeneous sheaf of Mukai vector $w$, we have

$$\pi^\star \Theta_v = \left(\det p_!(m^\star F \otimes q^\star E)\right)^{-1},$$

with $p$, $q$ and $m$ standing for the projection and addition morphism. Then

$$c_1(\pi^\star \Theta_v) = -\chi'\lambda - \chi\lambda'.$$

We obtain

$$\chi(\mathfrak{M}_w, \Theta_v) = \frac{1}{\chi'^2}\chi(A, \pi^\star\Theta_v) = \frac{1}{2\chi'^2}(\chi'\lambda + \chi\lambda')^2 = \frac{1}{2\chi'^2}\left(\chi'^2\lambda^2 + \chi^2\lambda'^2 - 2\chi\chi'(r\chi' + r'\chi)\right)$$
$$= \frac{1}{\chi'^2}(\chi^2 d_w + \chi'^2 d_v) = d_v.$$

This completes the proof of the Theorem.

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