

ON THE STRANGE DUALITY CONJECTURE FOR ABELIAN SURFACES II

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ABSTRACT. In the prequel to this paper, two versions of Le Potier's strange duality conjecture for sheaves over abelian surfaces were studied. A third version is considered here. In the current setup, the isomorphism involves moduli spaces of sheaves with fixed determinant and fixed determinant of the Fourier-Mukai transform on one side, and moduli spaces where both determinants vary, on the other side. We first establish the isomorphism in rank one using the representation theory of Heisenberg groups. For product abelian surfaces, the isomorphism is then shown to hold for sheaves with fiber degree 1 via Fourier-Mukai techniques. By degeneration to product geometries, the duality is obtained generically for a large number of numerical types. Finally, it is shown in great generality that the Verlinde sheaves encoding the variation of the spaces of theta functions are locally free over moduli.

1. INTRODUCTION

Three versions of Le Potier's strange duality conjecture were formulated in [MO1] for a polarized abelian surface (X, H) . We recall them briefly.

For a sheaf $E \rightarrow X$, we write

$$v(E) = \text{ch } E \in H^{2*}(X, \mathbb{Z})$$

for its Mukai vector. For two Mukai vectors

$$v = (v_0, v_2, v_4), \quad w = (w_0, w_2, w_4) \in H^{2*}(X, \mathbb{Z}),$$

the Mukai pairing is given by

$$\langle v, w \rangle = \int_X v_2 w_2 - v_0 w_4 - v_4 w_0.$$

We also set standardly

$$v^\vee = (v_0, -v_2, v_4) \in H^{2*}(X, \mathbb{Z}).$$

Let M_v be the moduli space of Gieseker H -semistable sheaves with Mukai vector v . When v is primitive and the polarization H is generic, the moduli space M_v consists of stable sheaves only, and is smooth projective of dimension

$$\dim M_v = 2d_v + 2, \quad \text{where } d_v = \frac{1}{2} \langle v, v \rangle.$$

We will make this assumption about the moduli spaces throughout the paper, unless specified otherwise. We furthermore consider three subspaces of M_v :

- the space M_v^+ of sheaves with a fixed determinant line bundle;
- the space M_v^- of sheaves with fixed determinant of their Fourier-Mukai transform;
- the space K_v of sheaves for which both the determinant and the determinant of their Fourier-Mukai transform is fixed.

In introducing the spaces M_v^- , K_v , we use the Fourier-Mukai transform

$$\mathbf{RS} : \mathbf{D}(X) \longrightarrow \mathbf{D}(\widehat{X})$$

with respect to the standardly normalized Poincaré line bundle

$$\mathcal{P} \rightarrow X \times \widehat{X}.$$

The moduli space K_v is precisely the fiber of the Albanese map

$$\mathbf{a} : M_v \rightarrow X \times \widehat{X}.$$

The morphism \mathbf{a} is defined up to the choice of a reference sheaf E_0 of type v . Specifically,

$$\mathbf{a}(E) = (\det \mathbf{RS}(E) \otimes \det \mathbf{RS}(E_0)^\vee, \det E \otimes \det E_0^\vee).$$

Consider now two Mukai vectors v and w , orthogonal in the sense that

$$\langle v^\vee, w \rangle = -\chi(X, v \cdot w) = 0.$$

A sheaf $F \rightarrow X$ with Mukai vector

$$w = \text{ch}(F) \in H^{2*}(X, \mathbb{Z})$$

gives rise to a line bundle

$$\Theta_F \rightarrow M_v$$

by the standard determinant construction described in [LeP2], [Li]. Specifically, if a universal family $\mathcal{E} \rightarrow M_v \times X$ exists, we set

$$(1) \quad \Theta_F = \det \mathbf{R}p_!(\mathcal{E} \otimes q^*F)^{-1} \rightarrow M_v,$$

where p, q are the two projections. By restriction, one gets line bundles on each of the subspaces M_v^+ , M_v^- , K_v .

Within a fixed Mukai class w , for each of the four moduli spaces considered, the dependence of the determinant line bundle on F takes a particular form, as explained in [MO1]:

- on K_v , the line bundle $\Theta_F = \Theta_w$ depends only on the Mukai class w of F ;
- on M_v^+ , the line bundle Θ_F is constant as long as the determinant of F is fixed;

- on \mathbf{M}_v^- , the line bundle Θ_F is constant as long the determinant of the Fourier-Mukai transform of F is fixed;
- on \mathbf{M}_v , the line bundle Θ_F is constant as long as F has both its determinant and its FM-transform determinant fixed.

Keeping these variations in mind, we write Θ_w for the determinant line bundle on each of the four moduli spaces, suitably understood. The distinctions above are further highlighted by the numerical equalities, cf. [MO1]:

$$(2) \quad \begin{aligned} \chi(\mathbf{K}_v, \Theta_w) &= \chi(\mathbf{M}_w, \Theta_v) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}, \\ \chi(\mathbf{M}_v^+, \Theta_w) &= \chi(\mathbf{M}_w^+, \Theta_v) = \frac{1}{2} \frac{c_1(v \otimes w)^2}{d_v + d_w} \binom{d_v + d_w}{d_v}, \\ \chi(\mathbf{M}_v^-, \Theta_w) &= \chi(\mathbf{M}_w^-, \Theta_v) = \frac{1}{2} \frac{c_1(\hat{v} \otimes \hat{w})^2}{d_v + d_w} \binom{d_v + d_w}{d_v}. \end{aligned}$$

Here, \hat{v} and \hat{w} denote the cohomological Fourier-Mukai transforms of v and w .

We would like to understand these numerical coincidences geometrically. The last two equalities were studied in [MO2]. Following Le Potier's original strange duality proposal, formulated for sheaves over the projective plane [LeP1], in the abelian case one sets up the Brill-Noether divisors

$$\Theta^+ = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathbf{M}_v^+ \times \mathbf{M}_w^+$$

and

$$\Theta^- = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathbf{M}_v^- \times \mathbf{M}_w^-.$$

The loci Θ^\pm are cut out by sections of the split line bundles

$$\mathcal{O}(\Theta^\pm) = \Theta_w \boxtimes \Theta_v$$

over $\mathbf{M}_v^\pm \times \mathbf{M}_w^\pm$; the see-saw theorem is used to justify the splitting above. In turn, these sections, viewed as elements in

$$H^0(\mathbf{M}_v^\pm \times \mathbf{M}_w^\pm, \Theta_w \boxtimes \Theta_v) = H^0(\mathbf{M}_v^\pm, \Theta_w) \otimes H^0(\mathbf{M}_w^\pm, \Theta_v),$$

yield maps

$$D^\pm : H^0(\mathbf{M}_v^\pm, \Theta_w)^\vee \longrightarrow H^0(\mathbf{M}_w^\pm, \Theta_v).$$

The main result of [MO2] is that D^\pm are isomorphisms for infinitely many Mukai vectors v and w and for abelian surfaces (X, H) which are products of elliptic curves.

In this paper we focus on the third possible geometry, associated with the divisor

$$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathbf{K}_v \times \mathbf{M}_w,$$

yielding the duality map

$$D : H^0(K_v, \Theta_w)^\vee \rightarrow H^0(M_w, \Theta_v).$$

The current setting is particularly interesting since it exhibits the fixed versus unfixed determinant asymmetry also present for moduli spaces of bundles over curves [Bea]. In this asymmetric setup, we establish the duality isomorphism generically for a large class of Mukai vectors v and w , as captured in our main Theorem 3 below. We now explain the salient points of the argument and state the most important results along the way.

The starting point is the case when v and w are Mukai vectors of rank 1. For each integer $a > 0$, we let $X^{[a]}$ be the Hilbert scheme of a points on X , and let

$$K^{[a]} \subset X^{[a]}$$

be the generalized Kummer variety of a points adding to zero on X . When rank $v =$ rank $w = 1$, we have

$$K_v \simeq K^{[a]}, \quad M_w \simeq X^{[b]} \times \widehat{X},$$

for suitable a, b . In this setup, we prove

Theorem 1. *Let $L \rightarrow X$ be an ample line bundle on an arbitrary abelian surface. Write $\chi(X, L) = \chi = a + b$ for positive integers a and b . The divisor*

$$\Theta_L = \{(I_Z, I_W, y) \text{ with } H^0(I_Z \otimes I_W \otimes y \otimes L) \neq 0\} \subset K^{[a]} \times X^{[b]} \times \widehat{X}$$

induces an isomorphism

$$D_L : H^0(K^{[a]}, \Theta_v)^\vee \longrightarrow H^0(X^{[b]} \times \widehat{X}, \Theta_w).$$

The analogous isomorphism when both sides involve the Hilbert schemes $X^{[a]}$ and $X^{[b]}$ and the theta bundles over them was shown to hold for all surfaces in [MO3]. By contrast, Theorem 1 is a subtler statement specific to abelian surfaces. Its proof requires new ideas and is obtained using the representation theory of the Heisenberg group.

Paralleling [MOY] and [MO2], the above result implies strange duality for product abelian surfaces via Fourier-Mukai techniques. Specifically, for moduli spaces of sheaves which are stable with respect to a suitable polarization in the sense of Friedman [F], we show

Theorem 2. *Let $X = B \times F$ be a product abelian surface. Assume v and w are two orthogonal Mukai vectors of ranks $r, r' \geq 2$ with*

$$c_1(v) \cdot f = c_1(w) \cdot f = 1.$$

Then, the locus

$$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathbf{K}_v \times \mathbf{M}_w$$

is a divisor, and induces an isomorphism

$$D : H^0(\mathbf{K}_v, \Theta_w)^\vee \rightarrow H^0(\mathbf{M}_w, \Theta_v).$$

In order to move from the product geometry of Theorem 2 to a generic abelian surface, we study the Verlinde sheaves

$$\mathbb{V}, \mathbb{W} \rightarrow \mathcal{A}.$$

These are defined in Section 4, and encode the spaces of generalized theta functions $H^0(\mathbf{K}_v, \Theta_w)$ and $H^0(\mathbf{M}_w, \Theta_v)$ respectively, as the pair (X, H) varies in its moduli space \mathcal{A} . We need to ensure that the Verlinde sheaves are generically locally free of expected rank given by the holomorphic Euler characteristics (2):

$$\text{rank } \mathbb{V} = \text{rank } \mathbb{W} = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

We establish this in our situation by showing that for surfaces of Néron-Severi rank 1 the theta line bundles are big and nef, and therefore carry no higher cohomology. This yields the following generic strange duality statement, which constitutes our main result.

Theorem 3. *Assume (X, H) is a generic primitively polarized abelian surface, and v, w are two orthogonal Mukai vectors of ranks $r, r' \geq 2$ with*

- (i) $c_1(v) = c_1(w) = H$;
- (ii) $\chi(v) < 0, \chi(w) < 0$.

Then, the locus

$$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathbf{K}_v \times \mathbf{M}_w$$

is a divisor, and induces an isomorphism

$$D : H^0(\mathbf{K}_v, \Theta_w)^\vee \longrightarrow H^0(\mathbf{M}_w, \Theta_v).$$

While the statements of Theorems 2 and 3 mirror the $K3$ and abelian cases studied in [MOY] and [MO2], different arguments are needed in the current *asymmetric* abelian setup. Several technical assumptions present in [MOY] and [MO2] are in addition removed, yielding stronger results.

Finally, in Section 6 we show in great generality that the Verlinde sheaves

$$\mathbb{V}, \mathbb{W} \rightarrow \mathcal{A}$$

are in fact locally free over the entire moduli space \mathcal{A} even though the higher cohomology of theta line bundles may not vanish. Specifically, this is implied by the following

Theorem 4. *Let (X, H) be a polarized abelian surface. Assume that*

$$v = (r, dH, \chi), \quad w = (r', d'H, \chi')$$

are orthogonal primitive Mukai vectors of ranks $r, r' \geq 2$ such that

- (i) $d, d' > 0$;
- (ii) $\chi < 0, \chi' < 0$.

Assume furthermore that if $(d, \chi) = (1, -1)$, then (X, H) is not a product of two elliptic curves. We have

$$h^0(\mathbf{K}_v, \Theta_w) = \chi(\mathbf{K}_v, \Theta_w) = \frac{d_v^2}{d_v + d_w} \begin{pmatrix} d_v + d_w \\ d_v \end{pmatrix}.$$

Moreover, for any representative $F \in \mathbf{K}_w$,

$$h^0(\mathbf{M}_v, \Theta_F) = \chi(\mathbf{M}_v, \Theta_F) = \frac{d_w^2}{d_v + d_w} \begin{pmatrix} d_v + d_w \\ d_v \end{pmatrix}.$$

The proof uses Bridgeland stability conditions, and relies on recent results concerning wall-crossing as stability varies. As walls are crossed, the dimensions of the space of sections do not change. Crucially, we show that we can move away from the Gieseker chamber to a chamber for which the theta line bundles become big and nef. In order to control the wall-crossings and complete the argument, we make use of the explicit description of the movable cone of the moduli space recently obtained in [Y2]; see also [BM].

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2. THE RANK ONE CASE

2.1. Notation and preliminaries. We let X be an arbitrary abelian surface and consider two Mukai vectors v and w with

$$\text{rank } v = \text{rank } w = 1.$$

Specifically, letting $L \rightarrow X$ be an ample line bundle, and writing $\chi(L) = a + b$ for positive integers a and b , we set

$$v = (1, 0, -a), \quad w = (1, c_1(L), a).$$

We then have

$$\mathbf{K}_v \simeq K^{[a]}, \quad \mathbf{M}_w \simeq X^{[b]} \times \widehat{X},$$

and the strange duality divisor is

$$(3) \quad \Theta_L = \{(I_Z, I_W, y) \text{ with } H^0(I_Z \otimes I_W \otimes y \otimes L) \neq 0\} \subset K^{[a]} \times X^{[b]} \times \widehat{X}.$$

Conforming to standard notation, we next set

$$L^{[a]} = \det \mathbf{R}p_* (\mathcal{O}_{\mathcal{Z}} \otimes q^* L) \text{ on } X^{[a]},$$

where $\mathcal{Z} \subset X^{[a]} \times X$ is the universal subscheme, and p, q are the projections to $X^{[a]}$ and X respectively. Throughout this section we also use

$$L^{[a]} \rightarrow K^{[a]}$$

to denote the restriction of the determinant line bundle to $K^{[a]} \subset X^{[a]}$.

The divisor

$$(4) \quad \Theta_L^+ = \{(I_Z, I_W) \text{ with } H^0(I_Z \otimes I_W \otimes L) \neq 0\} \subset X^{[a]} \times X^{[b]},$$

with associated line bundle

$$\mathcal{O}(\Theta_L^+) = L^{[a]} \boxtimes L^{[b]} \text{ over } X^{[a]} \times X^{[b]}$$

induces an isomorphism

$$(5) \quad D_L^+ : H^0(X^{[a]}, L^{[a]})^\vee \longrightarrow H^0(X^{[b]}, L^{[b]}).$$

This constitutes the simplest instance of the strange duality phenomenon on surfaces; the isomorphism is described in [MO3] and holds uniformly irrespective of the choice of surface.

Relative to this standard rank one setup, the divisor Θ_L represents a twist specific to the abelian geometry. In particular, the associated line bundle takes the more complicated form

$$(6) \quad \mathcal{O}(\Theta_L) = L^{[a]} \boxtimes L^{[b]} \boxtimes \widehat{L} \otimes (\mathbf{a}, \text{id})^* \mathcal{P} \text{ on } K^{[a]} \times X^{[b]} \times \widehat{X},$$

where $\mathcal{P} \rightarrow X \times \widehat{X}$ is the Poincaré line bundle, and

$$\mathbf{a} : X^{[b]} \rightarrow X$$

denotes the addition of points using the group law. We have also set

$$\widehat{L} = \det \mathbf{R}\mathcal{S}(L)^{-1} \text{ on } \widehat{X}.$$

Expression (6) is obtained by restricting to each factor and using Mumford's see-saw theorem; a detailed explanation is found in Example 1 of [O]. Establishing that the induced map on the spaces of sections

$$(7) \quad D_L : H^0(K^{[a]}, L^{[a]})^\vee \longrightarrow H^0(X^{[b]} \times \widehat{X}, L^{[b]} \boxtimes \widehat{L} \otimes (\mathbf{a}, \text{id})^* \mathcal{P})$$

is an isomorphism requires new ideas which we now describe.

2.2. Proof of Theorem 1. To begin, note that both sides of (7) have equal dimensions given by the Euler characteristics (2). For the left hand side, this follows from either Lemma 3 or Example 9 in [O]: both show the vanishing of the higher cohomology of

$$L^{[a]} \rightarrow K^{[a]}$$

under the assumption that $L \rightarrow X$ is ample. For the right hand side, we can invoke Proposition 5 of Section 6 which applies to the current context as well. A direct argument is also possible making use of the étale pullbacks of the proof below.

We rephrase the statement of the theorem in two steps. To start, let

$$\varphi_L : X \longrightarrow \widehat{X}, \quad \varphi_L(x) = t_x^* L \otimes L^{-1}$$

be the Mumford homomorphism; we also make use of $\varphi_{\widehat{L}} : \widehat{X} \longrightarrow X$. Consider now the diagram

$$\begin{array}{ccc} K^{[a]} \times X^{[b]} \times \widehat{X} & & \\ \downarrow \widehat{\Phi} & \searrow \Psi & \\ K^{[a]} \times X^{[b]} \times X & \xrightarrow{\Gamma} & X^{[a]} \times X^{[b]} \\ \downarrow \Phi & & \\ K^{[a]} \times X^{[b]} \times \widehat{X} & & \end{array}$$

where

$$\begin{aligned} \Phi(I_Z, I_W, x) &= (I_Z, t_x^* I_W, \varphi_L(x)), \\ \widehat{\Phi}(I_Z, I_W, y) &= (I_Z, I_W, \varphi_{\widehat{L}}(y)), \\ \Gamma(I_Z, I_W, x) &= (t_{-x}^* I_Z, I_W), \\ \Psi(I_Z, I_W, y) &= (t_{-\varphi_{\widehat{L}}(y)}^* I_Z, I_W) \implies \Psi = \Gamma \circ \widehat{\Phi}. \end{aligned}$$

All four maps are étale:

- Φ and $\widehat{\Phi}$ have degree $\chi^2 = \chi(L)^2 = \chi(\widehat{L})^2$;
- Γ has degree a^4 since it can be viewed as quotienting by the group of a -torsion points on X ;
- $\Psi = \Gamma \circ \widehat{\Phi}$ has degree $a^4 \chi^2$.

We now pull back the divisor $\Theta_L \subset K^{[a]} \times X^{[b]} \times \widehat{X}$ twice, first by Φ and then by $\widehat{\Phi}$.

2.2.1. *Pullback under Φ .* At the first stage, we obtain

$$\begin{aligned}\Phi^*\Theta_L &= \{(I_Z, I_W, x) \text{ with } H^0(I_Z \otimes t_x^* I_W \otimes \varphi_L(x) \otimes L) \neq 0\} \\ &= \{(I_Z, I_W, x) \text{ with } H^0(I_Z \otimes t_x^* I_W \otimes t_x^* L) \neq 0\} \\ &= \{(I_Z, I_W, x) \text{ with } H^0(t_{-x}^* I_Z \otimes I_W \otimes L) \neq 0\} \\ &= \Gamma^*\Theta_L^+.\end{aligned}$$

By contrast with expression (6), the line bundle associated with $\Phi^*\Theta_L$ has the simpler form

$$\mathcal{O}(\Phi^*\Theta_L) = \mathcal{O}(\Gamma^*\Theta_L^+) = \Gamma^*(L^{[a]} \boxtimes L^{[b]}) = L^{[a]} \boxtimes L^{[b]} \boxtimes L^a \text{ on } K^{[a]} \times X^{[b]} \times X.$$

The pullback divisor induces the map Φ^*D_L for which the diagram

$$\begin{array}{ccc} D_L : H^0(K^{[a]}, L^{[a]})^\vee & \longrightarrow & H^0(X^{[b]} \times \widehat{X}, L^{[b]} \boxtimes \widehat{L} \otimes (\mathbf{a}, \text{id})^*\mathcal{P}) \\ \parallel & & \downarrow \Phi^* \\ \Phi^*D_L : H^0(K^{[a]}, L^{[a]})^\vee & \longrightarrow & H^0(X^{[b]} \times X, L^{[b]} \boxtimes L^a) \end{array}$$

commutes. To show the original duality map D_L is injective (and thus by equality of dimensions an isomorphism), it suffices to show that the simpler Φ^*D_L in the above diagram is injective.

2.2.2. *Pullback under $\widehat{\Phi}$.* The second pullback, under $\widehat{\Phi}$, yields the divisor

$$\widetilde{\Theta}_L = \widehat{\Phi}^*\Phi^*\Theta_L$$

associated with the line bundle

$$\mathcal{O}(\widetilde{\Theta}_L) = \widehat{\Phi}^*(L^{[a]} \boxtimes L^{[b]} \boxtimes L^a) = L^{[a]} \boxtimes L^{[b]} \boxtimes \varphi_L^* L^a \text{ on } K^{[a]} \times X^{[b]} \times \widehat{X}.$$

Crucially, by our previous interpretation of $\Phi^*\Theta_L$, we also have

$$\widetilde{\Theta}_L = \widehat{\Phi}^*\Phi^*\Theta_L = \widehat{\Phi}^*\Gamma^*\Theta_L^+ = \Psi^*\Theta_L^+.$$

By the same argument as before, to show that the original duality map (7) is an isomorphism, it suffices to show that

Proposition 1. *The morphism $\widetilde{D}_L : H^0(K^{[a]}, L^{[a]})^\vee \longrightarrow H^0(X^{[b]} \times \widehat{X}, L^{[b]} \boxtimes \varphi_L^* L^a)$ induced by $\widetilde{\Theta}_L$ is injective.*

Proof. We interpret the duality map representation-theoretically, using the theory of discrete Heisenberg groups. \widetilde{D}_L is better suited for such an interpretation than the seemingly simpler morphism Φ^*D_L obtained at the previous stage.

We have seen above that

$$\Psi^* \mathcal{O}(\Theta_L^+) = \Psi^*(L^{[a]} \boxtimes L^{[b]}) = L^{[a]} \boxtimes L^{[b]} \boxtimes \varphi_L^* L^a.$$

Up to numerical equivalence on \widehat{X} , we have $\varphi_L^* L = \widehat{L}^\chi$. Thus, there exists $y \in \widehat{X}$ such that

$$\varphi_L^* L = t_y^* \widehat{L}^\chi.$$

We define

$$M = L \otimes y$$

and calculate

$$\widehat{M} := \det \mathbf{R}\mathcal{S}(M)^{-1} \implies \widehat{M} = t_y^* \widehat{L} \implies \widehat{M}^{a\chi} = t_y^* \widehat{L}^{a\chi} = \varphi_L^* L^a.$$

Therefore,

$$\Psi^*(L^{[a]} \boxtimes L^{[b]}) = L^{[a]} \boxtimes L^{[b]} \boxtimes \widehat{M}^{a\chi}.$$

We let $\mathbf{G}(\widehat{M}^a)$ be the Heisenberg group of the line bundle $\widehat{M}^a \rightarrow \widehat{X}$, sitting in an exact sequence

$$1 \rightarrow \mathbb{C}^* \rightarrow \mathbf{G}(\widehat{M}^a) \rightarrow \mathbf{H}(\widehat{M}^a) \rightarrow 1,$$

where the quotient is the abelian group

$$\mathbf{H}(\widehat{M}^a) = \{y, t_y^* \widehat{M}^a \simeq \widehat{M}^a\} \subset \widehat{X}.$$

For an introduction to Heisenberg group actions in the theory of abelian varieties we refer the reader to [Mu], for instance.

Importantly, by construction, the étale morphism

$$\Psi : K^{[a]} \times X^{[b]} \times \widehat{X} \longrightarrow X^{[a]} \times X^{[b]}$$

can be viewed precisely as quotienting by the abelian group $\mathbf{H}(\widehat{M}^a)$. The latter acts on $K^{[a]} \times \widehat{X}$ via

$$\eta \cdot (I_Z, y) = (t_{\varphi_{\widehat{M}}^*(\eta)}^* I_Z, y + \eta)$$

and trivially on $X^{[b]}$. Thus, as a pullback of $L^{[a]} \rightarrow X^{[a]}$ under the quotienting map Ψ , the line bundle

$$L^{[a]} \boxtimes \widehat{M}^{a\chi} \rightarrow K^{[a]} \times \widehat{X}$$

is $\mathbf{H}(\widehat{M}^a)$ -equivariant, in other words it is $\mathbf{G}(\widehat{M}^a)$ -equivariant, such that the center acts with weight 0. Independently, it is clear that the line bundle $\widehat{M}^{a\chi} \rightarrow \widehat{X}$ is $\mathbf{G}(\widehat{M}^a)$ -equivariant, the center acting with weight χ . It follows that

$$L^{[a]} \rightarrow K^{[a]}$$

is also $\mathbf{G}(\widehat{M}^a)$ -equivariant, so that the center acts with weight $-\chi$. The spaces of sections

$$H^0(\widehat{X}, \widehat{M}^{a\chi}) \text{ and } H^0(K^{[a]}, L^{[a]})$$

are in turn acted on with weights χ and $-\chi$ respectively, and furthermore, we can write

$$H^0(X^{[a]}, L^{[a]}) = \left(H^0(\widehat{X}, \widehat{M}^{a\chi}) \otimes H^0(K^{[a]}, L^{[a]}) \right)^{\mathbf{H}(\widehat{M}^a)}.$$

Taking into account the long-known isomorphism

$$D_L^+ : H^0(X^{[a]}, L^{[a]})^\vee \longrightarrow H^0(X^{[b]}, L^{[b]})$$

of equation (5), we see that the dual of the linear map \widetilde{D}_L is the natural

$$\widetilde{D}_L^\vee : \left(H^0(\widehat{X}, \widehat{M}^{a\chi}) \otimes H^0(K^{[a]}, L^{[a]}) \right)^{\mathbf{H}(\widehat{M}^a)} \otimes H^0(\widehat{X}, \widehat{M}^{a\chi})^\vee \longrightarrow H^0(K^{[a]}, L^{[a]}),$$

which pairs the vector space $H^0(\widehat{X}, \widehat{M}^{a\chi})$ and its dual. To conclude the proposition, we show now that this map is surjective.

Let $\{\mathbf{S}_\alpha\}_{\alpha \in I}$ denote the irreducible representations of $\mathbf{G}(\widehat{M}^a)$ with the center acting with weight $-\chi$. Decomposing into irreducibles, we write

$$H^0(K^{[a]}, L^{[a]}) = \bigoplus_{\alpha} \mathbf{S}_\alpha \otimes \mathbb{C}^{m_\alpha}, \quad H^0(\widehat{X}, \widehat{M}^{a\chi})^\vee = \bigoplus_{\alpha} \mathbf{S}_\alpha \otimes \mathbb{C}^{n_\alpha},$$

and the duality map \widetilde{D}_L^\vee is

$$\widetilde{D}_L^\vee : \left(\bigoplus_{\alpha} (\mathbb{C}^{n_\alpha})^\vee \otimes \mathbb{C}^{m_\alpha} \right) \otimes \left(\bigoplus_{\beta} \mathbf{S}_\beta \otimes \mathbb{C}^{n_\beta} \right) \longrightarrow \bigoplus_{\alpha} \mathbf{S}_\alpha \otimes \mathbb{C}^{m_\alpha},$$

given explicitly by the natural pairing of the multiplicity spaces $(\mathbb{C}^{n_\alpha})^\vee$ and \mathbb{C}^{n_α} .

We conclude \widetilde{D}_L^\vee fails to be surjective only if there is an irreducible \mathbf{S}_α which appears with nonzero multiplicity $m_\alpha \neq 0$ in $H^0(K^{[a]}, L^{[a]})$, but fails to appear in $H^0(\widehat{X}, \widehat{M}^{a\chi})$, so $n_\alpha = 0$. This is precluded by the following result, which in level 2 is Proposition 3.7 in [I]. This ends the proof of the proposition, and therefore of Theorem 1. \square

Lemma 1. *Let A be an abelian surface and $M \rightarrow A$ an ample line bundle. For any integer $k \geq 0$, all irreducible representations with central weight k of the Heisenberg group $\mathbf{G}(M)$ appear in the $\mathbf{G}(M)$ -module $H^0(A, M^k)$ with nonzero multiplicity.*

For the benefit of the reader, we give the quick argument, which we lifted from [I]. Consider the natural homomorphism $\mathbf{G}(M) \rightarrow \mathbf{G}(M^k)$ and write

$$\mathbf{K} \cong \mathbf{G}(M)/\mu_k$$

for its image. Fix \mathcal{S} a representation of the Heisenberg group $\mathbf{G}(M)$ of weight k . Certainly, \mathcal{S} is a representation of \mathbf{K} with weight 1. The induced representation

$$\mathbf{R} = \mathrm{Ind}_{\mathbf{K}}^{\mathbf{G}(M^k)} \mathcal{S}$$

of the Heisenberg group $\mathbf{G}(M^k)$ has weight 1, hence it splits as a sum of copies of the unique irreducible representation $H^0(A, M^k)$ of weight 1:

$$\mathbf{R} = H^0(A, M^k) \oplus \dots \oplus H^0(A, M^k).$$

We restrict this decomposition to $\mathbf{G}(M)$. By definition, the induced representation \mathbf{R} must contain a copy of \mathcal{S} as a \mathbf{K} -submodule, and therefore also as a $\mathbf{G}(M)$ -submodule. We conclude that \mathcal{S} must appear in the $\mathbf{G}(M)$ -module $H^0(A, M^k)$, as claimed.

3. PRODUCT ABELIAN SURFACES

Relying on the rank one case just established, Theorem 2 is derived by techniques developed in [MOY] and [MO2]. Specifically, we let

$$X = B \times F \rightarrow B$$

be a product of elliptic curves, which we view as an abelian surface elliptically fibered over B . We write f for the class of the fiber over the origin, and σ for the zero section of the fibration. As in [MO2], stability of sheaves over X is with respect to a polarization

$$H = \sigma + Nf$$

for N large enough. This polarization is suitable in the sense of Friedman [F]. Assuming v and w are vectors with

$$c_1(v) \cdot f = c_1(w) \cdot f = 1,$$

we show that

$$\mathbf{D} : H^0(\mathbf{K}_v, \Theta_w)^\vee \rightarrow H^0(\mathbf{M}_w, \Theta_v)$$

is an isomorphism.

As in [MO2], we use a fiberwise Fourier-Mukai transform

$$\mathbf{RS}^\dagger : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$$

to move from the rank 1 situation to higher rank Mukai vectors. The kernel of \mathbf{RS}^\dagger is given by the pullback of the normalized Poincaré sheaf

$$\mathcal{P}_F \rightarrow F \times F$$

to the product $X \times_B X \cong F \times F \times B$. The Fourier-Mukai transform gives rise to two birational isomorphisms

$$\mathbf{K}_v \dashrightarrow K^{[d_v]}$$

and

$$\mathbf{M}_w \dashrightarrow X^{[d_w]} \times \widehat{X}$$

which are regular in codimension 1. Explicitly, for any $E \in \mathbf{K}_v$ and $F \in \mathbf{M}_w$, away from codimension two loci, Proposition 1 of [MO2] in conjunction with Theorem 1.1 of [Br1] shows that

$$(8) \quad \mathbf{RS}^\dagger(E^\vee) = I_Z(r\sigma - \chi f)[-1],$$

$$(9) \quad \mathbf{RS}^\dagger(F) = I_W^\vee \otimes \mathcal{O}(-r'\sigma + \chi' f) \otimes y^{-1},$$

for subschemes

$$Z \in K^{[d_v]}, W \in X^{[d_w]}, \text{ and a line bundle } y \in \widehat{X}.$$

Here, we wrote

$$r = \text{rank}(v), \chi = \chi(v), r' = \text{rank}(w), \chi' = \chi(w).$$

We set

$$L = \mathcal{O}((r + r')\sigma - (\chi + \chi')f) \implies \chi(L) = d_v + d_w.$$

Now, the key to finishing the proof is the calculation:

$$\begin{aligned} \mathbb{H}^0(E \otimes^{\mathbf{L}} F) &= \text{Hom}_{\mathbf{D}(X)}(E^\vee, F) = \text{Hom}_{\mathbf{D}(X)}(\mathbf{RS}^\dagger(E^\vee), \mathbf{RS}^\dagger(F)) \\ &= \text{Ext}^1(I_Z \otimes y \otimes L, I_W^\vee) = \text{Ext}^1(I_W^\vee, I_Z \otimes y \otimes L)^\vee \\ &= \mathbb{H}^1(I_W \otimes^{\mathbf{L}} I_Z \otimes y \otimes L)^\vee. \end{aligned}$$

On the locus (of codimension 2 complement) of non-overlapping (Z, W) , the last hypercohomology group coincides with the regular cohomology group,

$$\mathbb{H}^1(I_W \otimes^{\mathbf{L}} I_Z \otimes y \otimes L) = H^1(I_W \otimes I_Z \otimes y \otimes L).$$

Thus under the birational map

$$\mathbf{K}_v \times \mathbf{M}_w \dashrightarrow K^{[d_v]} \times X^{[d_w]} \times \widehat{X},$$

the two theta divisors

$$\Theta = \{(E, F) : \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathbf{K}_v \times \mathbf{M}_w,$$

and

$$\Theta_L = \{(I_Z, I_W, y) : H^0(I_Z \otimes I_W \otimes y \otimes L) \neq 0\} \subset K^{[d_v]} \times X^{[d_w]} \times \widehat{X}$$

coincide, and the theta line bundles on each factor match up as well. Since in rank 1, Θ_L induces a strange duality isomorphism by Theorem 1, the same must be true about the divisor Θ inducing the map

$$D : H^0(K_v, \Theta_w)^\vee \longrightarrow H^0(M_w, \Theta_v).$$

This completes the proof. \square

Remark. The assumption that the rank is at least 3 is made in [MO2] to justify that equations (8) and (9) hold in codimension 1. This assumption is however not needed, as we now show. The reader wishing to go on to the proof of generic strange duality contained in the next section may choose to skip this argument.

To begin, we note that identity (9) follows from (8) via Grothendieck duality. In turn, equation (8) is a consequence of the fact that $\mathbf{RS}^\dagger(E^\vee)[1]$ is torsion free, cf. Proposition 1 in [MO2]. We will explain that this assertion holds in codimension 1, in rank 2. To this end, regard the kernel of \mathbf{RS}^\dagger , namely the Poincaré sheaf

$$\mathcal{P} \rightarrow X \times_B X,$$

as an object over $X \times X$ via the diagonal embedding

$$X \times_B X \rightarrow X \times X.$$

We will prove

Lemma 2. *For all sheaves E away from a codimension 2 locus in the moduli space, the set*

$$T_E = \{x \in X : \mathrm{Hom}(E, \mathcal{P}|_{X \times \{x\}}) \neq 0\} \subset X$$

is finite.

Assuming the lemma, we show that for all E such that T_E is a finite set, the transform $\mathbf{RS}^\dagger(E^\vee)[1]$ is a torsion free sheaf. To see this, consider a locally free resolution

$$(10) \quad 0 \rightarrow V \rightarrow W \rightarrow E \rightarrow 0$$

such that $W = \mathcal{O}_X(-mH)^{\oplus k}$ for sufficiently large m . Then

$$\mathrm{Ext}^1(W, \mathcal{P}|_{X \times \{x\}}) = \mathrm{Ext}^2(W, \mathcal{P}|_{X \times \{x\}}) = 0,$$

for all $x \in X$. As a consequence, the sheaf

$$\widehat{W} := \mathbf{RS}^\dagger(W^\vee)$$

is locally free. Next,

$$\mathrm{Ext}^2(E, \mathcal{P}|_{X \times \{x\}}) = \mathrm{Hom}(\mathcal{P}|_{X \times \{x\}}, E)^\vee = 0,$$

using that E is torsion free and $\mathcal{P}|_{X \times \{x\}}$ is of rank 0. From the exact sequence induced by the resolution (10), we conclude that

$$\mathrm{Ext}^1(V, \mathcal{P}|_{X \times \{x\}}) = \mathrm{Ext}^2(V, \mathcal{P}|_{X \times \{x\}}) = 0$$

for all $x \in X$. Therefore,

$$\widehat{V} := \mathbf{RS}^\dagger(V^\vee)$$

is locally free as well. The same resolution also shows that we have an exact triangle

$$\mathbf{RS}^\dagger(E^\vee) \rightarrow \widehat{W} \rightarrow \widehat{V} \rightarrow \mathbf{RS}^\dagger(E^\vee)[1]$$

which induces an exact sequence in cohomology sheaves

$$0 \rightarrow \mathcal{H}^0(\mathbf{RS}^\dagger(E^\vee)) \rightarrow \widehat{W} \xrightarrow{\phi} \widehat{V} \rightarrow \mathcal{H}^1(\mathbf{RS}^\dagger(E^\vee)) \rightarrow 0.$$

Note that $\phi|_{\{x\}}$ is injective whenever $x \notin T_E$. Then our assumption implies that ϕ is injective as a morphism of sheaves. Furthermore, $\mathrm{Coker} \phi$ is torsion free, as claimed. \square

Proof of Lemma 2. Consider the set

$$\Sigma = \{E : \text{there exists a fiber } f \text{ such that } E|_f \text{ contains a subbundle of slope } > 1\}.$$

This set has codimension at least 2 in the moduli space by Lemma 5.4 of [BH]. (A shift by 1 in the slope is necessary to align with the numerical conventions of [BH].) We will assume that E is chosen outside Σ . Furthermore, we may assume that there is at most one point of the surface where E fails to be locally free. This is always true in the moduli space away from codimension 2.

We claim that in this situation T_E consists of finitely many points. Indeed, let $x \in T_E$. Three cases need to be considered.

- (a) First, we rely on the fact that the polarization is suitable. In this case, the restriction of E to a generic fiber is stable. If x lies on such a generic fiber, then as a consequence of stability, we obtain the vanishing

$$\mathrm{Hom}(E, \mathcal{P}|_{X \times \{x\}}) = 0.$$

Therefore in this case $x \notin T_E$.

- (b) Assume now that x lies on a fiber f over which the restriction of E is locally free but unstable. In this situation, $E|_f$ splits as

$$E|_f = S_0 \oplus S_1$$

where S_0 is a degree zero line bundle over f , while S_1 has degree 1. Any other splitting type is not allowed by the definition of Σ . Now,

$$\mathrm{Hom}(E, \mathcal{P}|_{X \times \{x\}}) = \mathrm{Hom}(S_0, \mathcal{P}|_{X \times \{x\}}) \neq 0 \implies S_0 = \mathcal{P}|_{X \times \{x\}}.$$

This shows that x must be the point corresponding to the line bundle S_0 . Since by (a), there are only finitely many unstable fibers, we conclude that there are only finitely many choices for x .

- (c) Finally, we analyze the case when x lies on a fiber over which E is not locally free. Let s be the unique point where E fails to be locally free, and let f_s be the fiber through s . Then

$$E|_{f_s} = \mathbb{C}_s \oplus F,$$

where F is a rank 2 degree 0 vector bundle over f_s . If F is semistable, there exists an extension

$$0 \rightarrow S \rightarrow F \rightarrow S \rightarrow 0$$

where S is a line bundle of degree 0 over f_s . We have

$$\begin{aligned} \mathrm{Hom}(E, \mathcal{P}|_{X \times \{x\}}) = \mathrm{Hom}(F, \mathcal{P}|_{X \times \{x\}}) \neq 0 &\implies \mathrm{Hom}(S, \mathcal{P}|_{X \times \{x\}}) \neq 0 \\ &\implies S = \mathcal{P}|_{X \times \{x\}}. \end{aligned}$$

This proves that x is the point of the fiber through s corresponding to S .

To complete the argument, it suffices to show that the situation when F is not semistable corresponds to a codimension 2 subset of the moduli space. To this end, consider the codimension 1 locus \mathcal{Z} of sheaves in the moduli space which fail to be locally free at exactly one point. This is an irreducible subset. Indeed, any sheaf in \mathcal{Z} sits in an exact sequence

$$0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow \mathbb{C}_s \rightarrow 0,$$

with $M = E^{\vee\vee}$ stable locally free of Mukai vector

$$v^{\vee\vee} = v + (0, 0, 1).$$

Letting \mathcal{M} denote the moduli space of such locally free sheaves, there exists a fibration

$$\pi : \mathcal{Z} \rightarrow \mathcal{M}$$

whose fibers over M are Quot schemes of length 1 quotients $q : M \rightarrow \mathbb{C}_s \rightarrow 0$. The sheaf E is recovered uniquely as the kernel of the pair (M, q) . Since the fibers of π are irreducible of dimension 3, \mathcal{Z} must be irreducible as well.

Now, for locally free sheaves $M \in \mathcal{M}$, there are finitely many fibers for which $M|_f$ is unstable. Consider

$$\mathcal{Z}^\circ \hookrightarrow \mathcal{Z}$$

the set of pairs $(M, q : M \rightarrow \mathbb{C}_s \rightarrow 0)$ where s does not lie on an unstable fiber. The restriction of $M|_{f_s}$ is the Atiyah bundle of rank 2 and degree 1. The kernel of q is a torsion free sheaf E which is not locally free at s . In fact, we calculate

$$E|_{f_s} = \mathbb{C}_s \oplus F$$

where F is a subsheaf of degree 0 of the Atiyah bundle $M|_{f_s}$. Since $M|_{f_s}$ is stable, all its proper subbundles have slope ≤ 0 . It follows that F is semistable. Thus, to get F 's which are not semistable, we need to select (M, q) from $\mathcal{Z} \setminus \mathcal{Z}^\circ$. Clearly,

$$\mathcal{Z} \setminus \mathcal{Z}^\circ \rightarrow \mathcal{M}$$

has projective fibers of dimension 2. Thus, $\mathcal{Z} \setminus \mathcal{Z}^\circ$ has codimension 1 in \mathcal{Z} , as claimed. This completes the proof of Lemma 2 and ends the remark. \square

4. GENERIC STRANGE DUALITY

The isomorphism we established for product abelian surfaces implies strange duality for generic abelian surfaces. This is achieved via degeneration; see also Section 3 of [MOY].

Specifically, we let \mathcal{A} denote the moduli stack of pairs (X, H) with $H^2 = 2n$, where H is a primitive ample line bundle over X . Consider the universal family

$$\pi : (\mathcal{X}, \mathcal{H}) \rightarrow \mathcal{A}.$$

Fix integers χ, χ' and ranks $r, r' \geq 2$. For each $t \in \mathcal{A}$ representing a polarized abelian surface $(\mathcal{X}_t, \mathcal{H}_t)$, consider two orthogonal Mukai vectors

$$v_t = (r, c_1(\mathcal{H}_t), \chi), \quad w_t = (r', c_1(\mathcal{H}_t), \chi').$$

We form the relative moduli spaces of \mathcal{H}_t -semistable sheaves of type v_t and w_t

$$\pi : \mathfrak{K}[v] \rightarrow \mathcal{A}, \quad \pi : \mathfrak{M}[w] \rightarrow \mathcal{A}.$$

The product

$$\pi : \mathfrak{K}[v] \times_{\mathcal{A}} \mathfrak{M}[w] \rightarrow \mathcal{A}$$

carries the relative Brill-Noether locus

$$\Theta[v, w] = \{(X, H, E, F) : \mathbb{H}^0(X, E \otimes^{\mathbf{L}} F) \neq 0\}$$

obtained as the vanishing of a section of the relative theta line bundle

$$\Theta[w] \boxtimes \Theta[v] \rightarrow \mathfrak{K}[v] \times_{\mathcal{A}} \mathfrak{M}[w].$$

Pushing forward to \mathcal{A} via the natural projections π , we obtain the sheaves

$$\mathbb{V} = \pi_* (\Theta[w]), \quad \mathbb{W} = \pi_* (\Theta[v]),$$

as well as a section D of $\mathbb{V} \otimes \mathbb{W}$. The constructions are explained in detail in [MO4].

Crucial to the specialization procedure which yields generic strange duality is the statement that \mathbb{V} and \mathbb{W} are generically vector bundles of equal rank

$$\frac{d_v^2}{d_v + d_w} \begin{pmatrix} d_v + d_w \\ d_v \end{pmatrix}$$

whose fibers are the spaces of generalized theta functions. This is established in Proposition 2 below. Assuming this result, we let $\mathcal{A}^\circ \hookrightarrow \mathcal{A}$ denote the maximal open locus where the generic rank is achieved. Consider also the Humbert locus

$$\mathcal{S} \hookrightarrow \mathcal{A}$$

of split abelian surfaces

$$(X, H) = (B \times F, L_B \boxtimes L_F),$$

for line bundles $L_B \rightarrow B, L_F \rightarrow F$ of degrees 1 and n . Just as in Section 3 of [MOY], Theorem 2 can be rephrased as the statement that

$$\mathcal{S} \hookrightarrow \mathcal{A}^\circ$$

and that furthermore

$$D : \mathbb{V}^\vee \rightarrow \mathbb{W}$$

is an isomorphism along \mathcal{S} . To make the above claim, we need to exchange stability relative to a suitable polarization required by Theorem 2 with stability relative to the polarization H (which may lie on a wall). The next section, in particular Proposition 3, shows that the ensuing moduli spaces agree in codimension 1. We need to pass to the moduli stacks to invoke the proposition, but the corresponding spaces of sections do not change, as explained in Section 3 of [MOY].

As a consequence, D is an isomorphism generically over \mathcal{A}° . Since the generic fibers of \mathbb{V} and \mathbb{W} over \mathcal{A}° are spaces of generalized theta functions, we conclude that generic strange duality holds as in Theorem 3. \square

We now turn to Proposition 2 which was used in the argument above. A general local-freeness statement for the Verlinde sheaves will be proven in Section 6, but in its context, the proposition gives stronger positivity results with a simpler proof. We show

Proposition 2. *Let X be an abelian surface of Picard rank 1, with H the generator of the Néron-Severi group of X . Let*

$$v = (r, H, \chi), \quad w = (r', d'H, \chi')$$

be two orthogonal vectors of positive rank such that $\chi \neq 0$, $\chi' \leq 0$. Then, for any $F \in \mathbf{K}_w$, the line bundle

$$\Theta_w := \Theta_F \rightarrow \mathbf{M}_v$$

is big and nef, hence without higher cohomology. If $\chi' < 0$, then the above line bundle is ample. By restriction, the same results hold for $\Theta_w \rightarrow \mathbf{K}_v$.

Proof. In the $K3$ case, reflections along rigid sheaves were used to conclude that $\Theta_w \rightarrow \mathbf{M}_v$ is big and nef, hence without higher cohomology, cf. Proposition 4 of [MOY]. Unlike $K3$ surfaces, abelian surfaces do not admit rigid sheaves. A different argument will be given.

The starting point is the following well-known result of Jun Li [Li2]. Specifically, setting

$$w_0 = (0, rH, -2n),$$

the line bundle $\Theta_{w_0} \rightarrow \mathbf{M}_v$ is big and nef. We will moreover show that for the vector

$$w_1 = (2n, -\chi H, 0),$$

the line bundle $\Theta_{w_1} \rightarrow \mathbf{M}_v$ is also big and nef. Since for $\chi(w) \leq 0$, w is a linear combination with non-negative coefficients of w_0 and w_1 , the conclusion follows.

To prove the claim about w_1 , we consider two cases depending on the sign of $\chi(v)$. Let us first assume that $\chi(v) < 0$. By Proposition 3.5 of [Y1], the shifted Fourier-Mukai transform Φ with kernel

$$\mathcal{P}[1] \rightarrow X \times \widehat{X}$$

induces an isomorphism of moduli spaces

$$\Phi : \mathbf{M}_v \simeq \mathbf{M}_{\widehat{v}} \quad \text{where } \widehat{v} = (-\chi, \widehat{H}, -r) \text{ is a vector on } \widehat{X}.$$

For $\widehat{w} = (0, -\chi \widehat{H}, -2n)$, the bundle

$$\Theta_{\widehat{w}} \rightarrow \mathbf{M}_{\widehat{v}}$$

is big and nef, again by Jun Li's result. To conclude, it remains to observe that

$$\Phi^* \Theta_{\widehat{w}} = \Theta_{w_1},$$

hence the latter line bundle is also big and nef.

When $\chi(v) > 0$, the argument is similar. By Proposition 3.2 of [Y1], we have an isomorphism

$$\Psi : \mathbb{M}_v \simeq \mathbb{M}_{\widehat{v}}, \quad \widehat{v} = (\chi, \widehat{H}, r)$$

induced by the composition of the Fourier-Mukai transform with kernel \mathcal{P} with the dualization. Under this isomorphism, Jun Li's bundle $\Theta_{\widehat{w}}$, where $\widehat{w} = (0, \chi\widehat{H}, -2n)$, corresponds to Θ_{w_1} . \square

5. VARIATION OF POLARIZATION FOR THE MODULI SPACE OF GIESEKER SHEAVES

Let X be an arbitrary abelian surface, and fix a Mukai vector

$$v := (r, \xi, a) \in H^*(X, \mathbb{Z})$$

with $r > 0$. For an ample divisor H on X , denote by

$$\mathfrak{M}(v), \mathfrak{M}_H(v)^{ss} \text{ and } \mathfrak{M}_H(v)^{\mu-ss}$$

the stacks of all sheaves, of Gieseker H -semistable sheaves, and of slope H -semistable sheaves respectively – all of type v .

We are concerned with moduli spaces of sheaves when Gieseker stability varies: we show that they agree in codimension 1 each time a wall is crossed. This fact was used in the degeneration argument of Section 4 to exchange the suitable polarization with the polarization determined by the first Chern class.

First, for generic polarizations, the dimension of the moduli space is given by the following Lemma 4.3.2 in [MMY2]:

Lemma 3. *If H is general with respect to v , that is, H does not lie on a wall with respect to v , then*

$$(11) \quad \dim \mathfrak{M}_H(v)^{ss} = \begin{cases} \langle v, v \rangle + 1, & \langle v, v \rangle > 0 \\ \langle v, v \rangle + \ell, & \langle v, v \rangle = 0 \end{cases},$$

where $\ell = \gcd(r, \xi, a)$.

For the purposes of Section 4, we also need to analyze the situation when the polarization may lie on a wall. To this end, let H_1 be an ample divisor on X which belongs to a wall W with respect to v and H an ample divisor which belongs to an adjacent chamber. Then Gieseker H -semistable sheaves are slope H_1 -semistable

$$\mathfrak{M}_H(v)^{ss} \hookrightarrow \mathfrak{M}_{H_1}^{ss}(v) \hookrightarrow \mathfrak{M}_{H_1}(v)^{\mu-ss}.$$

All these stacks have dimension $\langle v, v \rangle + 1$ by Lemma 3.8 of [KY]. We estimate the codimension of

$$\mathfrak{M}_{H_1}(v)^{ss} \setminus \mathfrak{M}_H(v)^{ss}.$$

Specifically, we prove

Proposition 3. *Assume that v is a Mukai vector of positive rank with the property that there are no isotropic vectors u of positive rank such that $\langle v, u \rangle = 1$ or 2. Then,*

$$(12) \quad (\langle v, v \rangle + 1) - \dim(\mathfrak{M}_{H_1}(v)^{ss} \setminus \mathfrak{M}_H(v)^{ss}) \geq 2.$$

Therefore, in this situation, $\mathfrak{M}_H(v)^{ss}$ is independent of the choice of ample line bundle H (generic or on a wall) away from codimension 2.

The same statement holds true for the moduli stack $\mathfrak{K}_H(v)^{ss}$ of sheaves with fixed determinant and fixed determinant of the Fourier-Mukai.

Proof. The proof is essentially contained in Proposition 4.3.4 of [MMY2], but since specific aspects of the argument are used below, we give an outline for the benefit of the reader. Let E be a Gieseker H_1 -semistable sheaf, which is however not Gieseker H -semistable. In particular E is slope H_1 -semistable. Consider the Harder-Narasimhan filtration relative to H

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E.$$

By definition, the reduced H -Hilbert polynomials of F_i/F_{i-1} are strictly decreasing. In particular, the H -slopes are decreasing as well. In turn, this implies

$$\mu_{H_1}(F_1) \geq \mu_{H_1}(F_2/F_1) \geq \cdots \geq \mu_{H_1}(F_s/F_{s-1}),$$

and therefore

$$\mu_{H_1}(F_1) \geq \mu_{H_1}(F_2) \geq \cdots \geq \mu_{H_1}(F_s) = \mu_{H_1}(E).$$

Since E is slope H_1 -semistable, we must have equality throughout

$$\mu_{H_1}(F_1) = \mu_{H_1}(F_2) = \cdots = \mu_{H_1}(E).$$

Equivalently, writing

$$v(F_i/F_{i-1}) = v_i \text{ so that } v = \sum_{i=1}^s v_i,$$

we obtain

$$(13) \quad \frac{c_1(v_i) \cdot H_1}{\text{rk } v_i} = \frac{c_1(v) \cdot H_1}{\text{rk } v}, \quad 1 \leq i \leq s.$$

Let $\mathcal{F}_H(v_1, v_2, \dots, v_s)$ be the stack of the Harder-Narashimhan filtrations

$$(14) \quad 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E, \quad E \in \mathfrak{M}(v)$$

such that the quotients F_i/F_{i-1} , $1 \leq i \leq s$ are semistable with respect to H and

$$(15) \quad v(F_i/F_{i-1}) = v_i.$$

Thus

$$\mathfrak{M}_{H_1}(v)^{\mu-ss} \setminus \mathfrak{M}_H(v)^{ss} = \cup_{v_1, \dots, v_s} \mathcal{F}_H(v_1, v_2, \dots, v_s),$$

where (13) is satisfied. Then Lemma 5.3 in [KY] implies

$$(16) \quad \dim \mathcal{F}_H(v_1, v_2, \dots, v_s) = \sum_{i=1}^s \dim \mathfrak{M}_H(v_i)^{ss} + \sum_{i < j} \langle v_i, v_j \rangle.$$

Write $v_i = l_i v'_i$ where v'_i is a primitive Mukai vector. It is shown in Proposition 4.3.4 of [MMY2] that for all i, j we have

$$\langle v'_i, v'_j \rangle \geq 3$$

unless either v'_i or v'_j is isotropic, and in this case $\langle v'_i, v'_j \rangle \geq 1$. We estimate

$$\begin{aligned} (\langle v, v \rangle + 1) &- \dim(\mathfrak{M}_{H_1}(v)^{\mu-ss} \setminus \mathfrak{M}_H(v)^{ss}) \\ &= (\langle v, v \rangle + 1) - \sum_{i < j} \langle v_i, v_j \rangle - \sum_{i=1}^s \dim \mathfrak{M}_H(v_i)^{ss} \\ &= \sum_{i > j} \langle v_i, v_j \rangle - \sum_{i=1}^s (\dim \mathfrak{M}_H(v_i)^{ss} - \langle v_i, v_i \rangle) + 1 \\ &\geq \sum_{i > j} l_i l_j \langle v'_i, v'_j \rangle - \sum_{i=1}^s l_i + 1 \geq \sum_{i > j} l_i l_j - \sum_i l_i + 1 \geq 2. \end{aligned}$$

Indeed, the above inequality is satisfied for $s \geq 4$. The cases $s = 2$ and $s = 3$ need to be considered separately. The detailed analysis is contained in Proposition 4.3.4 of [MMY2]. The only possible exceptions correspond to

- $s = 2$, $l_1 = 1$, $l_2 = l$, v'_2 isotropic, $\langle v'_1, v'_2 \rangle = 1$;
- $s = 2$, $l_1 = 1$, $l_2 = 1$, v'_1 isotropic, $\langle v'_1, v'_2 \rangle = 2$;
- $s = 3$, $l_1 = l_2 = l_3 = 1$, $v = v'_1 + v'_2 + v'_3$, v'_i isotropic, $\langle v'_i, v'_j \rangle = 1$.

In all cases, taking $u = v'_1$, we obtain $\langle v, u \rangle = 1$ or 2 , which contradicts our assumption.

For the final claim about the moduli space $\mathfrak{K}_H(v)^{ss}$, we repeat the proof above. The only modification is the dimension estimate (16) which follows by going over the argument in [KY]. \square

Lemma 4. *Assume that*

$$\langle v, v \rangle > 4 \operatorname{rank}(v).$$

Then no isotropic vector u of positive rank satisfying $\langle v, u \rangle = 1$ or 2 occurs as Mukai vector of a quotient in a Harder-Narasimhan filtration of a sheaf of type v . Therefore, the moduli spaces

$$\mathfrak{M}_H(v)^{ss} \text{ and } \mathfrak{K}_H(v)^{ss}$$

are independent of the polarization H in codimension 1.

Proof. Assume that there exists an isotropic vector u as above such that $\langle v, u \rangle = 1$ or 2 . In this situation, we have

$$\frac{c_1(u) \cdot H_1}{\text{rk } u} = \frac{c_1(v) \cdot H_1}{\text{rk } v} \implies \left(\frac{c_1(u)}{\text{rk } u} - \frac{c_1(v)}{\text{rk } v} \right) \cdot H_1 = 0.$$

Using the Hodge index theorem, we conclude that

$$\left(\frac{c_1(u)}{\text{rk } u} - \frac{c_1(v)}{\text{rk } v} \right)^2 \leq 0.$$

By direct calculation, or via Lemma 1.1 of [KY], we obtain

$$\begin{aligned} \langle v, u \rangle &= -\frac{\text{rk}(v) \cdot \text{rk}(u)}{2} \left(\frac{c_1(u)}{\text{rk } u} - \frac{c_1(v)}{\text{rk } v} \right)^2 + \frac{\text{rk } (u)}{\text{rk } (v)} \cdot \frac{\langle v, v \rangle}{2} + \frac{\text{rk } (v)}{\text{rk } (u)} \cdot \frac{\langle u, u \rangle}{2} \\ &\geq \frac{\text{rk } (u)}{\text{rk } (v)} \cdot \frac{\langle v, v \rangle}{2} > 2 \text{ rk } (u) \geq 2. \end{aligned}$$

This contradiction completes the proof. \square

Remark. The Lemma above applies to the particular situation of a product abelian surface $X = B \times F$ considered in Section 4. We assume here that B, F are not isogenous, so that the section σ and the fiber class f generate the Néron-Severi group. Then, for Mukai vectors

$$v = (r, \sigma + nf, \chi), \quad w = (r', \sigma + nf, \chi')$$

with $\chi, \chi' < 0$ we obtain

$$\langle v, v \rangle = 2n - 2r\chi > -2r\chi \geq 4r,$$

as required in order to apply the Lemma.

The only exception may be the case $\chi = -1$ which will be treated separately. In this situation, we claim that there are no walls between the polarizations

$$H = \sigma + nf, \quad H' = \sigma + Nf,$$

where N is taken sufficiently large to ensure that H' is suitable. Indeed, assuming otherwise, consider a wall defined by an isotropic Mukai vector u such that

$$\langle v, u \rangle = 1 \text{ or } 2.$$

In fact, possibly doubling u , it suffices to analyze the case $\langle v, u \rangle = 2$. Let

$$H_0 = \sigma + kf, \quad k \geq n,$$

be an ample divisor on this wall, where $k \in \mathbb{Q}$. By definition, the vector u appears as the Mukai vector of a quotient in the Harder-Narasimhan filtration for H_0 . Setting $u = (p, \eta, q)$ with $p > 0$, we obtain from (13) that

$$\left(\frac{\eta}{p} - \frac{H}{r}\right) \cdot H_0 = 0 \implies (r\eta - pH) \cdot H_0 = 0.$$

Writing

$$r\eta - pH = a\sigma + bf,$$

we calculate

$$(r\eta - pH) \cdot H_0 = (a\sigma + bf) \cdot (\sigma + kf) = 0 \implies b = -ak.$$

Consequently,

$$(17) \quad (r\eta - pH)^2 = (a\sigma + bf)^2 = 2ab = -2a^2k \leq -2k,$$

unless $a = b = 0$. This particular situation can be analyzed by exactly the same methods; we leave the verification to the reader. In any case, the conditions that u is isotropic and $\langle v, u \rangle = 2$ translate into

$$\eta^2 = 2pq, \quad \eta \cdot H = -p + qr + 2,$$

respectively. With this understood, we compute the left hand side of (17)

$$(r\eta - pH)^2 = r^2\eta^2 + p^2H^2 - 2pr(\eta \cdot H) = 2np^2 + 2pr(p - 2) \geq 0 > -2k$$

with the only possible exception $p = 1$. In this case, the above calculation yields

$$(r\eta - pH)^2 = 2n - 2r.$$

By orthogonality,

$$2n = -r'\chi - r\chi' \geq r + r' > r$$

which implies

$$(r\eta - pH)^2 = 2n - 2r > -2n \geq -2k.$$

This contradicts (17), showing that there is no wall separating H from a suitable polarization.

□

6. THE VERLINDE SHEAVES ARE LOCALLY FREE

The goal of this section is to prove Theorem 4. We show that for any (X, H) , the dimension of the space of sections of the theta line bundles is given by the expected formula (2) for a very general class of Mukai vectors. This holds even without knowing the vanishing of higher cohomology. As a consequence, the Verlinde sheaves \mathbb{V} and \mathbb{W} used in the degeneration argument of Section 4 are in fact locally free over the entire moduli space \mathcal{A} of pairs (X, H) .

The result should be compared to Proposition 2 of Section 4. The *generic* local-freeness yielded by Proposition 2 was sufficient for proving our main Theorem 3. By contrast, Theorem 4 gives *global* local-freeness in great generality, and will be useful for future strange duality studies.

We split the theorem into two statements with proofs of different flavors. First, we show

Proposition 4. *Let (X, H) be a polarized abelian surface. Assume that*

$$v = (r, dH, \chi), \quad w = (r', d'H, \chi')$$

are orthogonal primitive Mukai vectors of ranks $r, r' \geq 2$ such that

- (i) $d, d' > 0$;
- (ii) $\chi < 0, \chi' < 0$.

Assume furthermore that if $(d, \chi) = (1, -1)$, then (X, H) is not a product of two elliptic curves. We have

$$h^0(\mathbb{K}_v, \Theta_w) = \chi(\mathbb{K}_v, \Theta_w) = \frac{d_v^2}{d_v + d_w} \begin{pmatrix} d_v + d_w \\ d_v \end{pmatrix}.$$

In the same context, the Proposition implies the requisite statement for the moduli space \mathbb{M}_v :

Proposition 5. *In the setup of Proposition 4, for any representative $F \in \mathbb{K}_w$ we have*

$$h^0(\mathbb{M}_v, \Theta_F) = \chi(\mathbb{M}_v, \Theta_F) = \frac{d_w^2}{d_v + d_w} \begin{pmatrix} d_v + d_w \\ d_v \end{pmatrix}.$$

6.1. Proof of Proposition 4. We begin by explaining the strategy of the proof when \mathbb{K}_v is smooth. The key point is Lemma 5 below which shows that $\Theta_w \rightarrow \mathbb{K}_v$ is movable, hence (big and) nef on a smooth birational model of \mathbb{K}_v , cf. Theorem 7 of [HT]. The birational models of \mathbb{K}_v arise as moduli spaces of Bridgeland stable objects. The dimension calculation is carried out on the moduli space of Bridgeland stable objects, where the higher cohomology vanishes. The Proposition follows since wall-crossings do

not change the dimension of the space of sections. The case when K_v may be singular requires first to desingularize the moduli space. The above argument can then be repeated on a symplectic resolution.

Let us elaborate the discussion. As already remarked, the proof uses moduli spaces of Bridgeland stable objects. Specifically, we consider stability conditions $\sigma = \sigma_{s,t} = (Z_{s,t}, \mathcal{A}_{s,t})$, for $t > 0$, corresponding to central charges

$$Z_{s,t}(E) = \langle \exp((s + it)H), v(E) \rangle.$$

The heart $\mathcal{A}_{s,t}$ has as objects certain 2-step complexes, and is obtained as a tilt of the abelian category of coherent sheaves on X at a certain torsion pair; the exact definition will not be used below, but we refer the reader to [Br2] for details. We form the moduli spaces $M_v(\sigma)$ of σ -semistable objects of type v . The moduli space comes equipped with the Albanese map

$$a : M_v(\sigma) \rightarrow X \times \widehat{X},$$

and we write $K_v(\sigma)$ for the Albanese fiber.

We begin by analyzing the case K_v smooth. The following observations (a)-(c) are useful for the argument.

- (a) In the large volume limit $t \gg 0$, Bridgeland stability with respect to $\sigma_{s,\infty} := \sigma_{s,t}$ coincides with Gieseker stability, cf. [Br2], Section 14.

The next remarks (b)-(c) are contained in the recent papers [MMY1] and [Y2]. For $K3$ surfaces, the similar statements are found in [BM].

- (b) The space of stability conditions admits a wall and chamber decomposition, so that the moduli spaces are constant in each chamber, but they undergo explicit birational transformations as walls are crossed. These birational transformations are regular in codimension 1.

For the next remark, observe that the theta map (1) gives an isomorphism

$$\Theta : (v^\vee)^\perp \rightarrow \text{Pic}(K_v(\sigma)),$$

in such a fashion that the Beauville-Bogomolov form on the right hand side corresponds to the Mukai pairing on the left hand side. Two basic (real) cones of divisors are necessary for our purposes. First, the positive cone

$$\text{Pos}(K_v(\sigma)) \hookrightarrow \text{Pic}(K_v(\sigma))_{\mathbb{R}}$$

can be expressed via the Beauville-Bogomolov form

$$\text{Pos}(K_v(\sigma)) = \{x : \langle x, x \rangle > 0, \langle x, A \rangle > 0 \text{ for a fixed ample divisor } A \text{ over } K_v(\sigma)\}.$$

Second, the movable cone

$$\mathrm{Mov}(\mathbf{K}_v(\sigma)) \hookrightarrow \mathrm{Pic}(\mathbf{K}_v(\sigma))_{\mathbb{R}}$$

is generated by divisors whose stable base locus has codimension 2 or higher. Positive movable divisors are big and nef on some smooth birational models, cf. Theorem 7 of [HT]. In our context, we have the following result obtained via the study of the movable cone in [MMY2]:

(c) A positive movable divisor

$$\mathrm{Mov}(\mathbf{K}_v(\sigma_{s,\infty})) \cap \mathrm{Pos}(\mathbf{K}_v(\sigma_{s,\infty}))$$

is identified, under the birational wall crossings of (b), with a big and nef divisor on a smooth moduli space $\mathbf{K}_v(\sigma_{s,t})$ of Bridgeland stable objects:

$$\{\Theta_w \rightarrow \mathbf{K}_v(\sigma_{s,\infty})\} \longleftrightarrow \{\Theta_w \rightarrow \mathbf{K}_v(\sigma_{s,t})\}.$$

(Note that the Mukai vector w labeling the theta line bundle may undergo Weyl reflections when crossing divisorial walls in $(v^\vee)^\perp$. However, since there are no divisorial walls within the movable chamber, w does not change in the present setting.)

The essential ingredient is then provided by the following

Lemma 5. *For v and w as in Proposition 4, the line bundle $\Theta_w \rightarrow \mathbf{K}_v(\sigma_{s,\infty})$ belongs to the positive movable cone.*

As a consequence of remarks (a)-(c) and of the lemma, we note

$$h^0(\mathbf{K}_v(\sigma_{s,\infty}), \Theta_w) = h^0(\mathbf{K}_v(\sigma_{s,t}), \Theta_w) = \chi(\mathbf{K}_v(\sigma_{s,t}), \Theta_w).$$

By the same argument as for the usual Gieseker stability, as in Proposition 1 of [MO1], we further have

$$\chi(\mathbf{K}_v(\sigma_{s,t}), \Theta_w) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

We conclude that

$$h^0(\mathbf{K}_v(\sigma_{s,\infty}), \Theta_w) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v},$$

as claimed in Proposition 4. □

Proof of Lemma 5. We begin by noting that Θ_w is positive in the Gieseker chamber. Indeed,

$$\langle \Theta_w, \Theta_w \rangle = \langle w, w \rangle > 0.$$

For the second inequality, an ample divisor on the moduli space $K_v(\sigma_{s,\infty})$ is constructed in [LeP2]; see also Remark 8.1.12 of [HL]. This divisor takes the form Θ_a for

$$a = (r, rmH, -2mnd - \chi), \quad \text{where } m \gg 0.$$

Recalling that $w = (r', d'H, \chi')$, we have

$$\langle \Theta_w, \Theta_a \rangle = \langle w, a \rangle = 2nm(d'r + dr') - r\chi' + r'\chi > 0,$$

as needed.

We will now show that for the vector

$$w_1 = (2nd, -\chi H, 0)$$

the line bundle Θ_{w_1} belongs to the closure of the movable cone for the Gieseker chamber.

We will combine this with a well-known result of Jun Li [Li2]. For the vector

$$w_0 = (0, rH, -2nd)$$

the associated theta line bundle

$$\Theta_{w_0} \rightarrow K_v(\sigma_{s,\infty})$$

is big and nef, so in particular it is in the closure of the movable cone. Notice now that the vector w is a positive linear combination of w_0 and w_1 ,

$$w = \frac{1}{2nd} (-\chi'w_0 + r'w_1),$$

hence Θ_w is movable.

To prove the claim about w_1 , we will use the description of the movable cone given in [MMY1] and [BM]. Specifically, we consider the hyperplanes in $\overline{\text{Pos}}(K_v(\sigma_{s,\infty}))$ given by

$$\Theta((u^\vee)^\perp \cap (v^\vee)^\perp), \quad 1 \leq \langle v, u \rangle \leq 2, \quad \langle u, u \rangle = 0.$$

The movable cone is cut out by these hyperplanes. To prove that Θ_{w_1} and Θ_{w_0} belong to the same chamber, it suffices to show that

$$\langle w_0, u^\vee \rangle \geq 0 \iff \langle w_1, u^\vee \rangle \geq 0,$$

whenever u is isotropic and $1 \leq \langle v, u \rangle \leq 2$. The first inequality above will in fact turn out strict for rank 3 or higher.

We assume $r > 2$ first. Let us write $u = (p, \eta, q)$ where

$$\eta^2 = 2pq, \quad p, q \in \mathbb{Z}.$$

Changing u into $-u$, we may furthermore assume that $p \geq 0$ and $\langle v, u \rangle = \pm 1, \pm 2$.

Recalling that $v = (r, dH, \chi)$, we calculate

$$(18) \quad \langle v, u \rangle = d(H \cdot \eta) - p\chi - qr = \pm 1 \text{ or } \pm 2.$$

We compute

$$(19) \quad \langle w_0, u^\vee \rangle \geq 0 \iff -r(H \cdot \eta) + 2ndp \geq 0.$$

Similarly,

$$(20) \quad \langle w_1, u^\vee \rangle \geq 0 \iff \chi(H \cdot \eta) - 2ndq \geq 0.$$

We therefore need to show that

$$-r(H \cdot \eta) + 2ndp \geq 0 \iff \chi(H \cdot \eta) - 2ndq \geq 0.$$

We consider first the case when $p = 0$. Then, replacing u by $-u$ we may assume that $H \cdot \eta \geq 0$. In fact, $H \cdot \eta = 0$ is impossible by (18) since $r > 2$. Therefore, $H \cdot \eta > 0$. In this situation, (19) is false. We argue that (20) is false as well. Assuming otherwise, we have

$$\chi(H \cdot \eta) \geq 2ndq \implies q < 0.$$

This is however incompatible with (18) which reads

$$d(H \cdot \eta) + r(-q) = \pm 1, \pm 2,$$

which is impossible for $r > 2$.

The crux of the argument is the case $p > 0$. In this situation, we distinguish the following subcases:

- (i) Assume $H \cdot \eta = 0$. By the Hodge index theorem $\eta^2 \leq 0$ hence

$$pq = \frac{\eta^2}{2} \leq 0 \implies q \leq 0.$$

This shows that both (19) and (20) are true at the same time.

- (ii) Assume $H \cdot \eta < 0$. In this case, (19) is true. We prove that (20) is true as well.

Assuming otherwise, we obtain that

$$\chi(H \cdot \eta) - 2ndq < 0.$$

In particular $q > 0$ and multiplying by $p > 0$ we see that

$$\frac{p\chi}{2nd}(H \cdot \eta) < pq = \frac{\eta^2}{2}.$$

By the Hodge index theorem, we have

$$\eta^2 \leq \frac{(H \cdot \eta)^2}{2n}.$$

The above inequality becomes

$$\frac{p\chi}{2nd}(H \cdot \eta) < \frac{(H \cdot \eta)^2}{4n} \implies (H \cdot \eta) < \frac{2p\chi}{d}.$$

We obtain therefore

$$d(H \cdot \eta) - p\chi - qr < 2p\chi - p\chi - qr = p\chi - qr < -2,$$

using $\chi < 0$ and $q > 0$. This contradicts (18). Thus (20) must be true as well.

- (iii) Assume $H \cdot \eta > 0$. Equation (18) implies that $q \geq 0$. In this case, the inequality (20) is false. We argue that (19) is false as well. Assume otherwise, so that

$$r(H \cdot \eta) \leq 2ndp \implies \frac{rq}{2nd}(H \cdot \eta) \leq pq = \frac{\eta^2}{2}.$$

Again by the Hodge index theorem, we have

$$\eta^2 \leq \frac{(H \cdot \eta)^2}{2n}$$

yielding

$$\frac{rq}{2nd}(H \cdot \eta) \leq pq = \frac{\eta^2}{2} \leq \frac{(H \cdot \eta)^2}{4n} \implies 2rq \leq d(H \cdot \eta).$$

We obtain

$$d(H \cdot \eta) - p\chi - rq \geq 2rq - p\chi - rq = rq - p\chi > 2$$

if $q > 0$, contradicting (18). When $q = 0$, equation (18) yields

$$d(H \cdot \eta) - p\chi = \pm 1, \pm 2,$$

which implies $d(H \cdot \eta) = 1, p\chi = -1$. Therefore $(d, \chi) = (1, -1)$ and $H \cdot \eta = 1, \eta^2 = 0$. In this case, (X, H) is a product of elliptic curves, which is not allowed.¹

When $r = 2$, the same argument goes through with the only exception corresponding to the case

$$p = 0, H \cdot \eta = 0.$$

Since $\eta^2 = 2pq = 0$ we obtain $\eta = 0$ by the Hodge index theorem. This yields the isotropic vector $u = (0, 0, 1)$. In fact, w_0 lies on the wall determined by u , hence we cannot pin down on which side of the wall w_1 lies. To remedy this problem, we replace w_0 by the vector

$$a = (r, rmH, -2mnd - \chi) = mw_0 + (r, 0, -\chi)$$

¹To see that X is a product, write $\tau = H - n \cdot \eta$. Therefore,

$$\eta^2 = \tau^2 = 0, \quad \eta \cdot \tau = 1.$$

In this situation, η and τ are represented by two elliptic curves E and F , cf. Proposition 2.3 in [K]. The sum morphism

$$s : E \times F \rightarrow X$$

must be an isogeny. The preimage of the origin corresponds to the intersection $E \cap F$, hence s must be an isomorphism.

which we have already seen to give an ample theta bundle for $m \gg 0$. For the vector $u = (0, 0, 1)$, direct computation shows

$$\langle a, u \rangle < 0, \quad \langle w_1, u \rangle < 0,$$

hence w_1 and a are also on the same side of the wall determined by u .

This completes the analysis, and therefore the proof when \mathbf{K}_v is smooth.

However, \mathbf{K}_v may be singular when the polarization H is not generic. In this situation, for any $\beta \in \text{NS}(X)_{\mathbb{Q}}$, we consider the moduli space of β -twisted H -semistable sheaves. Recall that a sheaf E is β -twisted H -semistable provided that

(i) for all subsheaves $F \subset E$, we have

$$\frac{c_1(F) \cdot H}{\text{rk}(F)} \leq \frac{c_1(E) \cdot H}{\text{rk}(E)};$$

(ii) if equality holds in (i), then

$$\frac{\chi(F) - c_1(F) \cdot \beta}{\text{rk}(F)} \leq \frac{\chi(E) - c_1(E) \cdot \beta}{\text{rk}(E)}.$$

We form the moduli space $\mathbf{K}_{\beta}(v)$ of β -twisted H -semistable sheaves. In fact, remark (a) above applies here as well, and consequently, $\mathbf{K}_{\beta}(v)$ can be viewed as a moduli space of Bridgeland's stable objects. In addition, if β is appropriately chosen, then the moduli space $\mathbf{K}_{\beta}(v)$ consists of stable sheaves only, and therefore is a smooth non-empty holomorphic symplectic manifold; see for instance Lemma 5.4 of [A]. Furthermore, Lemma 5.5 in [A] shows that there is a surjective morphism

$$\pi : \mathbf{K}_{\beta}(v) \rightarrow \mathbf{K}_v,$$

which is therefore a symplectic resolution. As a consequence of Proposition 1.3 of [Bea2] we have

$$\mathbf{R}\pi_* \mathcal{O}_{\mathbf{K}_{\beta}(v)} = \mathcal{O}_{\mathbf{K}_v}.$$

Now, as the moduli space $\mathbf{K}_{\beta}(v)$ consists of stable sheaves only, it carries a theta line bundle Θ_w . Furthermore, the line bundle Θ_w descends to the singular moduli space \mathbf{K}_v , which may contain strictly semistables. This is a consequence of Kempf's lemma and is shown to hold true in Theorem 8.1.5 of [HL]. The essential point is that $c_1(v) = dH$ is a multiple of the polarization. As a corollary,

$$H^0(\mathbf{K}_v, \Theta_w) = H^0(\mathbf{K}_{\beta}(v), \pi^* \Theta_w) = H^0(\mathbf{K}_{\beta}(v), \Theta_w).$$

We claim that Θ_w is movable over the smooth moduli space $\mathbf{K}_{\beta}(v)$. In fact, the argument we presented in the untwisted case carries over to the twisted situation. An essential ingredient of the proof is that Jun Li's line bundle is big and nef. This continues to hold

over $\mathbf{K}_\beta(v)$ by pullback, at least for β chosen as above. Alternatively, ample divisors are constructed in Lemma 5.5.2 of [MMY1]. Since Θ_w is movable, we conclude that

$$h^0(\mathbf{K}_\beta(v), \Theta_w) = \chi(\mathbf{K}_\beta(v), \Theta_w) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v},$$

as claimed. This completes the proof. \square

Remark. The argument above also remains valid in ranks 0 and 1. Consequently, the dimension calculation of Proposition 4 holds true for all primitive orthogonal Mukai vectors

$$v = (r, dH, \chi), \quad w = (r', d'H, \chi') \quad \text{with } r, r' \geq 0, \quad d, d' > 0, \quad \chi, \chi' < 0,$$

with the extra assumption that

$$- (X, H) \text{ is not a product when } (d, \chi) = (1, -1) \text{ or when } (r, d) = (1, 1).$$

6.2. Proof of Proposition 5. To prove the Proposition, we use the diagram

$$\begin{array}{ccc} \mathbf{K}_v \times X \times \widehat{X} & \xrightarrow{\Phi_v} & \mathbf{M}_v \\ \downarrow p & & \downarrow \mathbf{a} \\ X \times \widehat{X} & \xrightarrow{\Psi_v} & X \times \widehat{X} \end{array} .$$

Here, $\Phi_v : \mathbf{K}_v \times X \times \widehat{X} \rightarrow \mathbf{M}_v$ is defined as

$$\Phi_v(E, x, y) = t_x^* E \otimes y,$$

and

$$\mathbf{a} : \mathbf{M}_v \rightarrow X \times \widehat{X}$$

is the Albanese map. Both Φ_v and Ψ_v are étale of degree d_v^4 [Y1], [MO1]. In fact, it is proved in [Y1] that

$$\Psi_v(x, y) = (-\chi x - d\varphi_{\widehat{H}}(y), d\varphi_H(x) + ry),$$

where as usual

$$\widehat{H} \rightarrow \widehat{X}$$

is the inverse determinant of the Fourier-Mukai transform of H , and $\varphi_H, \varphi_{\widehat{H}}$ denote the Mumford homomorphisms. This explicit expression will however not be needed below.

Fix $F \in \mathbf{K}_w$. We have

$$\Phi_v^* \Theta_F = \Theta_w \boxtimes \mathcal{L}$$

for a line bundle $\mathcal{L} \rightarrow X \times \widehat{X}$. It is shown in Proposition 4 of [MO1] that

$$\chi(\mathcal{L}) = d_v^2 d_w^2.$$

In fact, by Lemma 1 in [O], up to numerical equivalence we have

$$(21) \quad \mathcal{L} = H^a \boxtimes \widehat{H}^b \otimes \mathcal{P}^c,$$

where $\mathcal{P} \rightarrow X \times \widehat{X}$ is the Poincaré bundle, and

$$a = -(\chi d' + \chi' d), \quad b = rd' + r'd, \quad c = dd'n + r'\chi = -dd'n - r\chi'.$$

In consequence of the assumptions $\chi, \chi' < 0$ and $d, d' > 0$, and also of the calculation

$$abn - c^2 = d_v d_w > 0,$$

we obtain the inequalities

$$a > 0, \quad b > 0, \quad abn > c^2.$$

These inequalities ensure that the line bundle \mathcal{L} is ample. To see this, we use the special form of the Nakai-Moishezon criterion for ampleness in the context of abelian varieties, as stated on page 77 of [BL]. Specifically, for abelian varieties, the criterion asserts that it is enough to check ampleness numerically on hyperplanes and intersections of hyperplanes under any fixed projective embedding, such as the one induced by $H + \widehat{H}$. A direct calculation then shows that a line bundle $\mathcal{L} \rightarrow X \times \widehat{X}$ of the form (21) is ample if and only if the three inequalities above are satisfied. In consequence, \mathcal{L} has no higher cohomology.

With this understood, we write with the aid of Proposition 4

$$(22) \quad h^0(\mathbb{K}_v \times X \times \widehat{X}, \Theta_w \boxtimes \mathcal{L}) = h^0(\mathbb{K}_v, \Theta_w) h^0(X \times \widehat{X}, \mathcal{L}) = \chi(\mathbb{K}_v, \Theta_w) \chi(X \times \widehat{X}, \mathcal{L}) \\ = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v} \cdot (d_v d_w)^2.$$

On the other hand,

$$(23) \quad h^0(\mathbb{K}_v \times X \times \widehat{X}, \Phi_v^* \Theta_F) = h^0(\mathbb{M}_v, (\Phi_v)_* \Phi_v^* \Theta_F) = \sum_{\tau} h^0(\mathbb{M}_v, \Theta_F \otimes \mathbf{a}^* \mathbb{L}_{\tau})$$

where

$$(\Psi_v)_* \mathcal{O} = \bigoplus_{\tau} \mathbb{L}_{\tau},$$

over $X \times \widehat{X}$. The line bundles \mathbb{L}_{τ} appearing in the decomposition above are indexed by the characters $\tau \in \widehat{\mathbb{G}}_v$ of the group

$$\mathbb{G}_v = \text{Ker } \Psi_v.$$

We claim that

Lemma 6. *For each character τ of \mathbf{G}_v , there exists an automorphism $f_\tau : \mathbf{M}_v \rightarrow \mathbf{M}_v$ such that*

$$\Theta_F \otimes \mathbf{a}^* \mathbb{L}_\tau = f_\tau^* \Theta_F.$$

By the lemma, we therefore have

$$h^0(\mathbf{M}_v, \Theta_F \otimes \mathbf{a}^* \mathbb{L}_\tau) = h^0(\mathbf{M}_v, \Theta_F)$$

hence by (23) we obtain

$$h^0(\mathbf{K}_v \times X \times \widehat{X}, \Phi_v^* \Theta_F) = \deg \Psi_v \cdot h^0(\mathbf{M}_v, \Theta_F) = d_v^4 \cdot h^0(\mathbf{M}_v, \Theta_F).$$

This implies via (22) that

$$h^0(\mathbf{M}_v, \Theta_F) = \frac{d_w^2}{d_v + d_w} \binom{d_v + d_w}{d_v},$$

establishing Proposition 5. □

Proof of Lemma 6. We consider the group

$$K(\mathcal{L}) \hookrightarrow X \times \widehat{X}$$

of pairs (x, y) leaving \mathcal{L} invariant by translation

$$t_{(x,y)}^* \mathcal{L} \simeq \mathcal{L}.$$

The group $K(\mathcal{L})$ has $\chi(\mathcal{L})^2 = (d_v d_w)^4$ elements.

For each pair $(x, y) \in K(\mathcal{L})$, we define the automorphism

$$f_{(x,y)} : \mathbf{M}_v \rightarrow \mathbf{M}_v$$

given by

$$f_{(x,y)}(E) = t_x^* E \otimes y.$$

We show that for $(x, y) \in K(\mathcal{L})$ we can find a line bundle $\mathbb{L}_\tau \in \widehat{\mathbf{G}}_v$ such that

$$(24) \quad f_{(x,y)}^* \Theta_F = \Theta_F \otimes \mathbf{a}^* \mathbb{L}_\tau.$$

Indeed, the two line bundles $f_{(x,y)}^* \Theta_F$ and Θ_F both restrict to Θ_w on each fiber of the Albanese map \mathbf{a} , hence for some line bundle $\mathbb{L} \rightarrow X \times \widehat{X}$ we have

$$f_{(x,y)}^* \Theta_F = \Theta_F \otimes \mathbf{a}^* \mathbb{L}.$$

It remains to explain that

$$\Psi_v^* \mathbb{L} = \mathcal{O},$$

or equivalently that

$$\Phi_v^* f_{(x,y)}^* \Theta_F = \Phi_v^* \Theta_F.$$

Direct calculation shows that over $K_v \times X \times \widehat{X}$ we have

$$f_{(x,y)} \circ \Phi_v = \Phi_v \circ (1, t_{(x,y)}).$$

Therefore

$$\begin{aligned} \Phi_v^* f_{(x,y)}^* \Theta_F &= (1, t_{(x,y)})^* \Phi_v^* \Theta_F = (1, t_{(x,y)})^* (\Theta_w \boxtimes \mathcal{L}) \\ &= \Theta_w \boxtimes t_{(x,y)}^* \mathcal{L} = \Theta_w \boxtimes \mathcal{L} = \Phi_v^* \Theta_F. \end{aligned}$$

As a consequence of (24), there exists a group homomorphism

$$\alpha : K(\mathcal{L}) \rightarrow \widehat{G}_v.$$

To complete the proof of the Lemma, we argue that α is surjective. Since

$$\text{order } K(\mathcal{L}) = (d_v d_w)^4, \text{ order } G_v = d_v^4$$

it suffices to prove that

$$\text{order } \text{Ker } \alpha = d_w^4.$$

In fact, we claim that

$$(25) \quad \text{Ker } \alpha \simeq G_w,$$

where G_w is the kernel of the morphism Ψ_w in the diagram

$$\begin{array}{ccc} K_w \times X \times \widehat{X} & \xrightarrow{\Phi_w} & M_w \\ \downarrow p & & \downarrow a \\ X \times \widehat{X} & \xrightarrow{\Psi_w} & X \times \widehat{X} \end{array}.$$

Here, $\Phi_w : K_w \times X \times \widehat{X} \rightarrow M_w$ is defined as

$$\Phi_w(G, x, y) = t_{-x}^* G \otimes y,$$

and

$$a : M_w \rightarrow X \times \widehat{X}$$

is the Albanese map

$$a(G) = (\det \widehat{G} \otimes \widehat{H}^{d'}, \det G \otimes H^{-d'}).$$

Furthermore, just as above, Φ_w and Ψ_w both have degree d_w^4 . To prove (25), note that

$$(x, y) \in \text{Ker } \alpha \iff f_{(x,y)}^* \Theta_F = \Theta_F \iff \Theta_{t_{-x}^* F \otimes y} = \Theta_F.$$

By [MO1], the last equality happens if and only if

$$\begin{aligned} \det(t_{-x}^* F \otimes y) &= \det F \text{ and } \det(t_{-x}^* \widehat{F} \otimes y) = \det \widehat{F} \\ \iff (a \circ \Phi_w)(F, x, y) &= 0 \iff \Psi_w(x, y) = 0 \iff (x, y) \in G_w, \end{aligned}$$

as claimed. The proof of the lemma is completed. \square

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