ON A CLASS OF SEMIHOMOGENEOUS VECTOR BUNDLES

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Abstract. We study a class of semihomogeneous vector bundles over the product of an abelian variety and its dual. For abelian surfaces, we connect these semihomogeneous bundles to the Verlinde bundles of generalized theta functions constructed from the moduli spaces of sheaves.

1. Introduction

1.1. **A class of semihomogeneous bundles.** Let A be an abelian variety over the complex numbers, and Θ a symmetric line bundle

$$(-1)^\star\Theta = \Theta,$$

inducing a principal polarization on A. In this article, we construct and study a class of semihomogeneous bundles

$$\mathbf{W}(\mathsf{P}) \to A \times \widehat{A}$$

indexed by certain triples of rational numbers

$$\mathsf{P} = (u, v, h).$$

Specifically, $\mathbf{W}(\mathsf{P})$ is the minimal symmetric semihomogeneous vector bundle over $A \times \widehat{A}$ of slope determined by P:

$$\mu(\mathbf{W}(\mathsf{P})) = \frac{\det \mathbf{W}(\mathsf{P})}{\text{rank } \mathbf{W}(\mathsf{P})} = u\Theta + v\widehat{\Theta} + h\mathcal{P} \in \text{Pic}(A \times \widehat{A}) \otimes \mathbb{Q},$$

with $\mathcal{P} \to A \times \widehat{A}$ denoting the normalized Poincaré bundle, and

$$\widehat{\Theta} = \det \mathbf{R}\mathcal{S}(\Theta)$$

denoting the determinant of the Fourier-Mukai transform with kernel $\mathcal{P}$. The construction and study of the bundles $\mathbf{W}(\mathsf{P})$ will be carried out in Section 2.

The theory of semihomogeneous vector bundles over abelian varieties was developed by Mukai in great generality [M2], and the bundles $\mathbf{W}(\mathsf{P})$ fit into Mukai's theory. The motivation for singling out this particular class comes from the conjectural relationship with the bundles of generalized theta functions

$$\mathbf{E} \to A \times \widehat{A},$$

when A is an abelian surface. We will now explain this connection.

1.2. **Verlinde bundles.** Let A be a polarized abelian surface, and consider the moduli spaces $\mathfrak{M}_v$ of Gieseker semistable sheaves $E \to A$ of fixed topological type encoded in a fixed Mukai vector

$$v = \mathrm{ch}(E) \in H^{\star}(A, \mathbb{C}).$$

The moduli space $\mathfrak{M}_v$ comes equipped with the following structures

– the Albanese morphism which takes sheaves to their determinants and determinants of the Fourier-Mukai transform

$$\alpha_v : \mathfrak{M}_v \to A \times \widehat{A}$$

– determinant line bundles $\Theta_w \to \mathfrak{M}_v$ which depend on the choice of a second Mukai vector w chosen orthogonal to v in K-theory in the sense that

$$\chi(v \cdot w) = 0.$$

The exact definitions and normalization conventions will be recalled in Section 3.

The sections of Θ_w either over $\mathfrak{M}_v$ or over the fibers of the Albanese morphism α_v are termed *generalized theta functions.* [1] With the latter interpretation, the Verlinde bundles, introduced in this setting in [O3], encode the space of generalized theta functions as the determinant and determinant of the Fourier-Mukai transform vary. Specifically, we set

$$\mathbf{E}(v, w) = (\alpha_v)_{\star} \Theta_w.$$

The local freeness of $\mathbf{E}(v, w)$ is noted in Proposition 2 below.

The above construction parallels the case of moduli of bundles over smooth curves initially studied by Popa [Po]. Succinctly, if C denotes a smooth projective curve, and $\mathfrak{M}_v^{\mathsf{C}}$ stands for the moduli space of semistable bundles $E \to C$ with

$$\mathrm{ch}(E) = v,$$

the structures introduced above take analogous form:

– the Albanese morphism maps bundles to their determinants

$$\alpha_v^{\mathsf{C}} : \mathfrak{M}_v^{\mathsf{C}} \to \mathrm{Jac}\ (C)$$

– there are determinant line bundles $\Theta_w \to \mathfrak{M}_v^{\mathsf{C}}$ which depend on the choice of a second vector w orthogonal to v.

Similarly, we set

$$\mathbf{E}^{\mathsf{C}}(v, w) = (\alpha_v^{\mathsf{C}})_{\star} \Theta_w.$$

The question we wish to address is the following:

[1] The terminology is motivated by the case of vectors of rank 1 over smooth projective curves which corresponds to classical theta functions over the Jacobian.

Question 1. *What are the Verlinde bundles $\mathbf{E}(v,w)$ over the abelian four-fold $A \times \widehat{A}$?*

The analogous question for curves was answered completely in [O2]; see Section 1.4 below.

1.3. **Strange duality and Fourier-Mukai symmetries.** Over curves, the motivation for the study of the Verlinde vector bundles comes from the strange duality conjecture, see [Po]. In [O3] we emphasized a number of common features that the curve and abelian strange duality share. We point out further analogies here.

Strange duality predicts a geometric isomorphism between spaces of generalized theta functions as the vectors v and w are interchanged. Specifically, write K_v and K_v^{c} the fibers of the Albanese morphisms α_v and α_v^{c} over the origin, in the abelian or curve case respectively. Under the numerical assumption (**A.2**) of Section 3, one expects a *geometrically induced isomorphism*

$$\mathsf{SD} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(K_w, \Theta_v), \tag{1}$$

and similarly

$$\mathsf{SD}^{\mathsf{c}} : H^0(\mathfrak{M}_v^{\mathsf{c}}, \Theta_w)^\vee \to H^0(K_w^{\mathsf{c}}, \Theta_v).$$

The curve isomorphism is already established, see [Bel1], [MO1], [Bel2]. The abelian situation is less understood, but [BMOY] proves the isomorphism generically for an infinite number of Mukai vectors.

Note however that the duality suffers from a slight asymmetry in the choice of moduli spaces involved on both sides. This asymmetry is corrected in the Verlinde bundles. In fact, it is explained in [Pol] and [O3] that the strange duality conjecture predicts the existence of specific geometric isomorphisms

$$\mathsf{SD} : \mathbf{E}(v,w)^\vee \to \widehat{\mathbf{E}(w,v)} \tag{2}$$

and

$$\mathsf{SD}^{\mathsf{c}} : \mathbf{E}^{\mathsf{c}}(v,w)^\vee \to \widehat{\mathbf{E}^{\mathsf{c}}(w,v)}$$

with the hat denoting the Fourier-Mukai transforms.

1.4. **Curves.** It will be shown in Proposition 2 below that the bundles $\mathbf{E}(v,w)$ and $\mathbf{E}^{\mathsf{c}}(v,w)$ are semihomogeneous over the corresponding base abelian varieties.

Regarding Question 1, in the curve context, the Verlinde bundles are determined explicitly in [O2]. Roughly speaking, the answer takes the form

$$\mathbf{E}^{\mathsf{c}}(v,w) = \bigoplus_{\ell} \mathbf{W}^{\mathsf{c}}(\mathsf{p}) \otimes \ell^{\oplus \mathsf{m}_\ell},$$

where

$$\mathbf{W}^{\mathsf{c}}(\mathsf{p}) \to \text{Jac }(C)$$

are certain indecomposable semihomogeneous vector bundles over the Jacobian. These bundles are generally constructed over any abelian variety endowed with a principal polarization Θ, and they depend on a rational number p which controls the slope:

$$\mu(\mathbf{W}^{\mathsf{c}}(\mathsf{p})) = \mathsf{p}\Theta.$$

In the setup here, $\mathsf{p} = \frac{\text{rk } w}{\text{rk } v}$. The bundles $\mathbf{W}^{\mathsf{c}}(\mathsf{p})$ were first pointed out in [M2] and further studied in [O2] in relation to the curve analogue of Question 1. They will be reviewed in Section 2; there, for convenience, they are denoted $\mathbf{W}_{a,b}$ for certain pairs of coprime positive integers $(a, b) = 1$. We read off a, b by writing in *lowest terms*

$$\mathsf{p} = \frac{\text{rk } w}{\text{rk } v} = \frac{b}{a}.$$

The line bundles $\ell \to \text{Jac}(C)$ are explicit torsion bundles over the Jacobian, and their multiplicities m_ℓ are explicit as well. We refer the reader to [O2] for the exact expressions.

1.5. **Abelian surfaces and overview of results.** Turning to abelian surfaces, the bundles $\mathbf{E}(v, w)$ were determined, and thus Question 1 was completely settled in [O3] when $c_1(v) = 0$; see also Lemma 11. The bundles $\mathbf{W}^{\mathsf{c}}(\mathsf{p})$ alluded to above appear in the answer.

In general however, the numerics are entangled in a more complicated fashion. Finding the correct framework to understand these numerics requires some effort. In some sense, to decouple the data, the bundles $\mathbf{W}(\mathsf{P})$ constructed in Section 2 are needed.

Specifically, in Section 3 we formulate the explicit Conjectures 1-3 relating the bundles of generalized theta functions $\mathbf{E}(v, w)$ to the bundles $\mathbf{W}(\mathsf{P})$, for a specific triple $\mathsf{P} = \mathsf{P}(v, w)$ depending on the two vectors v and w, and torsion points on the fourfold $A \times \widehat{A}$. Just as for curves, the conjecture *roughly* takes the form

$$\mathbf{E}(v, w) = \bigoplus_{\ell} \mathbf{W}(\mathsf{P}) \otimes \ell^{\oplus \mathsf{m}_\ell} \tag{3}$$

for explicit torsion points ℓ over $A \times \widehat{A}$, and explicit multiplicities m_ℓ. These conjectured multiplicities refine the Verlinde formula for abelian surfaces derived in [MO2]. In effect, Conjecture 3 recovers the Verlinde formula of [MO2] as the simplest special case. Alongside the study of the bundles $\mathbf{W}(\mathsf{P})$, the correct formulation of this conjecture, in particular the identification of the correct torsion points ℓ with exact multiplicities m_ℓ, is one of the main goals of this note.

From this perspective, the bundles $\mathbf{W}(\mathsf{P})$ thus arise as generalizations of the bundles $\mathbf{W}^{\mathsf{c}}(\mathsf{p})$ studied previously. We will see that the properties of the two classes of bundles, in the curve and abelian surface contexts, are also similar. In particular, Lemmas 3 – 5, Proposition 1 and Example 3 discuss the symmetries of these bundles and their behavior under Fourier-Mukai transforms. The proofs are however more difficult in the abelian situation, and require new ideas to deal with the more complicated numerics.

Next, in Section 3, we furthermore emphasize two important features of Conjecture 1:

- the statement is consistent with Fourier-Mukai symmetries. Specifically, (3) implies that the bundles in (2) are abstractly isomorphic, see Proposition 4:
$$\mathbf{E}(v,w)^{\vee} \simeq \widehat{\mathbf{E}(w,v)}.$$
This falls short of establishing strange duality, but it provides evidence in its favor;
- the conjecture specializes correctly in degree 0 to the expressions for the Verlinde bundle derived in [O3], and also in the case of coprime Mukai self-pairing; see Proposition 3 and Lemma 11.

As a modest application of our results, in Theorem 2, we prove the abelian analogue of the level 1 strange duality for curves established in [BNR].

1.6. **Variation in moduli.** Finally, we point out the following variation of the bundles of generalized theta functions. In the above construction, the abelian surface or the curve have been kept fixed. By varying these objects in moduli, while keeping the determinants and determinants of the Fourier-Mukai fixed to values determined by the polarization, one defines Verlinde bundles
$$\mathsf{E}(v,w) \to \mathcal{A}_{(d_1,d_2)} \text{ or } \mathsf{E}^{\mathsf{c}}(v,w) \to \mathcal{M}_g$$
over the moduli space of abelian surfaces endowed with a polarization of type (d_1, d_2), or over the moduli space of curves. The construction requires some care to kill off ambiguities; we refer the reader to [MO3] for details. In the curve case, the Chern characters are tautological
$$\mathrm{ch}(\mathsf{E}^{\mathsf{c}}(v,w)) \in \mathsf{R}^{\star}(\mathcal{M}_g).$$
In fact, explicit expressions were found and extended over the boundary, see [MOPPZ]. For abelian surfaces, it was shown in [BMOY] that
$$\mathsf{E}(v,w) \to \mathcal{A}_{(d_1,d_2)}$$
is indeed locally free under assumption (**A.2**) to be stated below. It would be interesting to find expressions for the Chern character or prove that
$$\mathrm{ch}(\mathsf{E}(v,w)) \in \mathsf{R}^{\star}(\mathcal{A}_{(d_1,d_2)}).$$

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2. Construction and study of semihomogeneous bundles

2.1. **Notation.** We begin by reviewing notation on abelian varieties X. We write $\mathcal{P}$ for the normalized Poincaré bundle over $X \times \widehat{X}$. Throughout the paper, we will consistently use the Fourier-Mukai transform with kernel $\mathcal{P}$:

$$\mathbf{R}\mathcal{S} : \mathbf{D}(X) \to \mathbf{D}(\widehat{X}), \quad \mathbf{R}\mathcal{S}(E) = \mathbf{R}p_!(\mathcal{P} \otimes q^\star E).$$

The following properties of Fourier-Mukai were established in [M1]:

$$\mathbf{R}\mathcal{S}(t_x^\star E) = \mathbf{R}\mathcal{S}(E) \otimes \mathcal{P}_{-x},$$

$$\mathbf{R}\mathcal{S}(E \otimes y) = t_y^\star \mathbf{R}\mathcal{S}(E),$$

for $x \in X, y \in \widehat{X}$. Recall from [M1] that a vector bundle $E \to X$ is said to satisy the index theorem with index i provided that

$$H^j(E \otimes y) = 0$$

for all degree 0 line bundles $y \in \widehat{X}$ and all $j \neq i$. In this context, the Fourier-Mukai transform of E is concentrated in degree i, and as customary, we write $\widehat{E}$ for the corresponding cohomology sheaf. If Θ is a line bundle inducing a principal polarization on X, we set

$$\widehat{\Theta} = \mathbf{R}\mathcal{S}(\Theta),$$

so that $-\widehat{\Theta}$ is the principal polarization of the dual abelian variety $\widehat{X}$. Finally, we write

$$\Phi : X \to \widehat{X}, \;\; \widehat{\Phi} : \widehat{X} \to X$$

for the morphisms induced by Θ and $\widehat{\Theta}$. We have

$$\Phi \circ \widehat{\Phi} = -1, \;\; \widehat{\Phi} \circ \Phi = -1.$$

2.2. **Semihomogeneous bundles.** The theory of semihomogeneous bundles over abelian varieties X was developed by Mukai in [M2] as a higher dimensional generalization of Atiyah's classification of vector bundles over elliptic curves.

We recall that a vector bundle $\mathbf{W} \to X$ is said to be semihomogeneous if for all $x \in X$, the translations take the form

$$t_x^\star \mathbf{W} = \mathbf{W} \otimes y$$

for some line bundle $y \to X$ that may depend on x. Simple semihomogeneous bundles are special; for instance, by [M2], their Chern characters are entirely determined by rank and slope

$$\mathrm{ch}\,(\mathbf{W}) = \mathrm{rank}\ \mathbf{W} \cdot \exp\left(c_1(\mu(\mathbf{W}))\right),$$

where we set

$$\mu(\mathbf{W}) = \frac{\det \mathbf{W}}{\mathrm{rank}\ \mathbf{W}} \in \mathrm{Pic}(X) \otimes \mathbb{Q}.$$

This is a consequence of the fact that up to suitable isogenies $f : Y \to X$, the pullback of $\mathbf{W}$ splits as a sum of line bundles

$$f^\star \mathbf{W} = \bigoplus_{i=1}^{\mathrm{rank}\ \mathbf{W}} \mathcal{L}.$$

Furthermore, for each simple semihomogeneous bundle $\mathbf{W}$, we consider the groups

$$K(\mathbf{W}) = \{x \in X : t_x^\star \mathbf{W} = \mathbf{W}\},\ \ \Sigma(\mathbf{W}) = \{y \in \widehat{X} : \mathbf{W} \otimes y = \mathbf{W}\}.$$

By [M2], the group $K(\mathbf{W})$ has order equal to $\chi(\mathbf{W})^2$, while $\Sigma(\mathbf{W})$ has order rank $(\mathbf{W})^2$.

Example 1. We recall a class of semihomogeneous bundles studied in [O2]. The motivation there was to answer questions similar to Question 1, but in the context of moduli of bundles over curves. Specifically, let (X, Θ) be a principally polarized abelian variety of dimension g, with Θ a symmetric *line bundle*:

$$(-1)^\star \Theta = \Theta.$$

For any pair of coprime integers (a, b), with a odd and positive, there exists a unique simple symmetric semihomogeneous vector bundle $\mathbf{W}_{a,b}$ with fixed slope:

$$\mathrm{rank}\ \mathbf{W}_{a,b} = a^g,\ \ \det \mathbf{W}_{a,b} = \Theta^{a^{g-1}b} \implies \mu(\mathbf{W}_{a,b}) = \frac{b\Theta}{a}.$$

(For a even and $g > 1$, uniqueness fails due to tensorization by $2a$-torsion points.) In the case at hand, it is not difficult to make the previous discussion explicit. Indeed, we recall from [O2] the following properties of the bundles $\mathbf{W}_{a,b}$:

- $K(\mathbf{W}_{a,b}) = X[b],\ \ \Sigma(\mathbf{W}_{a,b}) = \widehat{X}[a]$;
- the pullback under the multiplication by $a : X \to X$ splits as a sum

$$a^\star \mathbf{W}_{a,b} = \bigoplus_{i=1}^{a^g} \Theta^{ab};$$

- for $a, b > 0$ odd, the following Fourier-Mukai symmetry holds

$$\mathbf{R}\mathcal{S}(\mathbf{W}_{a,b}) = \mathbf{W}^\dagger_{b,a},$$

  where $\mathbf{W}^\dagger_{b,a}$ denotes the similar bundle with slope $\frac{b}{a}\widehat{\Theta}$ over $\widehat{X}$.

We rephrase the above discussion in a fashion that connects to the the results of the following sections. For each $\mathsf{p} \in \mathbb{Q}$, write $r(\mathsf{p})$ for the smallest positive denominator of p and set

$$\chi(\mathsf{p}) = r(\mathsf{p}) \cdot \mathsf{p}.$$

We say p is *admissible* if both $r(\mathsf{p})$ and $\chi(\mathsf{p})$ are odd integers. For admissible p there exists a unique symmetric simple semihomogeneous bundle $\mathbf{W}(\mathsf{p}) \to A$ with fixed slope

$$\mu(\mathbf{W}(\mathsf{p})) = \mathsf{p}\Theta \in \operatorname{Pic}(A) \otimes \mathbb{Q}.$$

We have

$$\operatorname{rank} \mathbf{W}(\mathsf{p}) = r(\mathsf{p})^g, \ \ \chi(\mathbf{W}(\mathsf{p})) = \chi(\mathsf{p})^g.$$

Furthermore, $\mathbf{W}(\mathsf{p})$ satisfies the index theorem and the Fourier-Mukai transform equals

$$\widehat{\mathbf{W}(\mathsf{p})} = \mathbf{W}(\mathsf{p}^{-1}).$$

2.3. **Admissible triples.** We will consider semihomogeneous bundles over product abelian varieties

$$X = A \times \widehat{A},$$

where A is an abelian variety over the complex numbers, and Θ a symmetric line bundle

$$(-1)^\star \Theta = \Theta,$$

inducing a principal polarization on A. The bundles $\mathbf{W}(\mathsf{P})$ below parallel and generalize the bundles in Subsection 2.2 which were defined over possibly nonsplit abelian varieties. In the current context, the details are more involved.

We begin with some terminology. For a triple $\mathsf{P} = (u, v, h)$ of rational numbers, we define the determinant

$$\det \mathsf{P} = uv + h^2.$$

The rank $r(\mathsf{P})$ is the smallest positive common denominator of the rational numbers u, v and $\det \mathsf{P}$, so that

$$ru, \ rv, \ \ r\det \mathsf{P}$$

are integers. Note that in particular

$$(rh)^2 = r \cdot r(uv + h^2) - (ru)(rv) \in \mathbb{Z}$$

hence the rational number $rh \in \mathbb{Z}$ as well. By the minimality of r, we must have

$$(r, ru, rv, r\det \mathsf{P}) = 1.$$

Define the Euler characteristic

$$\chi(\mathsf{P}) = r\det \mathsf{P} = r(uv + h^2) \in \mathbb{Z}.$$

A triple of rational numbers P is called *admissible* if $r(\mathsf{P})$ and $\chi(\mathsf{P})$ are odd integers. The parity conditions are necessary; in their absence, some of the results below fail.

For each admissible triple P we define the inverse triple

$$\mathsf{P}^{-1} = \frac{1}{\det \mathsf{P}} \cdot \mathsf{P}^{\star} = \left(\frac{u}{uv+h^2}, \frac{v}{uv+h^2}, -\frac{h}{uv+h^2} \right),$$

where we wrote $\mathsf{P}^{\star} = (u, v, -h)$. In particular

$$\det \mathsf{P}^{-1} = \frac{1}{\det \mathsf{P}}.$$

The rank of the the triple P^{-1} is easily seen to be

$$r(\mathsf{P}^{-1}) = \pm r(uv+h^2) = \pm \chi(\mathsf{P}),$$

where the sign is chosen so that the rank becomes positive. More succinctly,

$$r(\mathsf{P}^{-1}) = |\chi(\mathsf{P})|.$$

Up to the same sign as above (which we will not need explicitly), the Euler characteristic equals

$$\chi(\mathsf{P}^{-1}) = \pm r(uv+h^2) \det \mathsf{P}^{-1} = \pm r(\mathsf{P}).$$

The triple P^{-1} is clearly admissible.

By abuse of notation, for each triple P of rational numbers, we also consider the class

$$\mathsf{P} = u\Theta + v\widehat{\Theta} + h\mathcal{P} \in \operatorname{Pic}(A \times \widehat{A}) \otimes \mathbb{Q}.$$

Note that for a triple P of rank r, the line bundle

$$\mathcal{O}(r\mathsf{P}) = \left(\Theta^{ru} \boxtimes \widehat{\Theta}^{rv} \right) \otimes \mathcal{P}^{rh}$$

is well-defined over $A \times \widehat{A}$. There is an associated Mumford homomorphism

$$\Phi_{r\mathsf{P}} : A \times \widehat{A} \to A \times \widehat{A}, \ \ (x, y) \to t^{\star}_{(x,y)} \mathcal{O}(r\mathsf{P}) \otimes \mathcal{O}(-r\mathsf{P}).$$

A word on conventions may be needed here. Throughout the paper, we freely identify $A \times \widehat{A}$ with its dual in the usual way, via the Poincaré sheaf

$$\mathcal{Q} \to (A \times \widehat{A}) \times (A \times \widehat{A}).$$

Under this identification, a point $(\alpha, \beta) \in A \times \widehat{A}$ yields the line bundle $\beta \boxtimes \mathcal{P}_{\alpha}$ over $A \times \widehat{A}$. In particular, we may regard $\Phi_{r\mathsf{P}}(x, y)$ both as a point in $A \times \widehat{A}$ and as a line bundle over $A \times \widehat{A}$. Striving for simplicity, we will not distinguish notationally between the two meanings. The context should ensure that no confusion arises: translation presupposes that we think of $\Phi_{r\mathsf{P}}(x, y)$ as a point, tensorization indicates we consider the associated line bundle.

2.4. **A class of semihomogeneous vector bundles.** For each admissible triple P we construct the minimal semihomogeneous vector bundle $\mathbf{W}(\mathsf{P})$ over $A \times \widehat{A}$ of slope determined by P:

$$\mu(\mathbf{W}(\mathsf{P})) := \frac{\det \mathbf{W}(\mathsf{P})}{\operatorname{rank} \mathbf{W}(\mathsf{P})} = \mathsf{P}$$

where as before we write

$$\mathsf{P} := u\Theta + v\widehat{\Theta} + h\mathcal{P} \in \operatorname{Pic}(A \times \widehat{A}) \otimes \mathbb{Q}.$$

While it is more natural to think of semihomogeneous bundles in terms of their slope, the following result clarifies other relevant numerics:

Lemma 1. *For each admissible triple* P*, there exists a simple semihomogeneous bundle* $\mathbf{W}(\mathsf{P})$ *of slope* P*. The rank equals*

$$\operatorname{rank} \mathbf{W}(\mathsf{P}) = r(\mathsf{P})^g$$

and Euler characteristic is

$$\chi(\mathbf{W}(\mathsf{P})) = (-1)^g \chi(\mathsf{P})^g.$$

Proof. In general, Mukai proved the existence of a simple semihomogeneous bundle $\mathbf{W}(\mathsf{P})$ of any given slope, cf. Corollary 6.23 of [M2]. We need to show that the rank of $\mathbf{W}(\mathsf{P})$ equals r^g, for $r = \operatorname{rank} \mathsf{P}$. We let

$$\mathcal{L} = \left(\Theta^{ru} \boxtimes \widehat{\Theta}^{rv}\right) \otimes \mathcal{P}^{rh} = \mathcal{O}(r\mathsf{P})$$

and note

$$\mu(\mathbf{W}(\mathsf{P})) = \frac{\mathcal{L}}{r}.$$

Using Theorem 7.11 of [M2], the rank of $\mathbf{W}(\mathsf{P})$ can be calculated from the cardinality of the set

$$\# \left\{K(\mathcal{L}) \cap A[r] \times \widehat{A}[r]\right\} = u^2,$$

where $A[r]$ and $\widehat{A}[r]$ denote the groups of r-torsion points on the given abelian varieties. In fact,

$$\operatorname{rank} \mathbf{W}(\mathsf{P}) = \frac{r^{2g}}{u}.$$

We show that $u = r^g$. Indeed,

$$\begin{aligned}
(x, y) \in K(\mathcal{L}) \quad &\iff \quad t_x^\star \Theta^{ru} \boxtimes t_y^\star \widehat{\Theta}^{rv} \otimes t_{x,y}^\star \mathcal{P}^{rh} = \Theta^{ru} \boxtimes \widehat{\Theta}^{rv} \otimes \mathcal{P}^{rh} \\
&\iff \quad \Phi(rux) \boxtimes \widehat{\Phi}(rvy) = y^{-rh} \boxtimes \mathcal{P}_x^{-rh} \\
&\iff \quad \Phi(rux) = -rhy, \;\; \widehat{\Phi}(rvy) = -rhx.
\end{aligned}$$

We invoke Lemma A.2 of the Appendix to complete the proof. The lemma applies since

$$\left(r, ru, rv, \frac{(ru)(rv)+(rh)^2}{r}\right) = (r, ru, rv, r\det \mathsf{P}) = 1$$

by the definition of the rank.

Finally, the numerics of any simple semihomogeneous bundle are determined by its rank and slope

$$\mathrm{ch}\ \mathbf{W}(\mathsf{P}) = r^g \exp(\mathsf{P}).$$

Therefore,

$$\chi(\mathbf{W}(\mathsf{P})) = \int_{A\times\widehat{A}} \mathrm{ch}\ \mathbf{W}(\mathsf{P}) = r^g \frac{1}{(2g)!}\int_{A\times\widehat{A}} (u\Theta + v\widehat{\Theta} + h\mathcal{P})^{2g}$$
$$= (-1)^g\, r^g\, (uv+h^2)^g = (-1)^g\, \chi(\mathsf{P})^g.$$

Here, we made use of the intersections

$$\int_{A\times\widehat{A}} \frac{\Theta^i}{i!}\cdot\frac{\widehat{\Theta}^j}{j!}\cdot\frac{c_1(\mathcal{P})^k}{k!} = \begin{cases}(-1)^g\binom{g}{i} & \text{if } i=j,\ k=2g-2i\\ 0 & \text{otherwise}\end{cases}$$

which can be checked working in local coordinates. □

We next arrange our bundles be symmetric.

Theorem 1. *There is a unique simple semihomogeneous symmetric vector bundle* $\mathbf{W}(\mathsf{P})$ *of rank* r^g *and determinant* $\mathcal{O}(r^g\mathsf{P})$.

Proof. The proof uses the ideas of Section 2.1 of [O2]. We first tensor by suitable degree 0 line bundles, if needed, to achieve determinant $\mathcal{O}(r^g\mathsf{P})$. Next, to obtain symmetry, pick any possibly non-symmetric bundle $\mathbf{W}(\mathsf{P})$. The bundles $\mathbf{W}(\mathsf{P})$ and $(-1)^\star\mathbf{W}(\mathsf{P})$ are simple semihomogeneous and of the same slope. By Theorem 7.11 in [M2], we must have

$$(-1)^\star\mathbf{W}(\mathsf{P}) = \mathbf{W}(\mathsf{P})\otimes M$$

for some line bundle M of degree 0. In fact, $M^{r^g} = \mathcal{O}$ by comparing determinants. Pick a line bundle L of order dividing r^g with $L^2 = M$. This is possible since r is odd. Then $\mathbf{W}(\mathsf{P})\otimes L$ is symmetric semihomogeneous of the correct determinant.

To show uniqueness, assume $\mathbf{W}_1$ and $\mathbf{W}_2$ are two symmetric bundles as above, and apply again Theorem 7.11 of [M2] to write

$$\mathbf{W}_1 = \mathbf{W}_2\otimes M,$$

for some line bundle M. Then

$$\mathbf{W}_1 = (-1)^\star\mathbf{W}_1 \implies \mathbf{W}_2\otimes M = \mathbf{W}_2\otimes M^{-1} \implies M^2\in\Sigma(\mathbf{W}_2).$$

Proposition 7.1 of [M2] shows that $\Sigma(\mathbf{W}_2)$ has odd order, in fact equal to r^{2g}. Thus, the assignment

$$M \to M^2$$

is bijective on $\Sigma(\mathbf{W}_2)$. Hence $M^2 = L^2$ for some $L \in \Sigma(\mathbf{W}_2)$ which shows

$$M = L \otimes \xi \implies \mathbf{W}_1 = \mathbf{W}_2 \otimes M = \mathbf{W}_2 \otimes L \otimes \xi = \mathbf{W}_2 \otimes \xi$$

for a 2-torsion line bundle ξ. Comparing determinants we obtain $\xi^{r^g} = 1$. Since r is odd, we derive that $\xi = 1$. Therefore, $\mathbf{W}_1 = \mathbf{W}_2$.

□

Example 2. Let h be an integer, and write

$$u = \frac{b}{a}, \quad v = \frac{d}{c} \implies \mathsf{P} = \left(\frac{b}{a}, \frac{d}{c}, h\right),$$

where $(a, b) = (c, d) = 1$, and $a, c > 0$ are odd. Then $r = ac$, $\chi = bd + h^2ac$ and

$$\mathbf{W}(\mathsf{P}) = \mathbf{W}_{a,b} \boxtimes \mathbf{W}^{\dagger}_{c,d} \otimes \mathcal{P}^h.$$

Here $\mathbf{W}_{a,b}$ and $\mathbf{W}^{\dagger}_{c,d}$ are the unique simple symmetric semihomogeneous bundles over A and $\widehat{A}$ of slopes $\frac{b\Theta}{a}$ and $\frac{d\widehat{\Theta}}{c}$ respectively; their ranks equal a^g and c^g, see Section 2.2. This is in agreement with Lemma 1 which predicts rank $(ac)^g$. □

2.5. **Properties of semihomogeneous bundles.** In this subsection, we make precise some of Mukai's results in [M2] for the semihomogeneous bundles $\mathbf{W}(\mathsf{P})$ constructed above. In particular, we discuss:

- the calculation of the groups $\Sigma(\mathbf{W}(\mathsf{P}))$ and $K(\mathbf{W}(\mathsf{P}))$;
- the splitting of $\mathbf{W}(\mathsf{P})$ as sum of line bundles under pullbacks by explicit isogenies;
- the interaction of $\mathbf{W}(\mathsf{P})$ with the Fourier-Mukai functor.

These results should be compared with those reviewed in the last paragraph of Subsection 2.2 for the bundles $\mathbf{W}_{a,b}$. They will be used in Section 3.

Lemma 2. *For any admissible triple* P*, we have*

$$t_{rx}^{\star}\mathbf{W}(\mathsf{P}) \cong \mathbf{W}(\mathsf{P}) \otimes \Phi_{r\mathsf{P}}(x), \quad \textit{for } x \in A \times \widehat{A}.$$

Proof. This follows from Lemma 6.7 of [M2], where the result is shown in greater generality: for every semihomogeneous bundle $\mathbf{W}$ over an abelian variety X, of rank r and determinant $\mathcal{D}$, we have

$$t_{rx}^{\star}\mathbf{W} = \mathbf{W} \otimes \Phi_{\mathcal{D}}(x)$$

where $\Phi_{\mathcal{D}} : X \to \widehat{X}$ is the Mumford homomorphism. In our case, this yields

$$t_{r^g x}^{\star}\mathbf{W}(\mathsf{P}) = \mathbf{W}(\mathsf{P}) \otimes \Phi_{r^g\mathsf{P}}(x),$$

which is equivalent to the Lemma. Here we followed the conventions laid out at the end of Subsection 2.3: $\Phi_{r\mathsf{P}}(x)$ is understood as a line bundle over $A \times \widehat{A}$, but we did not introduce separate notation to indicate this fact. $\square$

Lemma 3. *The group*

$$\Sigma(\mathbf{W}(\mathsf{P})) = \{y \in A \times \widehat{A} : \mathbf{W}(\mathsf{P}) \otimes \mathcal{Q}_y = \mathbf{W}(\mathsf{P})\}$$

has r^{2g} elements, and can be identified with

$$\Sigma = \{y \in A[r] \times \widehat{A}\,[r] : \Phi_{\chi \mathsf{P}^{-1}}(y) = 0\}.$$

Here, $\mathcal{Q}_y$ denotes the line bundle over $A \times \widehat{A}$ associated to y.

Proof. The fact that Σ has order r^{2g} follows from Lemma A.2 of the Appendix. By Proposition 7.1 of [M2], the order of $\Sigma(\mathbf{W}(\mathsf{P}))$ also equals r^{2g}. To prove the lemma, we show that both Σ and $\Sigma(\mathbf{W}(\mathsf{P}))$ equal

$$\Sigma^+ = \{y = \Phi_{r\mathsf{P}}(x) : x \in A[r] \times \widehat{A}[r]\}.$$

Note that

$$\Sigma^+ = \Phi_{r\mathsf{P}}(A[r] \times \widehat{A}[r]) \simeq \frac{A[r] \times \widehat{A}[r]}{\operatorname{Ker} \Phi_{r\mathsf{P}} \cap (A[r] \times \widehat{A}[r])}$$

has order $\frac{r^{4g}}{r^{2g}} = r^{2g}$, again using Lemma A.2 to compute the order of $\operatorname{Ker} \Phi_{r\mathsf{P}} \cap (A[r] \times \widehat{A}[r])$. It suffices to prove the inclusions

$$\Sigma^+ \subset \Sigma(\mathbf{W}(\mathsf{P})), \;\; \Sigma^+ \subset \Sigma.$$

The first inclusion $\Sigma^+ \subset \Sigma(\mathbf{W}(\mathsf{P}))$ is an immediate consequence of Lemma 2. A direct calculation shows that

$$\Phi_{\chi \mathsf{P}^{-1}} \circ \Phi_{\mathsf{rP}} = -r^2 \det \mathsf{P}.$$

Thus if $y = \Phi_{r\mathsf{P}}(x) \in \Sigma^+$ for $x \in A[r] \times \widehat{A}[r]$, we obtain

$$\Phi_{\chi \mathsf{P}^{-1}}(y) = \Phi_{\chi \mathsf{P}^{-1}} \circ \Phi_{\mathsf{rP}}(x) = -r \det \mathsf{P} \cdot rx = 0.$$

This establishes the inclusion $\Sigma^+ \subset \Sigma$, completing the proof. $\square$

Lemma 4. *The group*

$$K(\mathbf{W}(\mathsf{P})) = \{x \in A \times \widehat{A} : t_x^\star \mathbf{W}(\mathsf{P}) = \mathbf{W}(\mathsf{P})\}$$

has order χ^{2g}, and can be identified with

$$K = \{x \in A\,[\chi] \times \widehat{A}\,[\chi] : \Phi_{r\mathsf{P}}(x) = 0\}.$$

Proof. This follows from Lemma 3, combined with the Fourier-Mukai symmetry of Proposition 1. We use here that Fourier-Mukai exchanges translation with tensorization. $\square$

Proposition 1. *For each admissible triple* P*, we have*

$$\widehat{\mathbf{W}(\mathsf{P})} \cong \mathbf{W}(\mathsf{P}^{-1}).$$

Proof. By Lemma 1, the semihomogeneous bundle $\mathbf{W}(\mathsf{P})$ has non-zero Euler characteristic. On general grounds, nondegenerate semihomogeneous bundles satisfy the index theorem (the index is moreover found in Corollary 1 below). The (shifted) Fourier-Mukai transform $\widehat{\mathbf{W}(\mathsf{P})}$ is thus locally free.

The slope in rational cohomology is found via Grothendieck-Riemann-Roch and a calculation in local coordinates

$$\mu(\widehat{\mathbf{W}(\mathsf{P})}) = \frac{1}{uv+h^2}(u\Theta + v\widehat{\Theta} - h\mathcal{P}) = \mathsf{P}^{-1}.$$

This matches the slope of $\mathbf{W}(\mathsf{P}^{-1})$. Both bundles are simple and semihomogeneous, as these properties are preserved under Fourier-Mukai. Therefore, by Proposition 6.17 in [M2], there exists ξ a degree 0 line bundle over $A \times \widehat{A}$ such that

$$\widehat{\mathbf{W}(\mathsf{P})} = \mathbf{W}(\mathsf{P}^{-1}) \otimes \xi.$$

Further arguments are necessary to prove that ξ may be taken to be trivial. Using the symmetry of the bundles involved, we conclude

$$(-1)^{\star}\widehat{\mathbf{W}(\mathsf{P})} = \widehat{\mathbf{W}(\mathsf{P})} \implies (-1)^{\star}\mathbf{W}(\mathsf{P}^{-1}) \otimes \xi^{-1} = \mathbf{W}(\mathsf{P}^{-1}) \otimes \xi \implies \xi^2 \in \Sigma(\mathbf{W}(\mathsf{P}^{-1})).$$

By Proposition 7.1 of [M2], $\Sigma(\mathbf{W}(\mathsf{P}^{-1}))$ has odd order, in fact equal to rank $\mathbf{W}(\mathsf{P}^{-1})^2 = \chi(\mathsf{P})^{2g}$. Thus, the map

$$\Sigma(\mathbf{W}(\mathsf{P}^{-1})) \ni \tau \to \tau^2 \in \Sigma(\mathbf{W}(\mathsf{P}^{-1}))$$

is an isomorphism. We may write $\xi^2 = \tau^2$, for some $\tau \in \Sigma(\mathbf{W}(\mathsf{P}^{-1}))$. Setting $\eta = \xi \otimes \tau^{-1}$, we have $\eta^2 = 1$ and furthermore

$$\widehat{\mathbf{W}(\mathsf{P})} = \mathbf{W}(\mathsf{P}^{-1}) \otimes \eta. \tag{4}$$

We claim that η is trivial.

This remaining part of the argument is somewhat roundabout. The proof we give below consists in two steps. We first reduce to the case of triples of rank 1 using a suitable isogeny. Then, a direct calculation will establish the claim.

Consider the isogeny $r : A \times \widehat{A} \to A \times \widehat{A}$ given by multiplication by r. Pushing forward (4), we obtain

$$r_{\star}\left(\widehat{\mathbf{W}(\mathsf{P})}\right) = r_{\star}\left(\mathbf{W}(\mathsf{P}^{-1}) \otimes \eta\right) = r_{\star}\left(\mathbf{W}(\mathsf{P}^{-1}) \otimes r^{\star}\eta\right) = r_{\star}\left(\mathbf{W}(\mathsf{P}^{-1})\right) \otimes \eta.$$

Here, we used that η is 2-torsion and r is odd, so that $\eta = r^\star\eta$. This rewrites as

$$r^\star\widehat{\mathbf{W}(\mathsf{P})} = r_\star\left(\mathbf{W}(\mathsf{P}^{-1})\right) \otimes \eta.$$

Taking determinants, we obtain

$$\det r^\star\widehat{\mathbf{W}(\mathsf{P})} = \det r_\star\left(\mathbf{W}(\mathsf{P}^{-1})\right) \otimes \eta^{r^{4g}\chi^g}.$$

We will show that

$$\det r^\star\widehat{\mathbf{W}(\mathsf{P})} = \det r_\star\left(\mathbf{W}(\mathsf{P}^{-1})\right). \tag{5}$$

This will prove that η is of order $r^{4g}\chi^g$, which is odd. Since η is also 2-torsion, it must be trivial, completing the proof. We will evaluate the two determinants in (5) explicitly.

First, we consider the pullback $r^\star\mathbf{W}(\mathsf{P})$ which has slope equal to

$$\mathcal{L} = r^\star\mathcal{O}(\mathsf{P}) = \Theta^{r^2u} \boxtimes \widehat{\Theta}^{r^2v} \otimes \mathcal{P}^{r^2h}.$$

The reason for making use of the isogeny r is evident here: we need $\mathcal{L}$ to be a genuine line bundle, as opposed to a fractional one. Since $r^\star\mathbf{W}(\mathsf{P})$ is semihomogeneous, by the classification theory contained in Propositions 6.18 and 6.2 in [M2], we conclude that

$$r^\star\mathbf{W}(\mathsf{P}) = \bigoplus_j \mathcal{L} \otimes \mathbf{U}_j \otimes \ell_j$$

where $\ell_j \to A \times \widehat{A}$ are line bundles of degree 0, while $\mathbf{U}_j$ are unipotent bundles, *i.e.* bundles admitting filtrations with trivial successive quotients. Comparing ranks, we find

$$\sum_j \text{rank } \mathbf{U}_j = r^g.$$

Taking determinants, we conclude

$$r^\star\mathcal{O}(r^g\mathsf{P}) = \mathcal{L}^{r^g} \otimes \left(\sum_j \text{rank } \mathbf{U}_j \cdot \ell_j\right) \implies \sum_j \text{rank } \mathbf{U}_j \cdot \ell_j = 0.$$

We now compute

$$r^\star\widehat{\mathbf{W}(\mathsf{P})} = \bigoplus_j \mathcal{L} \otimes \widehat{\mathbf{U}_j \otimes} \ell_j = \bigoplus_j t^\star_{\ell_j}\widehat{\mathcal{L} \otimes \mathbf{U}}_j.$$

To evaluate the determinant of the Fourier-Mukai transforms appearing in the last expression, we may assume that the unipotent bundles $\mathbf{U}_j$ are trivial; indeed, determinants are multiplicative in exact sequences. Thus

$$\det\left(r^\star\widehat{\mathbf{W}(\mathsf{P})}\right) = \bigotimes_j t^\star_{\ell_j}\left(\det\widehat{\mathcal{L}}\right)^{\text{rank } \mathbf{U}_j} = \left(\det\widehat{\mathcal{L}}\right)^{r^g}$$

after using the two identities derived above from the rank and determinant calculation.

Next, we consider the right hand side of (5). By definition, $\mathbf{W}(\mathsf{P}^{-1})$ has rank $|\chi|^g$ and determinant $\mathcal{O}(|\chi|^g\mathsf{P}^{-1})$. Recall that the absolute value is needed here to ensure that the rank of P^{-1} is chosen positive. Using Proposition 3.4 of [NR], we conclude that for r odd

$$\det r_\star\left(\mathbf{W}(\mathsf{P}^{-1})\right) = \det r_\star\left(\mathcal{O}(|\chi|^g\mathsf{P}^{-1})\right).$$

By Lemma 2.1 in [NR] we have

$$r^\star r_\star\left(\mathcal{O}(|\chi|^g\mathsf{P}^{-1})\right) = \bigoplus_\alpha t_\alpha^\star\,\mathcal{O}(|\chi|^g\mathsf{P}^{-1})$$

where the sum runs over all r-torsion points $\alpha \in A \times \widehat{A}$. Taking determinants, and using that the sum of the r-torsion points is trivial, we conclude

$$r^\star \det r_\star\left(\mathcal{O}(|\chi|^g\mathsf{P}^{-1})\right) = \mathcal{O}(r^{4g}|\chi|^g\mathsf{P}^{-1}).$$

Thus, up to some r-torsion point β, we have

$$\det r_\star\left(\mathcal{O}(|\chi|^g\mathsf{P}^{-1})\right) = \mathcal{O}(r^{4g-2}|\chi|^g\mathsf{P}^{-1}) \otimes \beta.$$

Since both sides are symmetric, it follows that β is 2-torsion. Since β is also r-torsion and r is odd, we find $\beta = 0$. Thus

$$\det r_\star\left(\mathbf{W}(\mathsf{P}^{-1})\right) = \det r_\star\left(\mathcal{O}(|\chi|^g\mathsf{P}^{-1})\right) = \mathcal{O}(r^{4g-2}|\chi|^g\mathsf{P}^{-1}).$$

To complete the proof of (5) it suffices to show that

$$\det \widehat{\mathcal{L}} = \mathcal{O}(r^{3g-2}|\chi|^g\mathsf{P}^{-1}).$$

In fact, we will show more generally that if

$$\mathcal{L} = \Theta^a \boxtimes \widehat{\Theta}^b \otimes \mathcal{P}^c,$$

where a, b, c are integers, then

$$\det \mathbf{R}\mathcal{S}(\mathcal{L}) = \left(\Theta^{a(ab+c^2)^{g-1}} \boxtimes \widehat{\Theta}^{b(ab+c^2)^{g-1}} \otimes \mathcal{P}^{-c(ab+c^2)^{g-1}}\right)^{(-1)^g}. \tag{6}$$

To conclude, we need to take into account the index ι of the line bundle $\mathcal{L} = \mathcal{O}(r^2\mathsf{P})$. This is determined in Corollary 1. The possible values are $\iota = 0, g, 2g$, in such a fashion that

$$(-1)^{\iota+g} = \left(\frac{|\chi|}{\chi}\right)^g.$$

This justifies the appearance of the absolute value in the expression above.

Equation (6) is equivalent to the Proposition for triples P of rank 1. The result might be known, but we include a possible argument for completeness. We match both sides under the assumption that a, b are odd positive integers, and c is even. On general grounds, the determinant in (6) is a polynomial expression in a, b and c, so this case will suffice.

Let a, b be odd and positive, and let c be even. Certainly, equality (6) holds at the level of Chern classes in rational cohomology by a Grothendieck-Riemann-Roch calculation in local coordinates. As both sides of (6) are symmetric, they differ by a 2-torsion point γ, which we show to be trivial. To this end, we let

$$\iota : \widehat{A} \to A \times \widehat{A}$$

be the inclusion of the zero section. We prove

$$\iota^\star \gamma = 1.$$

A similar argument for the inclusion

$$j : A \to A \times \widehat{A}$$

shows that $j^\star \gamma = 1$. This implies that γ is trivial, completing the proof. The claim that $\iota^\star \gamma = 1$ is precisely the statement that

$$\iota^\star \det \mathbf{R}\mathcal{S}(\mathcal{L}) = \widehat{\Theta}^{(-1)^g b(ab+c^2)^{g-1}} \iff \det \mathbf{R}\mathcal{S}\left(\mathbf{R}\pi_\star \mathcal{L}\right) = \widehat{\Theta}^{\,(-1)^g b(ab+c^2)^{g-1}}, \tag{7}$$

where the projection $\pi : A \times \widehat{A} \to A$ is the transpose of the inclusion ι.

Note that

$$\mathbf{R}\pi_\star \mathcal{L} = \Theta^a \otimes \mathbf{R}\pi_\star(\widehat{\Theta}^b \otimes \mathcal{P}^c) = \Theta^a \otimes c^\star\, \mathbf{R}\widehat{\mathcal{S}}\,(\widehat{\Theta}^b) = \Theta^a \otimes c^\star \mathbf{W}_{b,1}[-g].$$

Here, $\mathbf{R}\widehat{\mathcal{S}}$ denotes the Fourier-Mukai on the dual abelian variety. For the last equality, Proposition 2 of [O2] was used. To carry out the remainder of the proof, we use the same ideas as above. First, write $\frac{c^2}{b}$ in lowest terms $\frac{\beta}{\alpha}$ so that α is odd and β is even, and let $\mathbf{W}_{\alpha,\beta}$ be the simple semihomogeneous bundle of this given slope constructed in [O2]. The bundle $c^\star \mathbf{W}_{b,1}$ is semihomogeneous of the same slope as $\mathbf{W}_{\alpha,\beta}$, hence by the classification theory of [M2] we must have

$$c^\star \mathbf{W}_{b,1} = \bigoplus_j \mathbf{W}_{\alpha,\beta} \otimes \mathbf{U}_j \otimes m_j$$

for some unipotent $\mathbf{U}_j$ and some degree zero line bundles m_j. We next claim that $\mathbf{U}_j$ must be trivial. To this end, pull back both sides by $b\alpha$. The left hand side splits as a sum of line bundles

$$(b\alpha)^\star\, c^\star\, \mathbf{W}_{b,1} = (\alpha c)^\star\, b^\star \mathbf{W}_{b,1} = \bigoplus_{i=1}^{b^{2g}} (\alpha c)^\star \Theta^b.$$

The pullback of the right hand side must split as well. Note that

$$(b\alpha)^\star \mathbf{W}_{\alpha,\beta} = \bigoplus_{i=1}^{\alpha^{2g}} b^\star \Theta^{\alpha\beta} = \bigoplus_{i=1}^{\alpha^{2g}} (\alpha c)^\star \Theta^b.$$

From here it follows that $(b\alpha)^\star \mathbf{U}_j$ splits as sum of degree 0 line bundles. Since $\mathbf{U}_j$ is a direct summand of $(b\alpha)_\star (b\alpha)^\star \mathbf{U}_j$, we obtain that $\mathbf{U}_j$ must split as well. However $\mathbf{U}_j$ is unipotent, hence it must be trivial. Therefore

$$c^\star \mathbf{W}_{b,1} = \bigoplus_{j=1}^{N} \mathbf{W}_{\alpha,\beta} \otimes m_j$$

where the number of summands is $N = (b/\alpha)^g$ by comparing ranks. Comparing determinants, we find

$$\sum_{j=1}^{N} m_j = \mu,$$

where μ is α^g-torsion over A. Next,

$$\mathbf{R}\pi_\star \mathcal{L}\,[g] = \Theta^a \otimes c^\star \mathbf{W}_{b,1} = \Theta^a \otimes \mathbf{W}_{\alpha,\beta} \otimes \bigoplus_{j=1}^{N} m_j = \mathbf{W}_{\alpha,\beta+\alpha\cdot a} \otimes \bigoplus_{j=1}^{N} m_j.$$

Therefore,

$$\begin{aligned}
\det \mathbf{R}\mathcal{S}\left(\mathbf{R}\pi_\star \mathcal{L}\,[g]\right) &= \bigotimes_{j=1}^{N} \det \mathbf{R}\mathcal{S}\left(\mathbf{W}_{\alpha,\beta+\alpha\cdot a} \otimes m_j\right) = \bigotimes_{j} t_{m_j}^\star \det \mathbf{R}\mathcal{S}\left(\mathbf{W}_{\alpha,\beta+\alpha\cdot a}\right) \\
&= \bigotimes_{j=1}^{N} t_{m_j}^\star \left(\det \mathbf{W}^\dagger_{\beta+\alpha\cdot a,\alpha}\right) = \bigotimes_{j=1}^{N} t_{m_j}^\star \left(\widehat{\Theta}^{\alpha\cdot(\beta+\alpha\cdot a)^{g-1}}\right) \\
&= \widehat{\Theta}^{\alpha(\beta+\alpha\cdot a)^{g-1}\cdot N} \otimes \widehat{\Phi}(\alpha(\beta+\alpha\cdot a)^{g-1}\mu).
\end{aligned}$$

Furthermore,

$$\alpha(\beta+\alpha\cdot a)^{g-1}\cdot N = b(ab+c^2)^{g-1}.$$

The above equation finishes the proof. Indeed, the two sides of (7) differ by $\iota^\star\gamma$, which is known to be 2-torsion. We however exhibited the difference $\widehat{\Phi}(\alpha(\beta+\alpha\cdot a)^{g-1}\mu)$ which is α^g-torsion. Since α is odd, the difference must be trivial, completing the proof.

$\square$

We next consider the pullbacks of $\mathbf{W}(\mathsf{P})$ under a special class of isogenies.

Lemma 5. *We have*

$$\Phi_{r\mathsf{P}}^\star \mathbf{W}(\mathsf{P}^{-1}) = \mathcal{O}(-r^2\mathsf{P}) \otimes \mathbb{C}^{|\chi|^g}.$$

Similarly,

$$\Phi_{\chi\mathsf{P}^{-1}}^\star \mathbf{W}(\mathsf{P}) = \mathcal{O}(-\chi^2\mathsf{P}^{-1}) \otimes \mathbb{C}^{r^g}.$$

Proof. For each simple semihomogeneous bundle $\mathbf{W}$ over an abelian variety X, such as $X = A \times \widehat{A}$, Mukai introduced the scheme

$$\mathcal{Z}(\mathbf{W}) = \{(x, y) \in X \times \widehat{X} : t_x^\star \mathbf{W} = \mathbf{W} \otimes y\}.$$

Then, letting p denote the projection onto the first factor, he proved

$$p^\star \mathbf{W} = \mathbb{C}^{\text{rank } \mathbf{W}} \otimes \mathcal{L}$$

for some line bundle $\mathcal{L}$ over X, cf. Lemma 3.6 in [M2]. We apply this result to our situation. First, we compute using Proposition 1 and Lemma 3

$$\begin{aligned}\mathbf{R}\mathcal{S}(\mathbf{W}(\mathsf{P}^{-1}) \otimes \mathcal{Q}_{rx}) &= t_{rx}^\star \mathbf{R}\mathcal{S}(\mathbf{W}(\mathsf{P}^{-1})) = t_{rx}^\star \mathbf{W}(\mathsf{P}) = \mathbf{W}(\mathsf{P}) \otimes \Phi_{r\mathsf{P}}(x) \\ &= \mathbf{R}\mathcal{S}(\mathbf{W}(\mathsf{P}^{-1})) \otimes \Phi_{r\mathsf{P}}(x) = \mathbf{R}\mathcal{S}(t^\star_{\Phi_{r\mathsf{P}}(-x)} \mathbf{W}(\mathsf{P}^{-1})).\end{aligned}$$

On the first line, the shifts were omitted for ease of notation, and as before, for $\alpha \in A \times \widehat{A}$, we wrote $\mathcal{Q}_\alpha \to A \times \widehat{A}$ for the line bundle corresponding to α under the usual identifications. We derive

$$t^\star_{\Phi_{-r\mathsf{P}}(x)} \mathbf{W}(\mathsf{P}^{-1}) = \mathbf{W}(\mathsf{P}^{-1}) \otimes \mathcal{Q}_{rx}.$$

This gives a well-defined morphism

$$q : X \to \mathcal{Z} = \mathcal{Z}(\mathbf{W}(\mathsf{P}^{-1})), \ x \mapsto (\Phi_{r\mathsf{P}}(x), -rx)$$

such that

$$p \circ q = \Phi_{r\mathsf{P}}.$$

Mukai's result implies that

$$\Phi_{r\mathsf{P}}^\star \mathbf{W}(\mathsf{P}^{-1}) = \mathcal{L} \otimes \mathbb{C}^{|\chi|^g},$$

for some line bundle $\mathcal{L}$. We claim $\mathcal{L} = \mathcal{O}(-r^2\mathsf{P})$. Let

$$\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}(r^2\mathsf{P}).$$

Clearly, $\mathcal{L}'$ is symmetric, and comparing determinants we see that

$$\mathcal{O}(-r^2\mathsf{P})^{|\chi|^g} = \Phi_{r\mathsf{P}}^\star \det \mathbf{W}(\mathsf{P}^{-1}) = \mathcal{L}^{|\chi|^g} \implies \mathcal{L}'^{|\chi|^g} = 0.$$

The first equality follows from Lemma 6 below and $\det \mathbf{W}(\mathsf{P}^{-1}) = \mathcal{O}(|\chi|^g \mathsf{P}^{-1})$. Since χ is odd, symmetry and the above equation imply $\mathcal{L}'$ is trivial as desired. $\square$

Example 3. We illustrate the preceding results for the simplest triple

$$\mathsf{P} = \left(\frac{b}{a}, \frac{d}{c}, 0\right),$$

where $(a, b) = (c, d) = 1$ are odd positive integers. We have

$$\mathbf{W}(\mathsf{P}) = \mathbf{W}_{a,b} \boxtimes \mathbf{W}^\dagger_{c,d}.$$

- Lemmas 3 and 4 become

$$\Sigma(\mathbf{W}(\mathsf{P})) = \widehat{A}[a] \times A[c],\ K(\mathbf{W}(\mathsf{P})) = A[b] \boxtimes \widehat{A}[d].$$

- Proposition 1 states that

$$\mathbf{W}(\mathsf{P}^{-1}) = \mathbf{W}_{d,c} \boxtimes \mathbf{W}^{\dagger}_{b,a}$$

  is Fourier-Mukai dual to $\mathbf{W}(\mathsf{P}) = \mathbf{W}_{a,b} \boxtimes \mathbf{W}^{\dagger}_{c,d}$.
- Lemma 5 is implied by the following identification

$$(a,c)^{\star}\mathbf{W}(\mathsf{P}) = a^{\star}\mathbf{W}_{a,b} \boxtimes c^{\star}\mathbf{W}^{\dagger}_{c,d} = \bigoplus_{i=1}^{(ac)^g} \Theta^{ab} \boxtimes \widehat{\Theta}^{cd}.$$

These results are in agreement with those reviewed in Section 2.2. □

Finally, in relation to the Fourier-Mukai functor, we establish the following:

Corollary 1. *The bundle* $\mathbf{W}(\mathsf{P})$ *satisfies the index theorem, and*

(i) *if* $\det \mathsf{P} > 0$*, then index* $(\mathsf{P}) = g$*;*
(ii) *if* $\det \mathsf{P} < 0$*, index* (P) *is either* 0 *or* $2g$*. The first case occurs when* $u > 0$*, while the second case occurs when* $u < 0$*.*

Proof. It is more convenient to establish the similar claims for $\mathbf{W}(\mathsf{P}^{-1})$. To this end, we invoke Lemma 5, and the observation that the line bundle $\mathcal{O}(-r^2\mathsf{P})$ satisfies the index theorem with the appropriate indices.

The crucial claim is that the index of the line bundle

$$\mathcal{L} = \Theta^a \otimes \widehat{\Theta}^b \otimes \mathcal{P}^c$$

equals g when $ab + c^2 > 0$, and it equals 0 or $2g$ when $ab + c^2 < 0$, depending on the sign of a. The latter case is clear since the line bundle $\mathcal{L}$ is in fact ample or anti-ample as it follows from the Nakai-Moishezon criterion for abelian varieties, see for instance Section 6 of [BMOY] for a similar argument. When $ab + c^2 > 0$, we may assume $a > 0$ by Serre duality. The Leray spectral sequence for the projection $\pi : A \times \widehat{A} \to \widehat{A}$ yields

$$H^p(\widehat{A}, \mathbf{R}^q\pi_{\star}\,\mathcal{L}) \implies H^{p+q}(A \times \widehat{A}, \mathcal{L}).$$

Since $a > 0$, we have $\mathbf{R}^q\pi_{\star}\mathcal{L} = 0$ if $q \neq 0$ and

$$\mathbf{R}^0\pi_{\star}\mathcal{L} = c^{\star}\mathbf{R}\mathcal{S}(\Theta^a) \otimes \widehat{\Theta}^b = c^{\star}\mathbf{W}^{\dagger}_{a,1} \otimes \widehat{\Theta}^b.$$

By Section 2.2, the pullback of this bundle under the isogeny a splits as copies of

$$c^{\star}\widehat{\Theta}^a \otimes a^{\star}\widehat{\Theta}^b = \widehat{\Theta}^{a(ab+c^2)}.$$

Recalling that $\widehat{\Theta}^{-1}$ is ample and that $ab + c^2 > 0$, Serre duality shows that cohomology only occurs in degree $p = g$. Substituting into the spectral sequence we obtain the claim. $\square$

2.6. **Pullbacks under isogenies.** Here, we record auxiliary results already invoked in the proof of Lemma 5. In Section 3.3, these calculations will be used to refine the Lemma, thus giving the pullbacks of the bundles $\mathbf{W}(\mathsf{P})$, for certain triples P, under a different class of isogenies.

For matrices with integer entries

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we set

$$\rho = \rho_M : A \times \widehat{A} \to A \times \widehat{A},\ \ (x, y) \to (ax + b\,\widehat{\Phi}(y), c\,\Phi(x) + dy).$$

We determine the pullbacks of the three bundles $\{\Theta, \widehat{\Theta}, \mathcal{P}\}$ via the isogeny ρ_M.

Lemma 6. *The matrix of the transformation $\rho_M^\star$ acting on the lattice spanned by $\{\Theta, \widehat{\Theta}, \mathcal{P}\}$ equals*

$$R = \begin{bmatrix} a^2 & -c^2 & 2ac \\ -b^2 & d^2 & 2bd \\ -ab & -cd & ad - bc \end{bmatrix}.$$

Proof. We calculate

$$\rho^\star \Theta = (ax + b\widehat{\Phi}(y))^\star \Theta = \Theta^{a^2} \boxtimes \widehat{\Theta}^{-b^2} \otimes \mathcal{P}^{-ab}$$

via the see-saw theorem:

- the restriction to $A \times \{y\}$ equals

$$a^\star t^\star_{b\widehat{\Phi}(y)} \Theta = a^\star(\Theta \otimes \Phi(\widehat{\Phi}(by))) = \Theta^{a^2} \otimes \mathcal{P}_y^{-ab}$$

- the restriction to $\{0\} \times \widehat{A}$ equals $(b\widehat{\Phi})^\star \Theta = \widehat{\Theta}^{-b^2}$.

The remaining two identities are established similarly

$$\rho^\star \widehat{\Theta} = \Theta^{-c^2} \boxtimes \widehat{\Theta}^{d^2} \otimes \mathcal{P}^{-cd},$$

$$\rho^\star \mathcal{P} = \Theta^{2ac} \boxtimes \widehat{\Theta}^{2bd} \otimes \mathcal{P}^{ad-bc}.$$

$\square$

3. Applications to strange duality

The theory of the semihomogeneous bundles introduced in the previous section will now be applied in the setting of strange duality. The introduction, in particular Section 1.3, contains a brief discussion of strange duality over abelian surfaces. In the following sections, we study the Verlinde bundles of generalized theta functions, we formulate several conjectures regarding their general expressions, and offer supporting evidence. Conditionally on these conjectures, in Section 3.4, we investigate the Fourier-Mukai symmetry of the Verlinde bundles, as predicted by equation (2). Finally, in Section 3.5, as a modest application of our methods, we establish a particular instance of strange duality.

Throughout this section, A will be a complex abelian surface, so that $g = 2$.

3.1. **Bundles of generalized theta functions.** We begin by reviewing the construction of the Verlinde bundles of generalized theta functions. For a principally polarized abelian surface (A, Θ), fix the Mukai type

$$v = \mathrm{ch}(E) = (r, k\Theta, \chi)$$

of sheaves $E \to A$. We consider the moduli space $\mathfrak{M}_v$ of Θ-semistable sheaves of topological type v. The moduli space comes equipped with the Albanese morphism

$$\alpha_v : \mathfrak{M}_v \to A \times \widehat{A},$$

which takes sheaves E to their determinant and determinant of the Fourier-Mukai transform

$$\alpha_v(E) = (\det \mathbf{R}\mathcal{S}(E) \otimes \widehat{\Theta}^{-k}, \det E \otimes \Theta^{-k}).$$

The Albanese fiber will be denoted by K_v, thus parametrizing semistable sheaves with fixed determinant and fixed determinant of their Fourier-Mukai. We refer the reader to [Y1] for the study of the Albanese map and of the generalized Kummer variety.

We consider the natural theta line bundles over the above moduli spaces. Let

$$w = (r', k'\Theta, \chi')$$

be a Mukai vector orthogonal to v in the sense that in K-theory we have

$$\chi(v \cdot w) = 0 \iff r\chi' + r'\chi + 2kk' = 0. \tag{8}$$

Let o denote the identity in A. The complex

$$F = (r' - 1)\mathcal{O} + \Theta^{k'} + (\chi' - k'^2)\mathcal{O}_o$$

represents the Mukai vector w and maps to the origin under the Albanese map α_w. By Lemma 1 in [MO2], the constructions below do not depend on the specific choice of F

with these properties. Following [Li], [LP], we consider the inverse determinant of the Fourier-Mukai transform of F with kernel the universal sheaf $\mathcal{E} \to \mathfrak{M}_v \times A$:[2]

$$\Theta_w = \det \mathbf{R}p_!(\mathcal{E} \otimes^{\mathbf{L}} q^\star F)^{-1}.$$

Generalized theta functions are sections of Θ_w over the moduli spaces K_v or $\mathfrak{M}_v$.

The following assumptions will be made throughout:

($\mathbf{A.1}$) the vector $v = (r, k\Theta, \chi)$ is primitive, the moduli space K_v consists of stable sheaves only, and furthermore the Mukai self pairing

$$d_v = \frac{1}{2}\langle v, v\rangle := k^2 - r\chi$$

is an odd positive integer;

($\mathbf{A.2}$) Θ_w belongs to the movable cone of K_v.

In particular, Θ_w becomes big and nef on a smooth birational model of K_v. Such birational models are in fact moduli spaces of Bridgeland stable complexes $K_v(\sigma)$ [Y2], for suitable Bridgeland stability conditions σ. By [BMOY], condition ($\mathbf{A.2}$) is fulfilled for both vectors v and w simultaneously if

$$k, k' > 0, \;\; \chi, \chi' < -1.$$

When assumptions ($\mathbf{A1}$) – ($\mathbf{A2}$) are satisfied, we push forward Θ_w via the Albanese morphism α_v. This yields the Verlinde sheaf of generalized theta functions

$$\mathbf{E}(v, w) = (\alpha_v)_\star \Theta_w$$

over the abelian four-fold $A \times \widehat{A}$. Exchanging the roles of v and w, we similarly obtain

$$\mathbf{E}(w, v) \to A \times \widehat{A}.$$

In the context of curves, the Verlinde bundles have been introduced in [Po] and studied in [O1], [O2]. For abelian surfaces, the construction was first given in [O3]; there, the corresponding bundles were also determined in degree 0. The general case considered here is more involved.

3.2. **Slopes of the Verlinde bundles.** From now on, we fix two orthogonal primitive vectors

$$v = (r, k\Theta, \chi), \;\; w = (r', k'\Theta, \chi')$$

satisfying ($\mathbf{A.1}$) – ($\mathbf{A.2}$).

[2] or by descent from the Quot scheme in the absence of the universal sheaf

Proposition 2. *The sheaves* $\mathbf{E}(v,w)$ *and* $\mathbf{E}(w,v)$ *are locally free and semihomogeneous. Furthermore,*

$$\operatorname{rank}\mathbf{E}(v,w)=\frac{d_v^2}{d_v+d_w}\binom{d_v+d_w}{d_v},\ \ \operatorname{rank}\mathbf{E}(w,v)=\frac{d_w^2}{d_v+d_w}\binom{d_v+d_w}{d_v}.$$

Proof. The local freeness of the Verlinde sheaves follows by equation (10) below. Semihomogeneity also follows by equation (10). Indeed, $\mathbf{E}(v,w)$ splits as a sum of line bundles under pullback, and Lemma 5.4 of [M2] establishes the claim. The ranks of the two bundles are given by the Euler characteristics calculations in [MO2]. This is guaranteed by assumption (**A**.**2**), cf. Theorem 4 of [BMOY]:

$$\text{rank }\mathbf{E}(v,w)=\chi(K_v,\Theta_w)=\frac{d_v^2}{d_v+d_w}\binom{d_v+d_w}{d_v}.$$

□

We next investigate the slopes of $\mathbf{E}(v,w)$ and $\mathbf{E}(w,v)$. To this end, consider the triple

$$\mathsf{P}(v,w)=\frac{1}{d_v}\begin{bmatrix} rk'+r'k\\ \chi k'+\chi' k\\ r'\chi+kk'\end{bmatrix}. \tag{9}$$

Note that the first two entries are obtained from $c_1(v\otimes w)$ and $c_1(\widehat{v}\otimes\widehat{w})$. Similarly, $\mathsf{P}(w,v)$ is defined by reversing the roles of v and w.

Lemma 7. *In terms of* $\{\Theta,\widehat{\Theta},\mathcal{P}\}$*, the slopes of the Verlinde bundle have coordinates*

$$\mu(\mathbf{E}(v,w))=\mathsf{P}(v,w),\ \ \mu(\mathbf{E}(w,v))=\mathsf{P}(w,v)$$

in $Pic(A\times\widehat{A})\otimes\mathbb{Q}$.

Proof. We will use the standard étale diagram [Y1], [MO2]

$$\begin{array}{ccc} K_v\times A\times\widehat{A} & \xrightarrow{\tau} & \mathfrak{M}_v \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \alpha_v} \\ A\times\widehat{A} & \xrightarrow{\Psi_v} & A\times\widehat{A} \end{array}\quad .$$

The upper horizontal morphism

$$\tau: K_v\times A\times\widehat{A}\to\mathfrak{M}_v$$

is given by

$$\tau(E,x,y)=t_x^\star E\otimes y.$$

The lower horizontal morphism is computed in Lemma 4.3 of [Y1]:

$$\Psi_v(x,y)=(-\chi x+k\widehat{\Phi}(y),k\Phi(x)+ry).$$

By Section 2 of [O3], we have that

$$\tau^\star\Theta_w = \Theta_w \boxtimes \mathcal{L} \implies \Psi_v^\star\,\mathbf{E}(v,w) = H^0(K_v,\Theta_w)\otimes\mathcal{L} \tag{10}$$

where

$$\mathcal{L} = \left(\Theta^{-\chi'k-\chi k'} \boxtimes \widehat{\Theta}^{-rk'-r'k}\right)\otimes\mathcal{P}^{r'\chi+kk'}.$$

Lemma 6 gives the matrix R corresponding to the pullback via Ψ_v. Therefore, the slope of the Verlinde bundles $\mathbf{E}(v,w)$ has coordinates

$$R^{-1}\begin{bmatrix} -\chi k'-\chi' k \\ -rk'-r'k \\ r'\chi+kk' \end{bmatrix} = \begin{bmatrix} \chi^2 & -k^2 & -2\chi k \\ -k^2 & r^2 & 2kr \\ k\chi & -rk & -k^2-r\chi \end{bmatrix}^{-1}\begin{bmatrix} -\chi k'-\chi' k \\ -rk'-r'k \\ r'\chi+kk' \end{bmatrix} = \mathsf{P}(v,w)$$

in the basis $\{\Theta,\widehat{\Theta},\mathcal{P}\}$ over the rationals. The orthogonality equation (8) was used to simplify the answer. $\square$

Remark 1. The simplest candidates for bundles with these slopes are $\mathbf{W}(\mathsf{P}(v,w))$ and $\mathbf{W}(\mathsf{P}(w,v))$. We will check shortly in Lemma 9 that $\mathbf{W}(\mathsf{P}(v,w))$ does in fact pullback to a direct sum of copies of the line bundle $\mathcal{L}$ of (10) under the morphism Ψ_v.

Remark 2. It is useful to discuss the numerics of the triples $\mathsf{P}(v,w)$ and $\mathsf{P}(w,v)$. A straightforward calculation involving (8) shows that

$$\det\mathsf{P}(v,w) = -\frac{d_w}{d_v}.$$

Let

$$\Delta = (rk'+r'k,\ \chi k'+\chi' k,\ d_v,\ d_w).$$

Recalling the definition of $\mathsf{P}(v,w)$ in equation (9), and the definition of rank in Subsection 2.3, it follows that

$$\text{rank }\mathsf{P}(v,w) = \frac{d_v}{\Delta},\ \ \chi(\mathsf{P}(v,w)) = -\frac{d_w}{\Delta}. \tag{11}$$

Similarly,

$$\text{rank }\mathsf{P}(w,v) = \frac{d_w}{\Delta},\ \ \chi(\mathsf{P}(w,v)) = -\frac{d_v}{\Delta}.$$

By assumption (**A.1**), the triples $\mathsf{P}(v,w)$ and $\mathsf{P}(w,v)$ are admissible. By Lemma 1, we have

$$\text{rank }\mathbf{W}(\mathsf{P}(v,w)) = \left(\frac{d_v}{\Delta}\right)^2,\ \ \chi(\mathbf{W}(\mathsf{P}(v,w))) = \left(\frac{d_w}{\Delta}\right)^2,$$
$$\text{rank }\mathbf{W}(\mathsf{P}(w,v)) = \left(\frac{d_w}{\Delta}\right)^2,\ \ \chi(\mathbf{W}(\mathsf{P}(w,v))) = \left(\frac{d_v}{\Delta}\right)^2.$$

Lemma 8. *The following Fourier-Mukai symmetry holds*

$$\mathbf{W}(\widehat{\mathsf{P}(v,w)}) \cong \mathbf{W}(\mathsf{P}(w,v))^\vee.$$

Proof. A simple calculation shows

$$\mathsf{P}(v,w) = -\mathsf{P}(w,v)^{-1}.$$

Thus, by Proposition 1, the bundles $\mathbf{W}(\mathsf{P}(v,w))$ and $\mathbf{W}(\mathsf{P}(w,v))^\vee$ are connected by Fourier-Mukai transform. $\square$

Remark 3. To summarize, via Lemma 7, we provided evidence for the claim that the building blocks of the Verlinde bundles $\mathbf{E}(v,w)$ are the semihomogeneous bundles $\mathbf{W}(\mathsf{P}(v,w))$. Furthermore, these building blocks are exchanged under Fourier-Mukai, consistently with Proposition 4 to be discussed shortly. These statements are however only true up to torsion; this will be made precise in Conjectures 1 and 2 below.

Example 4. We consider the case when $k=0$ *i.e.* $c_1(v)=0$. Then, the slope of $\mathbf{E}(v,w)$ equals

$$\mathsf{P}(v,w) = \begin{bmatrix} -\frac{k'}{\chi} \\ \frac{-k'}{r} \\ -\frac{r'}{r} \end{bmatrix} = \begin{bmatrix} -\frac{k'}{\chi} \\ -\frac{k'}{r} \\ -h \end{bmatrix},$$

where we wrote $r' = rh, \chi' = -\chi h$. Since $(r,\chi)=1$, we have $h \in \mathbb{Z}$. Furthermore, $\Delta = ab$, where

$$a = (k',\chi),\ \ b = (k',r).$$

By Example 2, we have

$$\mathbf{W}(\mathsf{P}(v,w)) = \mathbf{W}_{-\frac{\chi}{a},\frac{k'}{a}} \boxtimes \mathbf{W}^\dagger_{\frac{r}{b},-\frac{k'}{b}} \otimes \mathcal{P}^{-h}.$$

$\square$

We now check that the bundles $\mathbf{W}(\mathsf{P}(v,w))$ split as direct sum of line bundles under the isogeny Ψ_v, as claimed above. This a refinement of Lemma 5 better suited to our purposes. The result will be useful below.

Lemma 9. *The pullback*

$$\Psi_v^\star \mathbf{W}(\mathsf{P}(v,w)) = \mathcal{L} \otimes \mathbb{C}^{(d_v/\Delta)^2}$$

splits as direct sum of line bundles, where

$$\mathcal{L} = \left(\Theta^{-\chi' k - \chi k'} \boxtimes \widehat{\Theta}^{-rk' - r'k}\right) \otimes \mathcal{P}^{r'\chi + kk'}.$$

Proof. We consider the dual vector $w^\vee = (r', -k'\Theta, \chi')$ and the morphism

$$\mathsf{j} : A \times \widehat{A} \to A \times \widehat{A},\ \ (x,y) \mapsto (x,-y).$$

A direct calculation shows that

$$\Phi_{\frac{d_v}{\Delta}\mathsf{P}(v,w)} \circ \Psi_v = \mathsf{j} \circ \Psi_{w^\vee} \circ \frac{d_v}{\Delta}.$$

It suffices to check that $\Psi_v^\star \mathbf{W}(\mathsf{P}(v,w))$ splits as direct sum of the same line bundle L, since then the line bundle is identified uniquely just as in Lemma 5, using invariance under (-1) and the determinant calculation. In particular, it suffices to check that

$$q : X \to \mathcal{Z}(\mathbf{W}(\mathsf{P}(v,w))) \hookrightarrow X \times \widehat{X}, \quad x \mapsto (\Psi_v(x), \mathsf{j} \circ \Psi_{w^\vee}(x))$$

is a well defined morphism, where $X = A \times \widehat{A}$. Here, $\mathsf{j} \circ \Psi_{w^\vee}(x)$ is interpreted as a line bundle over X. Then,

$$p \circ q = \Psi_v,$$

and we can invoke Lemma 3.6 of [M2] showing that the pullback under $p : \mathcal{Z} \to X$ already splits. It remains to prove

$$t^\star_{\Psi_v(x)} \mathbf{W}(\mathsf{P}(v,w)) = \mathbf{W}(\mathsf{P}(v,w)) \otimes \mathsf{j} \circ \Psi_{w^\vee}(x), \quad \text{for all } x \in X. \tag{12}$$

By the general study of the bundles $\mathbf{W}(\mathsf{P})$, in particular Lemma 2, we have

$$t^\star_{\frac{d_v}{\Delta} y} \mathbf{W}(\mathsf{P}(v,w)) = \mathbf{W}(\mathsf{P}(v,w)) \otimes \Phi_{\frac{d_v}{\Delta}\mathsf{P}(v,w)}(y). \tag{13}$$

Fix $x \in X$ and write

$$x = \frac{d_v}{\Delta} x', \quad \text{for } x' \in X.$$

Set $y = \Psi_v(x')$. We calculate

$$\Phi_{\frac{d_v}{\Delta}\mathsf{P}(v,w)}(y) = \Phi_{\frac{d_v}{\Delta}\mathsf{P}(v,w)} \circ \Psi_v(x') = \mathsf{j} \circ \Psi_{w^\vee} \circ \frac{d_v}{\Delta}(x') = \mathsf{j} \circ \Psi_{w^\vee}(x).$$

Now (12) follows from (13) by substitution. □

3.3. **Conjectural description of the Verlinde bundles.** In this subsection, we state our main conjectures expressing the Verlinde bundles in terms of the semihomogeneous vector bundles $\mathbf{W}(\mathsf{P})$.

3.3.1. *A finite group of symmetries.* In order to make the precise statement, we need to introduce a certain group of torsion points $\mathsf{Q}_{v,w}$. Recall the morphism

$$\Psi_v : A \times \widehat{A} \to A \times \widehat{A}, \quad \Psi_v(x,y) = (-\chi x + k\widehat{\Phi}(y), k\Phi(x) + ry).$$

We pullback line bundles over $A \times \widehat{A}$ by the isogeny Ψ_v. By Lemma 9 we have

$$\Sigma(\mathbf{W}(\mathsf{P}(v,w))) \subset \text{Ker } \Psi_v^\star.$$

By Lemma A.2 in the Appendix, it follows that Ψ_v has degree d_v^4. Using (11), we obtain that the quotient

$$\mathsf{Q}_{v,w} = \text{Ker } \Psi_v^\star / \Sigma(\mathbf{W}(\mathsf{P}(v,w)))$$

has $\frac{d_v^4}{(d_v/\Delta)^4} = \Delta^4$ elements. In fact, we will show that

Lemma 10. *There is an isomorphism*

$$\mathsf{Q}_{v,w} \simeq \widehat{A}\,[\Delta].$$

Proof. Let m and n be coprime integers such that

$$\chi m \equiv -kn \mod d_v, \ rn \equiv -km \mod d_v.$$

These exist by Lemma A.1 of the Appendix. Then

$$\xi^m \boxtimes \widehat{\Phi}^\star \xi^n \in \text{Ker } \Psi_v^\star,$$

for each $\xi \to A$ which is d_v-torsion. Indeed, we calculate

$$\Psi_v^\star \left(\xi^m \boxtimes \widehat{\Phi}^\star \xi^n\right) = (-\chi x + k\widehat{\Phi}(y), k\Phi(x) + ry)^\star \left(\xi^m \boxtimes \widehat{\Phi}^\star \xi^n\right)$$

$$= \xi^{-m\chi - nk} \boxtimes \widehat{\Phi}^\star \xi^{mk+nr} = \mathcal{O}.$$

The morphism

$$f : \widehat{A}\,[d_v] \to \text{Ker } \Psi_v^\star, \ \xi \to \xi^m \boxtimes \widehat{\Phi}^\star \xi^n$$

is injective since $(m,n) = 1$, and by comparing orders, it must then be an isomorphism. If ξ is in fact d_v/Δ-torsion, then

$$\xi^m \boxtimes \widehat{\Phi}^\star \xi^n \in \Sigma(\mathbf{W}(\mathsf{P}(v,w))).$$

This is a consequence of Lemma 3 and the following easily checked identity

$$\Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\widehat{\Phi}(\xi^n), \xi^m) = 0.$$

(The switch in the positions of the arguments takes into account the identification of $A \times \widehat{A}$ with its dual; this is required to apply Lemma 3.) By comparing orders again, we obtain the isomorphism

$$f : \widehat{A}\,[d_v/\Delta] \to \Sigma(\mathbf{W}(\mathsf{P}(v,w))).$$

This yields the identification claimed in the lemma

$$f : \frac{\widehat{A}\,[d_v]}{\widehat{A}\,[d_v/\Delta]} \simeq \widehat{A}\,[\Delta] \to \mathsf{Q}_{v,w}.$$

3.3.2. *An application.* We apply the results of the previous subsection to obtain the expression for the Verlinde bundle in a special case:

Proposition 3. *When $(d_v, d_w) = 1$ we have*

$$\mathbf{E}(v,w) = \bigoplus_{i=1}^{N} \mathbf{W}(\mathsf{P}(v,w)), \text{ where } N = \frac{1}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

Proof. To justify the splitting, note that Lemma 9 and equation (10) imply

$$\Psi_v^\star \mathbf{E}(v,w) = \Psi_v^\star \mathbf{W}(\mathsf{P}(v,w)) \otimes \mathbb{C}^N.$$

Pushing forward by Ψ_v, it follows that the Verlinde bundle sits a direct summand in

$$\mathbf{E}(v,w) \hookrightarrow \bigoplus_{i=1}^{N} \mathbf{W}(\mathsf{P}(v,w)) \otimes y$$

where $y \in \text{Ker } \Psi_v^\star$. Since $(d_v, d_w) = 1$ we must have $\Delta = 1$. By Lemma 10, we have that Ker $\Psi_v^\star = \Sigma(\mathbf{W}(\mathsf{P}(v,w)))$. As a result, tensoring by y leaves $\mathbf{W}(\mathsf{P}(v,w))$ invariant, and thus

$$\mathbf{E}(v,w) \hookrightarrow \bigoplus_{i=1}^{d_v^4 N} \mathbf{W}(\mathsf{P}(v,w)).$$

The claimed splitting follows since $\mathbf{W}(\mathsf{P}(v,w))$ is indecomposable. The number of summands is found by comparing ranks. $\square$

3.3.3. *Conjectures.* For general numerics, the argument of Proposition 3 gives the decomposition

$$\mathbf{E}(v,w) = \bigoplus_{\zeta \in \mathsf{Q}_{v,w}} \mathbf{W}(\mathsf{P}(v,w)) \otimes \zeta^{\oplus \mathsf{m}_\zeta},$$

for certain multiplicities m_ζ associated to different points $\zeta \in \mathsf{Q}_{v,w}$. We formulate conjectures regarding these multiplicities:

Conjecture 1. *Assume* $(\mathbf{A.1}) - (\mathbf{A.2})$ *hold for both pairs* (v,w) *and* (w,v)*. Write*

$$\mathbf{E}(v,w) = \bigoplus_{\zeta \in \mathsf{Q}_{v,w}} \mathbf{W}(\mathsf{P}(v,w)) \otimes \zeta^{\oplus \mathsf{m}_\zeta},$$

and the analogous expression for $\mathbf{E}(w,v)$*. Then*

(i) *the multiplicity* m_ζ *only depends on the order of* $\zeta \in \mathsf{Q}_{v,w}$*. Write* $\mathsf{m}_\omega(v,w) := \mathsf{m}_\zeta$ *for the value of the multiplicity corresponding to elements of order* $\omega =$ *order* ζ*;*

(ii) *the following symmetry holds*

$$\mathsf{m}_\omega(v,w) = \mathsf{m}_\omega(w,v).$$

Item (ii) makes sense by (i) and the abstract isomorphism given by Lemma 10:

$$\mathsf{Q}_{v,w} \simeq \mathsf{Q}_{w,v}.$$

We further speculate

Conjecture 2. *We have*

$$\mathsf{m}_{\omega}(v,w) = \frac{1}{d_v + d_w} \sum_{\delta \mid \Delta} \frac{\delta^4}{\Delta^2} \left\{ \frac{\Delta/\omega}{\delta} \right\} \binom{d_v/\delta + d_w/\delta}{d_v/\delta}.$$

The Jordan totient $\{\ \}$ appearing here is defined in terms of prime factorization. First, the Jordan totient is set to 1 if $h = 1$. For an integer $h \geq 2$, we decompose

$$h = p_1^{a_1} \dots p_n^{a_n}$$

into powers of primes. We set

$$\left\{ \frac{\lambda}{h} \right\} = \begin{cases} 0 & \text{if } p_1^{a_1 - 1} \dots p_n^{a_n - 1} \text{ does not divide } \lambda, \\ \prod_{i=1}^{n} \left(\epsilon_i - \frac{1}{p_i^4} \right) & \text{otherwise,} \end{cases}$$

where $\epsilon_i = 1$ when $p_i^{a_i} | \lambda$, and $\epsilon_i = 0$ otherwise. A similar arithmetic function enters the expressions of the Verlinde bundles over curves, see [O1] [O2].

3.3.4. *Some evidence.* To motivate the expression of the multiplicities we conjectured, we show

Lemma 11. *Conjecture 1 and Conjecture 2 hold true when $c_1(v) = 0$.*

Proof. We have $\Psi_v = (-\chi, r)$ so

$$\text{Ker } \Psi_v^{\star} = \widehat{A}[-\chi] \times A[r].$$

Writing

$$(k', \chi) = a, \ (k', r) = b,$$

we noted in Example 4 that

$$\mathbf{W}(\mathsf{P}(v,w)) = \mathbf{W}_{-\frac{\chi}{a}, \frac{k'}{a}} \boxtimes \mathbf{W}^{\dagger}_{\frac{r}{b}, -\frac{k'}{b}} \otimes \mathcal{P}^{-h}.$$

By Section 2.2, it follows that

$$\Sigma(\mathbf{W}(\mathsf{P}(v,w))) = \widehat{A}\left[-\frac{\chi}{a}\right] \times A\left[\frac{r}{b}\right].$$

Therefore

$$\mathsf{Q}_{v,w} \cong \widehat{A}[a] \times A[b].$$

There are precisely $\Delta^4 = (ab)^4$ possible torsion points in $\mathsf{Q}_{v,w}$. For each element $\zeta \in \widehat{A}[a] \times A[b]$, any choice of $\ell_\zeta \in \widehat{A} \times A$ such that

$$\left(-\frac{\chi}{a}, \frac{r}{b}\right) \ell_\zeta = \zeta$$

gives a well-defined element of $\mathsf{Q}_{v,w}$. The conjecture predicts that

$$\mathbf{E}(v,w)=\bigoplus_{\zeta}\mathbf{W}(\mathsf{P}(v,w))\otimes\ell_{\zeta}^{\oplus\mathsf{m}_{\zeta}}=\bigoplus_{\zeta}\left(\mathbf{W}_{-\frac{\chi}{a},\frac{k'}{a}}\boxtimes\mathbf{W}^{\dagger}_{\frac{r}{b},-\frac{k'}{b}}\otimes\mathcal{P}^{-h}\right)\otimes\ell_{\zeta}^{\oplus\mathsf{m}_{\zeta}}.$$

The line bundle ζ of order ω dividing (a,b) should contribute with multiplicity

$$\mathsf{m}_{\omega}=\frac{1}{d_v+d_w}\sum_{\delta|ab}\frac{\delta^4}{(ab)^2}\left\{\frac{ab/\omega}{\delta}\right\}\binom{d_v/\delta+d_w/\delta}{d_v/\delta}.$$

This statement is Theorem 3 in [O3], and was established via a trace calculation on the space of generalized theta functions

$$\mathrm{Trace}(\zeta,\chi(K_v,\Theta_w))=\sum_i(-1)^i\,\mathrm{Trace}(\zeta,H^i(K_v,\Theta_w)),$$

for all torsion points $\zeta\in\widehat{A}[a]\times A[b]$ and for a suitable action of ζ on the pair (K_v,Θ_w). The Jordan totient appears via inversion formulas solving for the multiplicities m_{ω} in terms of the traces. $\square$

3.3.5. *Further conjectures – Equivariant Verlinde numbers.* In general, it may be possible to establish Conjectures 1 and 2 using trace calculations similar to those of [O3]. This however requires specifying the correct setup, and it leads to questions related to equivariant Verlinde numbers.

To this end, consider the isogenies Ψ_v and Ψ_w and their kernels

$$\mathsf{G}_v=\mathrm{Ker}\ \Psi_v\subset A\times\widehat{A},\quad \mathsf{G}_w=\mathrm{Ker}\ \Psi_w\subset A\times\widehat{A}.$$

It is clear that G_v acts on K_v by translation and tensorization

$$E\mapsto t_x^{\star}E\otimes y$$

but we would like to single out a subgroup that also acts on $\Theta_w\to K_v$.

Recall the line bundle $\mathcal{L}$ of equation (10) which emerges by pulling back the theta bundle under Ψ_v. It is shown in the proof of Lemma 6 of [BMOY] that

$$\mathsf{G}_v\subset K(\mathcal{L}),\quad \mathsf{G}_w^{+}:=\mathsf{j}(\mathsf{G}_w)\subset K(\mathcal{L}),$$

where

$$\mathsf{j}:A\times\widehat{A}\to A\times\widehat{A},\ \ \mathsf{j}=(1,-1).$$

Set

$$\mathsf{G}=\mathsf{G}_v\cap\mathsf{G}_w^{+}\subset K(\mathcal{L}).$$

The group G arises naturally. Furthermore, the following hold true:

(a) G is isomorphic to $A[\Delta]$. An explicit isomorphism is given by

$$A[\Delta] \to \mathsf{G}, \;\; z \mapsto (mz, n\Phi(z))$$

for coprime integers (m, n) such that

$$\chi m + kn \equiv 0 \mod \Delta, \quad km + rn \equiv 0 \mod \Delta$$

$$-\chi' m + k' n \equiv 0 \mod \Delta, \quad -k' m + r' n \equiv 0 \mod \Delta.$$

The existence of m, n is a consequence of Lemma A.1;

(b) G is an isotropic subgroup of $K(\mathcal{L})$. That is, writing $e^{\mathcal{L}}$ for the commutator pairing, see for instance [Pol], Section 2.1, an immediate coordinate calculation shows that

$$e^{\mathcal{L}}|_{\mathsf{G}\times\mathsf{G}} \equiv 1.$$

Therefore, $\mathcal{L}$ descends to a line bundle $\mathcal{M}$ over the quotient

$$\pi : A \times \widehat{A} \to (A \times \widehat{A})/\mathsf{G},$$

see for instance [Pol], Theorem 10.5. As usual, a canonical choice of $\mathcal{M}$ can be made by requiring symmetry. In fact, it is not hard to write down an explicit expression for $\mathcal{M}$, but this will not be needed. With this understood, equation (10) becomes

$$\tau^{\star}\Theta_w = \Theta_w \boxtimes \pi^{\star}\mathcal{M}.$$

Since τ and π are both G-invariant, it follows that the corresponding pullbacks acquire natural G-actions. This yields a G-action on the bundle $\Theta_w \to K_v$, covering the action of G on K_v by translation and tensorization. We refer the reader to [O3] for the similar argument in the curve context.

We speculate that

Conjecture 3. *For each $\zeta \in \mathsf{G}$ of order δ, we have*

$$\mathrm{Trace}\,(\zeta, \chi(K_v, \Theta_w)) = \frac{d_v^2}{d_v + d_w}\binom{d_v/\delta + d_w/\delta}{d_v/\delta}.$$

When $c_1(v) = 0$ this is confirmed by Theorem 3 of [O3] as it was mentioned in the proof of Lemma 11.

If true, Conjecture 3 provides an equivariant extension of the Verlinde formulas for abelian varieties established in [MO2]. We also expect representation theoretic arguments to link Conjecture 2 and Conjecture 3. (The methods of [O3] should allow to switch from the Euler characteristics to global sections, under assumptions (**A1**) – (**A2**).) We will pursue these matters elsewhere.

3.4. **Fourier-Mukai symmetry.** We now take note of the symmetry of the multiplicities m_ω of Conjecture 1 (ii) under the exchange of v and w. This has the following consequence:

Proposition 4. *Conjecture 1 (ii) implies that*

$$\widehat{\mathbf{E}(v,w)} \cong \mathbf{E}(w,v)^\vee.$$

As explained in the Introduction, the Fourier-Mukai symmetry of the Proposition is predicted by the strange duality conjecture. However, strange duality is equivalent to the existence of a *specific geometric* isomorphism as above, while Proposition 4 shows that the two bundles are only *abstractly* isomorphic.

Proof. Fix $\zeta \in \mathsf{Q}_{v,w}$. We consider the summand $\mathbf{W}(\mathsf{P}(v,w)) \otimes \zeta$ appearing in the conjectured expression of the Verlinde bundle $\mathbf{E}(v,w)$. We calculate the Fourier-Mukai dual (up to shifts) making use of Lemma 8:

$$\mathbf{R}\mathcal{S}(\mathbf{W}(\mathsf{P}(v,w)) \otimes \zeta) = t_\zeta^\star \mathbf{R}\mathcal{S}(\mathbf{W}(\mathsf{P}(v,w))) = t_\zeta^\star\, \mathbf{W}(\mathsf{P}(w,v))^\vee.$$

(Here, we identified $A \times \widehat{A}$ with its dual, so that ζ is a line bundle on the left, and a translation point on the right. The calculations below keep the same abuse of notation.) Writing

$$\zeta = \frac{d_w}{\Delta} \cdot \zeta',$$

and using equation (13), the above expression becomes

$$\mathbf{R}\mathcal{S}(\mathbf{W}(\mathsf{P}(v,w)) \otimes \zeta) = t^\star_{\frac{d_w}{\Delta}\zeta'} \mathbf{W}(\mathsf{P}(w,v))^\vee = \left(\mathbf{W}(\mathsf{P}(w,v)) \otimes \Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\zeta')\right)^\vee = \left(\mathbf{W}(\mathsf{P}(w,v)) \otimes \widetilde{\zeta}\right)^\vee \tag{14}$$

where

$$\widetilde{\zeta} := \Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\zeta').$$

We claim that $\widetilde{\zeta}$ gives a well-defined element of $\mathsf{Q}_{w,v}$, and that the map

$$\mathsf{Q}_{v,w} \ni \zeta \to \widetilde{\zeta} \in \mathsf{Q}_{w,v}$$

is bijective. Thus, by (14), every summand appearing in $\widehat{\mathbf{E}(v,w)}$ gives rise to a corresponding summand in $\mathbf{E}(w,v)^\vee$, and by hypothesis, the multiplicities match as well. This establishes the isomorphism between the Verlinde bundle and the Fourier-Mukai dual.

Let us now verify that $\widetilde{\zeta}$ is well-defined in $\mathsf{Q}_{w,v}$. Indeed, by direct calculation we have

$$\Psi_w^\star \widetilde{\zeta} = \Psi_w^\star \Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\zeta') = \mathsf{j}^\star \Psi_v^\star \left(\frac{d_w}{\Delta}\zeta'\right) = \mathsf{j}^\star \Psi_v^\star \zeta = \mathcal{O},$$

where j was introduced in Lemma 9:

$$\mathsf{j} : A \times \widehat{A} \to A \times \widehat{A}, \;\; \mathsf{j}(x, y) = (x, -y).$$

To see that $\widetilde{\zeta}$ does not depend on the choice of ζ', it suffices to prove that

$$\frac{d_w}{\Delta}\alpha \in \Sigma(\mathbf{W}(\mathsf{P}(v,w))) \iff \Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\alpha) \in \Sigma(\mathbf{W}(\mathsf{P}(w,v))). \tag{15}$$

This is a consequence of Lemma 3. To apply the Lemma we recall equation (11) which gives the numerics of the triples $\mathsf{P}(v,w)$ and $\mathsf{P}(w,v)$. We have

$$\frac{d_w}{\Delta}\alpha \in \Sigma(\mathbf{W}(\mathsf{P}(v,w))) \iff \frac{d_v}{\Delta}\cdot\left(\frac{d_w}{\Delta}\alpha\right) = 0 \text{ and } \Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}\left(\frac{d_w}{\Delta}\alpha\right) = 0.$$

In a similar fashion,

$$\Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\alpha) \in \Sigma(\mathbf{W}(\mathsf{P}(w,v))) \iff \frac{d_w}{\Delta}\left(\Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\alpha)\right) = 0 \text{ and}$$

$$\Phi_{\frac{d_v}{\Delta}\mathsf{P}(v,w)}\left(\Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)}(\alpha)\right) = 0.$$

To see the equivalence claimed in (15), it suffices to observe that

$$\Phi_{\frac{d_v}{\Delta}\mathsf{P}(v,w)} \circ \Phi_{\frac{d_w}{\Delta}\mathsf{P}(w,v)} = \frac{d_v}{\Delta}\cdot\frac{d_w}{\Delta}.$$

Equation (15) also implies that the assignment $\mathsf{Q}_{v,w} \ni \zeta \to \widetilde{\zeta} \in \mathsf{Q}_{w,v}$ is injective and therefore, by comparing orders, an isomorphism. □

3.5. **An application to level 1 strange duality.** The integers d_v and d_w should be thought of as the rank and level in the theory of curves. As a very modest application of our results, we prove the following analogue of the level 1 strange duality for curves established in [BNR]. For $K3$ surfaces, with d_v interpreted as $\frac{1}{2}\langle v, v\rangle + 1$, the corresponding statement is proved in [BM], Section 15, but the argument requires different techniques.

Theorem 2. *When $d_v = 1$ or $d_w = 1$, the strange duality map is either zero or an isomorphism.*

Proof. When $d_v = 1$ or $d_w = 1$, we have $\Delta = 1$. By Proposition 3,

$$\mathbf{E}(v,w) \cong \mathbf{W}(\mathsf{P}(v,w)), \;\; \mathbf{E}(w,v) \cong \mathbf{W}(\mathsf{P}(w,v)).$$

The assumption $d_v = 1$ or $d_w = 1$ is used to note that we have unique indecomposable factors. By Proposition 1, both the Verlinde bundle $\mathbf{E}(v,w)^\vee$ and the Fourier-Mukai transform $\widehat{\mathbf{E}(w,v)}$ must coincide with $\mathbf{W}(\mathsf{P}(v,w))^\vee$, which being simple and semihomogeneous is stable with respect to any polarization, cf. [M2]. Any map between the two bundles, in particular the strange duality map, would have to be an isomorphism or zero.

By means of examples, as in [BM], one sees that the possibility that the map SD vanishes can occur. $\square$

Appendix A. Torsion points on abelian varieties

In this appendix, we collect elementary arithmetic results used repeatedly throughout the paper. We begin by proving:

Lemma A.1. *Consider a quadruple of integers such that* $\gcd(a, b, c, d) = 1$ *and* $b^2 \equiv ac \mod d$*. We can find* (m, n) *coprime integers such that*

$$cm \equiv bn \mod d, \ \ bm \equiv an \mod d.$$

Proof. We prove that $m = (a, b, d)m'$ and $n = (b, c, d)$ work for an appropriate choice of m'. We have

$$\begin{aligned} cm \equiv bn \mod d &\iff c(a, b, d)m' \equiv b(b, c, d) \mod d \\ &\iff \frac{c}{(b, c, d)}m' \equiv \frac{b}{(a, b, d)} \mod \frac{d}{(b, ac, d)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} bm \equiv an \mod d &\iff b(a, b, d)m' \equiv a(b, c, d) \mod d \\ &\iff \frac{b}{(b, c, d)}m' \equiv \frac{a}{(a, b, d)} \mod \frac{d}{(b, ac, d)}. \end{aligned}$$

We can find integers B, C and D such that

$$bB + cC + dD = (b, c, d).$$

We let

$$m' \equiv \frac{bC + aB}{(a, b, d)} \mod \frac{d}{(b, ac, d)}.$$

The condition $b^2 \equiv ac \mod d$ ensures that the pair (m, n) thus obtained satisfies both congruences.

It suffices to show we can find m' such that $(m, n) = 1$. It is easy to see that if $(u, v, w) = 1$, then the arithmetic progression $\{u + vx\}_{x\in\mathbb{Z}}$ contains a term coprime to w. We apply this in our situation to construct m' coprime to $n = (b, c, d)$. We need to verify that

$$\left(\frac{bC + aB}{(a, b, d)}, \frac{d}{(b, ac, d)}, (b, c, d)\right) = 1.$$

This is indeed the case, since if p is a prime dividing all three numbers above, we can find $\beta, \gamma, \delta \geq 1$ such that

$$p^\beta || b, \ \ p^\gamma || c, \ \ p^\delta || d.$$

We must have $(p, a) = 1$ because $\gcd(a, b, c, d) = 1$. Since

$$p|bC + aB \implies p|aB \implies p|B.$$

Moreover

$$p|\frac{d}{(b, ac, d)} \implies \delta > \min(\beta, \gamma).$$

Now,

$$b^2 \equiv ac \mod d \implies p^\delta|b^2 - ac \implies \beta < \gamma.$$

Indeed, if $\beta \geq \gamma$, we have $\delta \geq \gamma + 1$ and $2\beta \geq \gamma + 1$. But then p^γ divides $b^2 - ac$, but p^δ does not. Thus

$$\delta > \beta, \gamma > \beta.$$

Finally,

$$p^\beta||(b, c, d) = bB + cC + dD$$

but the right hand side is divisible by $p^{\beta+1}$ since c, d are, and B is divisible by p. This contradiction completes the proof. □

Lemma A.2. *Assume a, b, c, d are integers such that*

$$b^2 \equiv ac \mod d,$$

and

$$\gcd\left(a, c, d, \frac{b^2 - ac}{d}\right) = 1.$$

Then, for any g dimensional complex abelian variety A, we can find exactly d^{2g} pairs $(x, y) \in A[d] \times A[d]$ such that

$$ax = by, \ \ bx = cy.$$

Proof. We prove the Lemma in several steps. Fix $a, b, c,$ and let S_d be the set of solutions of the equations

$$ax = by, \ \ bx = cy, \ \ dx = dy = 0.$$

Let us write

$$s_d = \text{order of } \mathsf{S}_d.$$

We show that $s_d = d^{2g}$.

Step 1. We prove s_d is multiplicative. Assume $d = d_1 d_2$ with $(d_1, d_2) = 1$. Note that

$$\left(a, c, d, \frac{b^2 - ac}{d}\right) = 1 \iff \left(a, c, d_i, \frac{b^2 - ac}{d_i}\right) = 1, \ \ 1 \leq i \leq 2.$$

We claim that

$$\mathsf{S}_d \to \mathsf{S}_{d_1} \times \mathsf{S}_{d_2}, \ \ (x, y) \to ((d_2 x, d_2 y), (d_1 x, d_1 y))$$

is a bijection. Indeed, if we pick A and B such that

$$Ad_1 + Bd_2 = 1,$$

an inverse is given by

$$((z_1, w_1), (z_2, w_2)) \to (Bz_1 + Az_2, Bw_1 + Aw_2).$$

We conclude that

$$s_d = s_{d_1} s_{d_2}.$$

Step 2: We prove the lemma when $(a, b, c, d) = 1$. In fact, it suffices to assume $d = p^\delta$ for some prime p, by the step above.

We pick m and n as in Lemma A.1:

$$am \equiv bn \mod d, \ \ cn \equiv bm \mod d.$$

By the proof of the Lemma, we can assume that $(b, c, d)|m$ and $(a, b, d)|n$. Since $(m, n) = 1$ and d is a power of a prime, then either

$$(m, d) = 1 \text{ or } (n, d) = 1.$$

Without loss of generality assume the former case holds. As $(b, c, d)|m$, we obtain $(b, c, d) = 1$. Let $(x, y) \in \mathsf{S}_d$. Since $(m, d) = 1$, we can write

$$x = mz, \quad \text{for } z \in A[d].$$

Set $w = y - nz \in A[d]$. Clearly, we have

$$bw = b(y - nz) = by - amz = by - ax = 0,$$

$$cw = c(y - nz) = cy - bmz = cy - bx = 0,$$

$$dw = 0.$$

We conclude $(b, c, d)w = 0 \implies w = 0$. Thus $y = nz$, and the mapping

$$\pi : A[d] \to \mathsf{S}_d, \ \ z \to (mz, nz)$$

is surjective. The kernel of π is trivial since $(m, n) = 1$. Therefore π is an isomorphism, showing $s_d = d^{2g}$.

Step 3: Next we prove the statement when $d = b^2 - ac$. In this case, the equations $dx = dy = 0$ are redundant. Set $e = (a, b, c)$ and write

$$a = ea', \ b = eb', \ c = ec', \ \ (a', b', c') = 1.$$

We need to solve

$$a'x - b'y = \alpha, \ \ b'x - c'y = \beta,$$

where $(\alpha, \beta) \in A[e] \times A[e]$ runs over all $e^{4g} = e^{2g} \cdot e^{2g}$ pairs of e-torsion points in A.

We consider the morphism

$$f : A \times A \to A \times A,$$

given by

$$f(x, y) = (a'x - b'y, b'x - c'y).$$

By the previous step, we know the kernel of f has $(b'^2 - a'c')^{2g}$ elements since $(a', b', c') = 1$. Therefore, for each of the e^{4g} choices of pairs (α, β), the equation

$$f(x, y) = (\alpha, \beta)$$

has $(b'^2 - a'c')^{2g}$ solutions. We obtain

$$e^{4g}(b'^2 - a'c')^{2g} = (b^2 - ac)^{2g} = d^{2g}$$

possibilities for (x, y).

Step 4. We claim that for each (a, b, c, d) as in the statement of the lemma, we have

$$s_d \leq d^{2g}.$$

By the first step, it suffices then to assume $d = p^\delta$, for some prime p. We will induct on δ, the base case $\delta = 0$ being clear.

We may assume p divides a, b, c since otherwise $(a, b, c, d) = 1$, a case which has already been discussed in *Step 2.* Write

$$a = pa', b = pb', c = pc'.$$

If $\delta = 1$, then

$$\left(a, c, d, \frac{b^2 - ac}{d}\right) = p,$$

contradiction. Hence $\delta \geq 2$. Note that

$$d | b^2 - ac = p^2(b'^2 - a'c') \implies p^{\delta-2} | b'^2 - a'c'.$$

Write $d' = p^{\delta-2}$. Since $(a, c, d, \frac{b^2-ac}{d}) = 1$ we know that $b'^2 - a'c'$ is not divisible by $p^{\delta-1}$. In particular

$$d' | b'^2 - a'c', \text{ and } \left(a', c', d', \frac{b'^2 - a'c'}{d'}\right) = 1.$$

Furthermore,

$$(d, b'^2 - a'c') = p^{\delta-2} = d',$$

so we can write

$$d' = dU + (b'^2 - a'c')V.$$

For pairs $(x, y) \in \mathsf{S}_d$, we must have

$$a'x = b'y + \alpha, \quad b'x = c'y + \beta, \quad dx = dy = 0,$$

where $(\alpha, \beta) \in A[p] \times A[p]$. In fact, we obtain

$$(b'^2 - a'c')x = b'\beta - c'\alpha, \ dx = 0 \implies d'x = V(b'\beta - c'\alpha).$$

For each one of the $p^{2g} \cdot p^{2g} = p^{4g}$ pairs (α, β), we have at most $(d')^{2g}$ choices for (x, y) solving

$$a'x = b'y + \alpha, \ b'x = c'y + \beta, \ d'x = V(b'\beta - c'\alpha), \ d'y = V(a'\beta - b'\alpha).$$

Indeed, if a solution (x_0, y_0) exists to begin with, then for all the other pairs we have

$$(x - x_0, y - y_0) \in \mathsf{S}_{d'}(a', b', c')$$

which has at most $(d')^{2g}$ elements by induction. Therefore

$$s_d \le p^{4g} d'^{2g} = d^{2g},$$

as claimed.

Step 5. We can now prove that $s_d = d^{2g}$ for all d as in the lemma. Fix a, b, c and write

$$b^2 - ac = p_1^{\alpha_1} \dots p_r^{\alpha_r} q_1^{\beta_1} \dots q_\ell^{\beta_\ell},$$

where the primes $p_1, \dots, p_r$ also divide (a, c). Now clearly d is product of p's and q's, but if p_i divides d then in fact

$$p_i^{\alpha_i} || d,$$

by the gcd condition in the lemma. Thus we may write

$$d = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\gamma_1} \cdots q_\ell^{\gamma_\ell}, \quad \text{for } s \le r.$$

We have already observed that

$$s_{p_i^{\alpha_i}} \le p_i^{2g\alpha_i}, \ 1 \le i \le r$$

in the previous step. Also,

$$s_{q_j^{\gamma_j}} = q_j^{2g\gamma_j}, \ 1 \le j \le \ell$$

by the fact that $(a, c, q_j^{\gamma_j}) = 1$ and *Step* 2. Thus by the first step

$$s_{b^2 - ac} = s_{p_1^{\alpha_1}} \cdots s_{p_r^{\alpha_r}} \cdot s_{q_1^{\beta_1}} \cdots s_{q_\ell^{\beta_\ell}} \le (p_1^{\alpha_1} \cdots p_r^{\alpha_r} q_1^{\beta} \cdots q_\ell^{\beta_\ell})^{2g} = (b^2 - ac)^{2g}.$$

Since we proved in the third step that equality occurs, we conclude that

$$s_{p_i^{\alpha_i}} = p_i^{2g\alpha_i}, \ 1 \le i \le r,$$

same as for the powers $q_j^{\gamma_j}$. By the first step again, we must have $s_d = d^{2g}$.

$\square$

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