GENERIC STRANGE DUALITY FOR $K3$ SURFACES

ALINA MARIAN AND DRAGOS OPREA

Abstract. Strange duality is shown to hold over generic $K3$ surfaces in a large number of cases. The isomorphism for elliptic $K3$ surfaces is obtained first via Fourier-Mukai techniques. Applications to Brill-Noether theory for sheaves on $K3$s are also obtained. The appendix written by K. Yoshioka discusses the behavior of the moduli spaces under change of polarization, as needed in the argument.

1. Introduction

1.1. **The strange duality morphism.** We consider moduli spaces of sheaves over $K3$ surfaces, and the strange duality map on spaces of generalized theta functions associated to them.

To start, we recall the general geometric setting for strange duality phenomena. Let (X, H) be a smooth polarized complex projective surface. To give our exposition a uniform character, we assume that X is simply connected. Let v be a class in the topological K-theory $K_{\text{top}}(X)$ of the surface, and denote by $\mathfrak{M}_v$ the moduli space of Gieseker H-semistable sheaves on X of topological type v.

The moduli space $\mathfrak{M}_v$ carries natural line bundles which we now discuss. Consider the bilinear form on $K_{\text{top}}(X)$ given by

$$(v, w) = \chi(v \cdot w), \text{ for } v, w \in K_{\text{top}}(X), \tag{1}$$

where the product in K-theory is used. Let

$$v^{\perp} \subset K_{\text{top}}(X)$$

consist of the K-classes orthogonal to v relative to this form. When $\mathfrak{M}_v$ consists of stable sheaves only[1], there is a group homomorphism

$$\Theta : v^{\perp} \to \text{Pic}\, \mathfrak{M}_v, \;\; w \mapsto \Theta_w,$$

studied among others in [LeP], [Li2]. If $\mathfrak{M}_v$ carries a universal sheaf

$$\mathcal{E} \to \mathfrak{M}_v \times X,$$

[1]The homomorphism is defined in all generality from a slightly more restricted domain.

and w is the class of a vector bundle F, we have

$$\Theta_w = \det \mathbf{R}p_!(\mathcal{E} \otimes q^\star F)^{-1}.$$

Here p and q are the two projection maps from $\mathfrak{M}_v \times X$. The theta line bundle is also defined in the absence of a universal sheaf, by descent from the Quot scheme.

We consider now two classes v and w in $K_{\text{top}}(X)$ satisfying

$$(v, w) = 0.$$

If the conditions

$$H^2(E \otimes F) = 0, \ \ \text{Tor}^1(E, F) = \text{Tor}^2(E, F) = 0$$

hold outside a locus in $\mathfrak{M}_v \times \mathfrak{M}_w$ of codimension greater than 1, the locus

$$\Theta = \{(E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w \text{ such that } H^0(E \otimes F) \neq 0\} \tag{2}$$

should correspond to a divisor. Furthermore,

$$\mathcal{O}(\Theta) = \Theta_w \boxtimes \Theta_v,$$

so Θ induces a map

$$\mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v).$$

The main query concerning this map is

Question 1. *When nonzero, is* D *an isomorphism?*

While the question is too naive for an affirmative answer to be expected in this generality, the isomorphism was shown to hold for many pairs $(\mathfrak{M}_v, \mathfrak{M}_w)$ of moduli spaces of sheaves over either $K3$ or rational surfaces, cf. [A] [D1] [D2] [G] [OG2] [S]. In all examples however, one of the moduli spaces involved has small dimension and the other consists of rank 2 sheaves. A survey of some of the known results is contained in [MO]. In this paper, we establish the isomorphism on moduli spaces over generic $K3$ surfaces for a large class of topological types of the sheaves, allowing in particular for arbitrarily high ranks and dimensions. The precise statements are as follows.

1.2. **Results.** Let (X, H) be a polarized $K3$ surface. We use as customary the Mukai vector

$$v(E) = \text{ch}E\sqrt{\text{Todd}\, X} \ \in \ H^\star(X, \mathbb{Z})$$

to express the topological type of a sheaf E on X. We write

$$v = v_0 + v_2 + v_4$$

to distinguish cohomological degrees in v, and set

$$v^\vee = v_0 - v_2 + v_4.$$

Note also the Mukai pairing on cohomology:

$$\langle v, w\rangle = \int_S v_2 w_2 - v_0 w_4 - v_4 w_0.$$

In terms of the pairing (1), we have

$$(v, w) = \langle v, w^\vee \rangle = -\langle v^\vee, w\rangle.$$

We assume that the moduli space $\mathfrak{M}_v$ of Gieseker H-semistable sheaves of fixed Mukai vector v consists only of stable sheaves. In this case, $\mathfrak{M}_v$ is an irreducible holomorphic symplectic manifold whose dimension is simply expressed in terms of the Mukai self-pairing of v,

$$\dim \mathfrak{M}_v = \langle v, v\rangle + 2.$$

We show

Theorem 1. *Assume (X, H) is a generic polarized $K3$ surface with $\operatorname{Pic} X = \mathbb{Z}\, H$, and consider orthogonal Mukai vectors v and w of ranks $r, s \geq 2$ such that*

(i) $c_1(v) = c_1(w) = H$,
(ii) $\chi(v) \leq 0$, $\chi(w) \leq 0$,
(iii) $\langle v, v\rangle \geq 2r(r^2 - r + 1)$, $\langle w, w\rangle \geq 2s(s^2 - s + 1)$.

Then

$$\mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism. In particular, the locus Θ has codimension 1 in $\mathfrak{M}_v \times \mathfrak{M}_w$.

The genericity means that the statement holds on a nonempty open subscheme of the moduli space of polarized $K3$s. We expect the result to be true for all $K3$s and we will pursue this aspect in future work.

In rank 2, the available numerics bounds are sharper, and we have, more strongly

Theorem 1A. *Assume (X, H) is a generic polarized $K3$ surface of degree at least 8, and consider orthogonal Mukai vectors v and w of rank 2 such that*

(i) $c_1(v) = c_1(w) = H$,
(ii) $\chi(v) \leq 0$, $\chi(w) \leq 0$.

Then

$$\mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

The statement is obtained by degeneration to moduli spaces over a smooth elliptic $K3$ surface X with a section. Better results are in fact available here. Let us assume that the fibers have at worst nodal singularities and that the Néron-Severi group is

$$\mathrm{NS}\,(X) = \mathbb{Z}\sigma + \mathbb{Z}f,$$

where σ and f are the classes of the section and of the fiber respectively. In particular,

$$\sigma^2 = -2,\ \ f^2 = 0,\ \ \sigma \cdot f = 1.$$

We show

Theorem 2. *Let v and w be orthogonal Mukai vectors corresponding to sheaves of ranks r and s on X with $r, s \geq 2$. Assume further that*

(i) $c_1(v) \cdot f = c_1(w) \cdot f = 1$,
(ii) $\langle v, v\rangle + \langle w, w\rangle > 2(r+s)^2$.

Then the duality map

$$\mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

Along the way we establish the following Brill-Noether result for sheaves on $K3$ elliptic surfaces.

Theorem 3. *Under the assumptions of Theorem 2, the locus Θ has codimension 1 in the product of moduli spaces $\mathfrak{M}_v \times \mathfrak{M}_w$. In particular, for a generic sheaf $E \in \mathfrak{M}_v$,*

$$\Theta_E = \{F \in \mathfrak{M}_w : h^0(E \otimes F) \neq 0\}$$

is a divisor in $\mathfrak{M}_w$.

The proofs use the fact that the moduli spaces $\mathfrak{M}_v$ and $\mathfrak{M}_w$ are birational to Hilbert schemes of points on X,

$$\mathfrak{M}_v \dashrightarrow X^{[a]},\ \ \mathfrak{M}_w \dashrightarrow X^{[b]},\ \text{ with } a = \frac{\langle v, v\rangle}{2} + 1,\ \ b = \frac{\langle w, w\rangle}{2} + 1. \tag{3}$$

The birational maps (3) were described by [OG] and shown to be regular in codimension 1. The main theorem is then a consequence of the explicit identification of the theta divisor (2) with a divisor in the product $X^{[a]} \times X^{[b]}$ known to induce strange duality. Specifically, for any line bundle L on X with $\chi(L) = a + b$ and no higher cohomology, one can consider the divisor associated to the locus

$$\theta_{L,a,b} = \{(I_Z, I_W) \text{ such that } h^0(I_Z \otimes I_W \otimes L) \neq 0\} \subset X^{[a]} \times X^{[b]}.$$

Furthermore, observe the involution on the elliptic surface X, given by fiberwise reflection across the origin of the fiber:

$$p \in f \mapsto -p \in f$$

The involution is defined away from the codimension 2 locus of singular points of fibers of X. It induces an involution on any Hilbert scheme of points on X, defined outside a codimension 2 locus,

$$\iota : X^{[a]} \dashrightarrow X^{[a]}, \;\; Z \mapsto \widetilde{Z}.$$

Consider the pullback

$$\tilde{\theta}_{L,a,b} = (\iota \times 1)^\star \theta_{L,a,b},$$

under the birational map

$$\iota \times 1 : \; X^{[a]} \times X^{[b]} \dashrightarrow X^{[a]} \times X^{[b]}, \;\; (Z, W) \mapsto (\widetilde{Z}, W).$$

It can be shown that

$$(\iota \times 1)^\star \theta_{L,a,b} = (1 \times \iota)^\star \theta_{L,a,b}.$$

Viewing Θ as a locus in $X^{[a]} \times X^{[b]}$, we prove

Theorem 4. *There exists a line bundle L on X, such that*

$$\Theta = \tilde{\theta}_{L,a,b} \;\; \textit{in the product} \;\; X^{[a]} \times X^{[b]}.$$

$\theta_{L,a,b}$ is known to give an isomorphism on the associated spaces of sections on $X^{[a]}$ and $X^{[b]}$, cf. [MO]. Therefore so does $\tilde{\theta}_{L,a,b}$, yielding Theorem 2.

The identification of the two theta divisors of Theorem 4 is not immediate even though the O'Grady birational isomorphism with the Hilbert scheme is explicit. To achieve it, we interpret the O'Grady construction by means of Fourier-Mukai transforms. The difficulty of this approach lies in the fact that the Fourier-Mukai transforms of *generic* O'Grady sheaves are actually two-step complexes in the derived category. More importantly, a careful analysis is necessary to keep track of the dimensions of the special loci where the generic description may fail.

The same method gives results for arbitrary simply connected elliptic surfaces

$$\pi : X \to \mathbb{P}^1$$

with a section and at worst nodal fibers. The dimension of the two complementary moduli spaces will be taken large enough compared to the constant

$$\Delta = \chi(X, \mathcal{O}_X) \cdot \big((r+s)^2 + (r+s) + 2\big) - 2(r+s).$$

We continue to assume that the polarization is suitable. We prove

Theorem 5. *Assume v and w are two orthogonal topological types of rank $r, s \geq 2$, such that*

(i) $c_1(v) \cdot f = c_1(w) \cdot f = 1$
(ii) $\dim \mathfrak{M}_v + \dim \mathfrak{M}_w > \Delta$.

Then, Θ is a divisor in $\mathfrak{M}_v \times \mathfrak{M}_w$.

We also propose

Conjecture 1. *Under the assumptions of Theorem 5,*

$$\mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

The conjecture is in fact established up to the statement that, in this new setting, the birational isomorphism of $\mathfrak{M}_v$ and $\mathfrak{M}_w$ with Hilbert schemes of points holds *in codimension 1.* We believe this to be true.

The paper is structured as follows. The main part of the argument concerns the case of elliptic $K3$ surfaces and is presented in Section 2. The last part of Section 2 treats the case of arbitrary simply connected elliptic surfaces. Section 3 explains generic strange duality via a deformation argument. The appendix written by Kota Yoshioka contains a discussion of change of polarization for higher-rank moduli spaces of sheaves over $K3$s.

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2. The theta isomorphism for elliptic $K3$ surfaces

2.1. **O'Grady's construction.** Keeping the notations of the introduction, we let $\pi : X \to \mathbb{P}^1$ be an elliptic $K3$ surface with a section, whose fibers have at worst nodal singularities.

We are concerned with sheaves on X with Mukai vector of type

$$v = r + (\sigma + kf) + p\,\omega,$$

for some $k, p \in \mathbb{Z}$, with ω being the class of a point in X. We consider a v-*suitable* polarization

$$H = \sigma + mf \text{ for } m >> 0.$$

This means that H lies in a v-chamber of the ample cone of X adjacent to the class f of the fiber [OG]. The moduli space $\mathfrak{M}_v$ of H-semistable sheaves consists of slope-stable sheaves only, and the choice of H ensures that $E \in \mathfrak{M}_v$ is stable if and only if its restriction to a generic fiber is stable. The restriction to special fibers may be unstable, as we will show in Lemma 1 below.

As explained in [OG], we can inductively build the moduli spaces $\mathfrak{M}_v$ as follows. Note first that tensoring with $\mathcal{O}(f)$ gives an isomorphism

$$\mathfrak{M}_v \cong \mathfrak{M}_{\tilde{v}}, \text{ where } \tilde{v} = r + (\sigma + (k+r)f) + (p+1)\omega.$$

Such a twist raises the Euler characteristic by 1. We normalize the moduli space by requiring that $p = 1 - r$; when it has dimension $2a$ we refer to it as $\mathfrak{M}_r^a$. Points in $\mathfrak{M}_r^a$ are rank r sheaves with Mukai vector

$$v_{r,a} = r + (\sigma + (a - r(r-1))f) + (1-r)\omega.$$

The normalization amounts to imposing that

$$\chi(E) = 1 \text{ for } E \in \mathfrak{M}_v.$$

In rank 1, note the isomorphism

$$X^{[a]} \cong \mathfrak{M}_1^a, \quad I_Z \mapsto I_Z(\sigma + af).$$

For any r, the generic point E_r of $\mathfrak{M}_r^a$ has exactly one section [OG],

$$h^0(E_r) = 1,$$

as expected since the Euler characteristic is 1. Moreover, $h^0(E_r(-f)) = 0$ generically, and

$$h^0(E_r(-2f)) = 0 \text{ for } E_r \text{ outside a codimension 2 locus in } \mathfrak{M}_r^a.$$

In addition, stability forces the vanishing $h^2(E_r(-2f)) = 0$ for all sheaves in $\mathfrak{M}_r^a$, so

$$h^1(E_r(-2f)) = -\chi(E_r(-2f)) = 1 \tag{4}$$

outside a codimension 2 locus in $\mathfrak{M}_r^a$. In [OG], an open subscheme $U_r^a \subset \mathfrak{M}_r^a$ is singled out, on which (4) holds. For sheaves E_r in U_r^a there is a unique nontrivial extension

$$0 \to \mathcal{O} \to E_{r+1} \to E_r \otimes \mathcal{O}(-2f) \to 0. \tag{5}$$

The resulting middle term E_{r+1} is torsion-free, with Mukai vector $v_{r+1,a}$, and is stable unless E_r belongs to a divisor D_r in U_r^a. In the latter case, a stabilization procedure is required to ensure that the resulting rank $r+1$ sheaf also belongs to $\mathfrak{M}_{r+1}^a$. The assignment

$$E_r \mapsto E_{r+1}$$

identifies open subschemes

$$U_r^a \cong U_{r+1}^a,$$

giving rise to a birational map

$$\phi_r : \mathfrak{M}_r^a \dashrightarrow \mathfrak{M}_{r+1}^a, \tag{6}$$

and therefore a birational morphism away from codimension 2,

$$\Phi_r : X^{[a]} \cong \mathfrak{M}_1^a \dashrightarrow \mathfrak{M}_r^a. \tag{7}$$

It will not be necessary for us to dwell on the details of the semistable reduction process along the D_rs although this, together with the identification of the D_rs themselves as divisors on the Hilbert scheme $X^{[a]}$, constitutes the most difficult part of [OG]. We record here however, for future use, that

$$D_1 = Q \cup S, \text{ and } D_r = S \text{ for } r \geq 2. \tag{8}$$

Here, Q is the divisor on $X^{[a]}$ consisting of ideals I_Z such that

$$h^0(I_Z((a-1)f)) \neq 0.$$

Equivalently, Q is the divisor of cycles on $X^{[a]}$ with at least two points contained in the same elliptic fiber of X. Furthermore, S is the divisor of cycles in $X^{[a]}$ which intersect the section σ of the elliptic fibration.

2.2. **O'Grady's moduli space via Fourier-Mukai.** We will reinterpret here the birational map

$$\Phi_r : X^{[a]} \dashrightarrow \mathfrak{M}_r^a$$

as a Fourier-Mukai transform. This will be crucial for the identification of the theta divisor and the proof of Theorem 4.

We let $Y \to \mathbb{P}^1$ denote the dual elliptic $K3$ surface *i.e.,* the relative moduli space of rank 1 degree 0 sheaves over the fibers of $\pi : X \to \mathbb{P}^1$. In fact, X and Y are isomorphic. Writing

$$\mathcal{P} \to X \times_{\mathbb{P}^1} Y$$

for the universal sheaf, we consider the Fourier-Mukai transform

$$\mathsf{S}_{X\to Y} : \mathbf{D}(X) \to \mathbf{D}(Y),$$

with kernel $\mathcal{P}$, given by

$$\mathsf{S}_{X\to Y}(x) = \mathbf{R}q_! \left(\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathbf{L}p^\star x \right). \tag{9}$$

Here p and q are the two projections. We will normalize $\mathcal{P}$ such that

$$\mathsf{S}_{X\to Y}(\mathcal{O}) = \mathcal{O}_\sigma[-1].$$

In fact, we have

$$c_1(\mathcal{P}) = \Delta - p^\star\sigma - q^\star\sigma$$

where Δ is the diagonal in $X \times_{\mathbb{P}^1} Y$. In a similar fashion, we set

$$\mathcal{Q} = \mathcal{P}^\vee[1],$$

and use this as the kernel of the transform

$$\mathsf{T}_{Y\to X} : \mathbf{D}(Y) \to \mathbf{D}(X)$$

It was shown in [Br] that the functors $\mathsf{S}_{X\to Y}$ and $\mathsf{T}_{Y\to X}$ are equivalences of categories and that

$$\mathsf{S} \circ \mathsf{T} = \mathbf{1}_{\mathbf{D}(Y)}[-1], \quad \mathsf{T} \circ \mathsf{S} = \mathbf{1}_{\mathbf{D}(X)}[-1]. \tag{10}$$

Fix a cycle $Z \in X^{[a]}$, and let E_r denote the sheaf in $\mathfrak{M}_r^a$ corresponding to Z under the O'Grady isomorphism Φ_r. We will consider *generic* subschemes Z, in the sense that

(i) Z consists of distinct points,
(ii) no two points of Z lie in the same fiber,
(iii) Z is disjoint from the section,
(iv) Z does not contain any singular points of the fibers.

These requirements allow us to assume E_r is locally free. When $r \geq 3$ this holds on general grounds away from codimension 2. For $r = 2$, we will explain in Lemma 2 below that (ii) implies E_2 is locally free.

We determine the images of E_r and of its dual $E_r^\vee$ under the functor (9). The answer is simpler for the dual, which is in fact WIT_1 relative to $\mathsf{S}_{X\to Y}$. We show

Proposition 1. *For generic Z, we have*

$$\mathsf{S}_{X\to Y}(E_r^\vee) = I_Z(r\sigma + 2rf)[-1].$$

Furthermore,

$$\mathsf{S}_{X\to Y}(E_r^\vee(nf)) = \mathsf{S}_{X\to Y}(E_r^\vee) \otimes \mathcal{O}(nf).$$

To express the Fourier-Mukai transform of E_r, let

$$\widetilde{Z} = \iota^\star Z$$

be the cycle obtained by taking the inverses of all points in Z in the group law of their corresponding fibers. This makes sense even for singular fibers using the group law of the regular locus. As already remarked, O'Grady's construction gives rise to a rank 2 vector bundle $\widetilde{E_2}$ together with an exact sequence

$$0 \to \mathcal{O} \to \widetilde{E_2} \to I_{\widetilde{Z}}(\sigma + (a-2)f) \to 0.$$

The complex

$$\mathsf{C}_Z = \left[\widetilde{E_2} \to \mathcal{O}(\sigma + (a-2)f)\right]$$

is naturally constructed over Y using the identification $Y \cong X$.

Proposition 2. *For $r \geq 1$, we have*

$$\mathsf{S}_{X \to Y}(E_r) = \mathsf{C}_Z \otimes \mathcal{O}(-r\sigma - 2(r-1)f).$$

The rest of this section is devoted to the proofs of Propositions 1 and 2. We study first how the generic bundle E_r restricts to the fibers. Consider a fiber f of $\pi : X \to \mathbb{P}^1$ with origin $o = \sigma \cap f$, and let

$$\mathsf{W}_r \to f$$

be the unique rank r stable bundle on f with determinant $\mathcal{O}_f(o)$. The W_r's were constructed by Atiyah over smooth elliptic curves. His arguments extend verbatim to nodal genus 1 curves: we define W_r inductively as the unique nontrivial extension

$$0 \to \mathcal{O} \to \mathsf{W}_{r+1} \to \mathsf{W}_r \to 0, \quad \mathsf{W}_1 = \mathcal{O}_f(o). \tag{11}$$

Similarly, if p is any smooth point of the fiber f, we write

$$\mathsf{W}_{r,\,p} \to f$$

for the Atiyah bundle of determinant $\mathcal{O}_f(p)$ such that

$$0 \to \mathcal{O} \to \mathsf{W}_{r+1,\,p} \to \mathsf{W}_{r,\,p} \to 0. \tag{12}$$

The convention

$$\mathsf{W}_{0,\,p} = \mathcal{O}_p$$

is used throughout. We have the following

Lemma 1. *(i) If f is a fiber such that $Z \cap f = \emptyset$, then*

$$E_r|_f = \mathsf{W}_r.$$

(ii) If f is a fiber through $p \in Z$, then

$$E_r|_f = \mathsf{W}_{r-1,\,p} \oplus \mathcal{O}_f(o-p).$$

Proof. This is seen by induction starting with the case $r = 1$ when

$$E_1 = I_Z(\sigma + af).$$

The basic observation is that for $p \in X$ and I_p denoting its ideal sheaf in X, the restriction to the fiber f through p is

$$I_p|_f = \mathcal{O}_p \oplus \mathcal{O}_f(-p).$$

This gives the statement for E_1. The inductive step from r to $r+1$ follows from the exact sequence

$$0 \to \mathcal{O} \to E_{r+1} \to E_r(-2f) \to 0.$$

Its restriction to any fiber never splits as explained by Lemma $I.4.7$ [OG]. The restriction to a fiber avoiding Z must therefore coincide with the Atiyah bundle W_{r+1}, since the latter is the only nontrivial extension

$$0 \to \mathcal{O} \to \mathsf{W}_{r+1} \to \mathsf{W}_r \to 0.$$

The same argument holds for the fibers through points of Z, using that there is a unique extension

$$0 \to \mathcal{O} \to \mathsf{W}_{r,\,p} \oplus \mathcal{O}_f(o-p) \to \mathsf{W}_{r-1,\,p} \oplus \mathcal{O}_f(o-p) \to 0.$$

□

Letting f be a smooth elliptic fiber, we record now the Fourier-Mukai transforms of the Atiyah bundles relative to the standard Poincaré kernel on $f \times f$. We use hatted notation for the transforms, and as before we let

$$\iota : f \to f$$

denote reflection about the origin of f. We have

$$\widehat{\mathsf{W}_r} = \mathcal{O}_f(-r \cdot o), \tag{13}$$

$$\widehat{\mathsf{W}_{r,\,p}} = \mathcal{O}_f(-(r+1)\cdot o + \iota^\star p). \tag{14}$$

By [Muk], these two imply

$$\widehat{\mathsf{W}_r^\vee} = \mathcal{O}_f(r \cdot o)[-1], \tag{15}$$

$$\widehat{\mathsf{W}_{r,\,p}^\vee} = \mathcal{O}_f((r+1)\cdot o - p)[-1]. \tag{16}$$

The first transform (13) is obtained inductively by applying the Fourier-Mukai functor to the defining sequence (11). The base case $r = 1$ is obvious. Similarly (14) can be derived using sequence (12). An alternate argument starts by noticing

$$\mathsf{W}_{r,\,p} = \mathsf{W}_r \otimes M$$

where

$$M^r = \mathcal{O}_f(p - o) = \mathcal{O}_f(o - \iota^\star p).$$

Pick m such that the line bundle M corresponds to the point $m \in f$. Then $rm = p$ holds in the group law of the fiber. Using the properties of the Fourier-Mukai transform [Muk], we obtain

$$\widehat{\mathsf{W}_{r,p}} = t_m^\star \widehat{\mathsf{W}}_r = t_m^\star \mathcal{O}_f(-r \cdot o) = \mathcal{O}_f(-r \cdot m) = \mathcal{O}_f(-(r+1) \cdot o + \iota^\star p).$$

Equations (13), (14), (15), and (16) also hold for the singular fibers; this is explained by Lemma 2.13, Definition 2.15, and Remark 2.17 in [BK]. Note that the transforms in [BK] are stated for the functor $\mathsf{T}_{Y \to X}$, but the results for the functor $\mathsf{S}_{X \to Y}$ follow via (10).

Proof of Proposition 1. We will first check that the isomorphism

$$\mathsf{S}_{X \to Y}(E_r^\vee)[1] = I_Z(r\sigma + 2rf)$$

holds fiberwise. Derived restriction to fibers commutes with Fourier-Mukai [Br], and Lemma 1 gives the restriction of $E_r^\vee$ to each fiber. The Fourier-Mukai transform of the restriction to a general fiber is

$$\mathcal{O}_f(r \cdot o)[-1],$$

by (15). For a special fiber f containing a point $p \in Z$, equation (16) yields the transform

$$\mathcal{O}_f(r \cdot o - p)[-1] \oplus \mathcal{O}_p[-1].$$

The two formulas above give precisely the derived restriction of $I_Z(r\sigma + 2rf)[-1]$. We have therefore checked that the proposition holds on every fiber.

Since both sides are sheaves of rank 1, we complete the proof by checking equality of determinants. As X is simply connected, it is enough to match the first Chern classes. In general, let V be a rank r sheaf of Euler characteristic χ and

$$c_1(V) = l\sigma + mf.$$

Then, by Grothendieck-Riemann-Roch, we have

$$c_1(\mathsf{S}_{X \to Y}(V)) = q_! \left(p^\star \mathrm{ch}(V) \cdot \mathrm{Todd}(X \times_{\mathbb{P}^1} Y/Y) \cdot \mathrm{ch}\mathcal{P}\right)_{(2)}.$$

The Chern character of V is

$$\mathrm{ch}(V) = r + (l\sigma + mf) + (\chi - 2r)\omega,$$

where ω is the class of a point. Moreover,

$$\mathrm{Todd}(X \times_{\mathbb{P}^1} Y/Y) = p^\star(1 - f + 2\omega).$$

Hence

$$\begin{aligned} c_1(\mathsf{S}_{X\to Y}(V)) &= rc_1(\mathsf{S}_{X\to Y}(\mathcal{O})) + (\chi - 2r - l)q_!(p^\star\omega) + q_!\left(p^\star(l\sigma + mf)c_1(\mathcal{P})\right) \\ &= -r\sigma + (\chi - 2r - l)f + 2lf \\ &= -r\sigma + (\chi - 2r + l)f. \end{aligned}$$

For $V = E_r^\vee$ the Chern class calculation gives

$$c_1(\mathsf{S}_{X\to Y}(E_r^\vee)) = -r\sigma - 2rf,$$

which proves the first isomorphism. The calculation also shows that

$$c_1(\mathsf{S}_{X\to Y}(E_r^\vee(f)) = c_1(\mathsf{S}_{X\to Y}(E_r^\vee)) - f.$$

The claim about twisting by fibers follows by repeating the above argument for $E_r^\vee(nf)$ and comparing determinants. □

Proof of Proposition 2. Via (10), it suffices to check that

$$\mathsf{T}_{Y\to X}(\mathsf{C}_Z \otimes \mathcal{O}(-r\sigma - 2(r-1)f)) = E_r[-1].$$

We will first prove the case $r = 1$,

$$\mathsf{T}_{Y\to X}(\mathsf{C}_Z \otimes \mathcal{O}(-\sigma)) = \mathsf{T}_{Y\to X}\left(\left[\widetilde{E}_2(-\sigma) \to \mathcal{O}((a-2)f)\right]\right) = I_Z(\sigma + af)[-1].$$

We check that the equality holds fiberwise. Over the general fibers, the derived restriction of $\mathsf{C}_Z \otimes \mathcal{O}(-\sigma)$ becomes

$$[\mathsf{W}_2(-o) \to \mathcal{O}_f] \cong \mathcal{O}_f(-o)$$

whose Fourier-Mukai agrees with the right hand side. For the special fibers, passing through points $p \in Z$, set

$$q = \iota^\star p.$$

In this case, the restriction equals

$$[\mathcal{O}_f(-q) \oplus \mathcal{O}_f(q-o) \to \mathcal{O}_f]\,.$$

Applying the Fourier-Mukai functor T on the fiber we obtain

$$(\mathcal{O}_f(o-p) + \mathcal{O}_p)\,[-1].$$

This agrees with the derived restriction of the right hand side. Since both sides have rank 1, just as before, it suffices to compute the first Chern class

$$c_1(\mathsf{T}_{Y\to X}(\mathsf{C}_Z \otimes \mathcal{O}(-\sigma))) = \sigma + af$$

via Grothendieck-Riemann-Roch to conclude equality.

The arbitrary rank case follows by induction. Using the exact triangle

$$\mathcal{O}_\sigma[-1] \to \mathsf{C}_Z \otimes \mathcal{O}(-(r+1)\sigma - 2rf) \to \mathsf{C}_Z \otimes \mathcal{O}(-r\sigma - 2rf) \to \mathcal{O}_\sigma$$

we obtain

$$\mathcal{O}[-1] \to \mathsf{T}_{Y\to X}(\mathsf{C}_Z \otimes \mathcal{O}(-(r+1)\sigma - 2rf) \to E_r(-2f)[-1] \to \mathcal{O}.$$

This implies the inductive claim by the fact the E_{r+1} is the unique nontrivial extension

$$0 \to \mathcal{O} \to E_{r+1} \to E_r(-2f) \to 0.$$

□

2.3. **Line bundles and theta divisors over the Hilbert scheme of points.** The birational isomorphism (7) allows us to identify the Picard group of $\mathfrak{M}_r^a$ with that of the Hilbert scheme $X^{[a]}$.

For any smooth projective surface X and any line bundle L on it, we indicate by $L_{(a)}$ the line bundle on $X^{[a]}$ induced from the symmetric line bundle $L^{\boxtimes a}$ on the product $X \times \ldots \times X$. Letting p and q be the projections

$$p : X^{[a]} \times X \to X^{[a]}, \;\; q : X^{[a]} \times X \to X,$$

and letting $\mathcal{O}_{\mathcal{Z}}$ denote the universal structure sheaf on $X^{[a]} \times X$, we further set

$$L^{[a]} = \det\, p_!(\mathcal{O}_{\mathcal{Z}} \otimes q^\star L). \tag{17}$$

It is well known that the line bundles $L_{(a)}$ for $L \in \operatorname{Pic} X$ and $M = \mathcal{O}^{[a]}$ generate the Picard group of $X^{[a]}$, and that for any L on X,

$$L^{[a]} = L_{(a)} \otimes M.$$

We have, for instance,

$$\mathcal{O}(S) = \mathcal{O}(\sigma)_{(a)},$$

and

$$\mathcal{O}(Q) = \mathcal{O}((a-1)f)^{[a]}.$$

We note for future use the formulas of [EGL],

$$h^0(X^{[a]}, L_{(a)}) = \binom{h^0(X,L)+a-1}{a}, \;\; h^0(X^{[a]}, L^{[a]}) = \binom{h^0(X,L)}{a}. \tag{18}$$

Consider now two Hilbert schemes of points $X^{[a]}$ and $X^{[b]}$, and the rational morphism, defined away from codimension 2,

$$\tau : X^{[a]} \times X^{[b]} \dashrightarrow X^{[a+b]}, \quad (I_Z, I_W) \mapsto I_Z \otimes I_W. \tag{19}$$

Assume that L is a line bundle on X with no higher cohomology, and such that

$$\chi(L) = h^0(L) = a+b.$$

From (18), we note that

$$h^0(X^{[a+b]}, L^{[a+b]}) = \binom{h^0(X, L)}{a+b} = 1.$$

The unique section of $L^{[a+b]}$ vanishes on the locus

$$\theta_L = \{I_V \in X^{[a+b]}, \text{ such that } H^0(I_V \otimes L) \neq 0\}, \tag{20}$$

whose pullback under τ is the divisor

$$\theta_{L,a,b} = \{(I_Z, I_W) \in X^{[a]} \times X^{[b]} \text{ such that } H^0(I_Z \otimes I_W \otimes L) \neq 0\}.$$

We furthermore have

$$\mathcal{O}(\theta_{L,a,b}) = \tau^\star L^{[a+b]} = L^{[a]} \boxtimes L^{[b]} \text{ on } X^{[a]} \times X^{[b]}. \tag{21}$$

It was observed in [MO] that $\theta_{L,a,b}$ induces an isomorphism

$$\mathsf{D} : H^0(X^{[a]}, L^{[a]})^\vee \to H^0(X^{[b]}, L^{[b]}). \tag{22}$$

It will be important for our arguments to consider the following partial reflection of the divisor $\theta_{L,a,b}$:

$$\widetilde{\theta}_{L,a,b} = \{(Z, W) \in X^{[a]} \times X^{[b]} \text{ such that } h^0(I_Z \otimes I_{\widetilde{W}} \otimes L) \neq 0\}.$$

As usual, the subschemes

$$\widetilde{Z} = \iota^\star Z, \ \widetilde{W} = \iota^\star W$$

are obtained from the involutions

$$\iota : X^{[a]} \dashrightarrow X^{[a]}, \ \iota : X^{[b]} \dashrightarrow X^{[b]}$$

induced by the fiberwise reflection $\iota : X \dashrightarrow X$. There is a seeming asymmetry in the roles of Z and W in the definition of $\widetilde{\theta}_{L,a,b}$, but in fact we also have

$$\widetilde{\theta}_{L,a,b} = \{(Z, W) \in X^{[a]} \times X^{[b]} \text{ such that } h^0(I_{\widetilde{Z}} \otimes I_W \otimes L) \neq 0\}.$$

To explain this equality, note first that the line bundle L is invariant under ι

$$\iota^\star L = L.$$

Hence, so are the tautological line bundles $L^{[a]}$, $L^{[b]}$ and $L^{[a+b]}$. On $X^{[a+b]}$, the divisor θ_L of (20) corresponds to the unique section of $L^{[a+b]}$, therefore must be invariant under ι as well,

$$\iota^\star \theta_L = \theta_L.$$

The same is then true for the pullback

$$\theta_{L,a,b} = \tau^\star \theta_L,$$

which implies that

$$h^0(I_Z \otimes I_{\widetilde{W}} \otimes L) = 0 \text{ if and only if } h^0(I_{\widetilde{Z}} \otimes I_W \otimes L) = 0.$$

The above discussion also shows that $\tilde{\theta}_{L,a,b}$ is a section of the line bundle $L^{[a]} \boxtimes L^{[b]}$ and that furthermore it induces an isomophism

$$\widetilde{\mathsf{D}} : H^0(X^{[a]}, L^{[a]})^\vee \to H^0(X^{[b]}, L^{[b]}). \tag{23}$$

2.4. **The strange duality setup and the standard theta divisor.** We now place ourselves in the setting of Theorem 2 *i.e.,* we take X to be an elliptically fibered $K3$ surface with section, and consider two moduli spaces of sheaves $\mathfrak{M}_v$ and $\mathfrak{M}_w$ with Mukai vectors satisfying

(i) $\langle v, w^\vee \rangle = 0$,
(ii) $c_1(v) \cdot f = c_1(w) \cdot f = 1$,
(iii) $\langle v, v \rangle + \langle w, w \rangle > 2(r+s)^2$.

Equivalently, we consider two normalized moduli spaces $\mathfrak{M}_r^a$ and $\mathfrak{M}_s^b$ such that

$$r + s \,|\, a + b - 2, \text{ and moreover } -\nu =_{\text{def}} \frac{a+b-2}{r+s} - (r+s-2) > 2. \tag{24}$$

We also assume that $r, s \geq 2$. The divisibility condition and the definition of ν are so as to ensure that

$$\chi(E_r \cdot F_s(\nu f)) = 0, \text{ for sheaves } E_r \in \mathfrak{M}_r^a, F_s \in \mathfrak{M}_s^b.$$

Furthermore, the stability condition implies that

$$H^2(E_r \otimes F_s(\nu f)) = 0.$$

The vanishing

$$\text{Tor}^1(E_r, F_s) = \text{Tor}^2(E_r, F_s) = 0$$

is satisfied when E_r or F_s are locally free, which occurs away from codimension 2 in the product space.

We denote by $\Theta_{r,s}$ the locus

$$\{(E_r, F_s) \in \mathfrak{M}_r^a \times \mathfrak{M}_s^b \text{ such that } h^0(E_r \otimes F_s(\nu f)) \neq 0\}.$$

The condition defining $\Theta_{r,s}$ is divisorial, but it is not a priori clear that this locus actually has codimension 1. Nonetheless, using the explicit formulas of [OG], the *line bundle* $\mathcal{O}(\Theta_{r,s})$ on $\mathfrak{M}_r^a \times \mathfrak{M}_s^b$ can be expressed on the product $X^{[a]} \times X^{[b]}$ via the birational map

$$(\Phi_r, \Phi_s) : X^{[a]} \times X^{[b]} \dashrightarrow \mathfrak{M}_r^a \times \mathfrak{M}_s^b.$$

Letting

$$L = \mathcal{O}((r+s)\sigma + (2(r+s) - 2 - \nu)f) \text{ on } X, \tag{25}$$

it was shown in [MO] that

$$\mathcal{O}(\Theta_{r,s}) = L^{[a]} \boxtimes L^{[b]}. \tag{26}$$

We prove that

Theorem 4. *$\Theta_{r,s} = \tilde{\theta}_{L,a,b}$ on $X^{[a]} \times X^{[b]}$.*

2.5. **The theta divisor over the generic locus.** We first identify the theta divisor $\Theta_{r,s}$ on the locus corresponding to generic Z and W. Our genericity assumptions were specified in (i)-(iv) of Section 2.2.

On any Hilbert scheme of points of X, we consider then the following:

(i) the divisor R consisting of cycles with at least two coincident points,
(ii) the divisor Q of cycles with two points on the same fiber,
(iii) the divisor S of cycles which intersect the section.

Recall that along the divisors S and Q the extensions (5) have unstable middle terms needing to undergo semi-stable reduction.

We single out here only the nongeneric loci corresponding to divisors, as we can ignore higher codimension phenomena for the purpose of Theorem 2. Thus we will disregard the loci corresponding to

(iv) schemes whose supports contain singular points of fibers of X

We work with the rational morphism

$$\tau : X^{[a]} \times X^{[b]} \dashrightarrow X^{[a+b]},$$

and we will pullback the divisors R, S and Q to the product of Hilbert schemes and of moduli spaces $\mathfrak{M}_r^a$ and $\mathfrak{M}_s^b$. We set

$$\mathfrak{M} = \mathfrak{M}_r^a \times \mathfrak{M}_s^b \setminus (\tau^\star R \cup \tau^\star Q \cup \tau^\star S).$$

For $(E_r, F_s) \in \mathfrak{M}$, we explained in section 2.2 that we may assume E_r and F_s are both locally free.

Using E_r is locally free, we will show

$$H^0(E_r \otimes F_s(\nu f) = 0 \text{ if and only if } H^1(I_Z \otimes I_{\widetilde{W}} \otimes L) = 0. \tag{27}$$

The assumption that F_s is locally free would yield in turn

$$H^0(E_r \otimes F_s(\nu f) = 0 \text{ if and only if } H^1(I_{\widetilde{Z}} \otimes I_W \otimes L) = 0. \tag{28}$$

Both conditions correspond to the divisor $\tilde{\theta}_{L,a,b}$. In other words, we show

$$\Theta_{r,s} \setminus (\tau^\star R \cup \tau^\star Q \cup \tau^\star S) = \tilde{\theta}_{L,a,b} \setminus (\tau^\star R \cup \tau^\star Q \cup \tau^\star S). \tag{29}$$

To establish (27), we use the Fourier-Mukai functor $\mathsf{S}_{X\to Y}$ defined in (9), as well as Propositions 1 and 2. We calculate

$$\begin{aligned} H^i(E_r \otimes F_s(\nu f)) &= \operatorname{Ext}^i(E_r^\vee(-\nu f), F_s) = \operatorname{Hom}_{\mathbf{D}(X)}(E_r^\vee(-\nu f), F_s[i]) \\ &= \operatorname{Hom}_{\mathbf{D}(Y)}\left(\mathsf{S}_{X\to Y}(E_r^\vee(-\nu f)), \mathsf{S}_{X\to Y}(F_s[i])\right) \\ &= \operatorname{Hom}_{\mathbf{D}(Y)}\left(I_Z(r\sigma + 2rf - \nu f), \mathsf{C}_W \otimes \mathcal{O}(-s\sigma - 2(s-1)f)[i+1]\right) \\ &= \operatorname{Ext}^{i+1}(I_Z \otimes L, \mathsf{C}_W) \\ &= \operatorname{Ext}^{1-i}(\mathsf{C}_W, I_Z \otimes L)^\vee. \end{aligned}$$

Using the triangle

$$\mathsf{C}_W \to \widetilde{F_2} \to \mathcal{O}(\sigma + (b-2)f) \to \mathsf{C}_W[1],$$

we obtain

$$\begin{array}{ccccc} \operatorname{Ext}^1(\mathcal{O}(\sigma+(b-2)f), I_Z\otimes L) & \longrightarrow & \operatorname{Ext}^1(\widetilde{F_2}, I_Z \otimes L) & \longrightarrow & \operatorname{Ext}^1(\mathsf{C}_W, I_Z\otimes L) \to 0 \\ \downarrow \cong & & \downarrow \cong & \nearrow & \\ H^1(I_Z \otimes L \otimes \mathcal{O}(-\sigma - (b-2)f)) & \xrightarrow{\alpha} & H^1(I_Z \otimes L \otimes \widetilde{F_2}^\vee) & & \end{array}$$

The above identification uses the fact that $\widetilde{F_2}$ is locally free which is true away from the divisor Q by Lemma 2. The surjectivity of the last map follows from the vanishing of

$$\begin{aligned} \operatorname{Ext}^2(\mathcal{O}(\sigma+(b-2)f), I_Z \otimes L) &= H^2(I_Z \otimes L \otimes \mathcal{O}(-\sigma-(b-2)f)) \\ &= H^2(L(-\sigma-(b-2)f)) \\ &= H^0(L^\vee(\sigma+(b-2)f))^\vee = 0. \end{aligned}$$

Clearly we have

$$H^0(E_r \otimes F_s(\nu f)) = 0 \iff \operatorname{Ext}^1(\mathsf{C}_W, I_Z \otimes L) = 0 \iff \alpha \text{ is surjective.}$$

Dualizing the sequence

$$0 \to \mathcal{O} \to \widetilde{F_2} \to I_{\widetilde{W}} \otimes \mathcal{O}(\sigma + (b-2)f) \to 0,$$

we obtain

$$0 \to \mathcal{O}(-\sigma-(b-2)f) \to \widetilde{F_2}^\vee \to \mathcal{O} \to \mathcal{E}xt^1(I_{\widetilde{W}} \otimes O(\sigma+(b-2)f), \mathcal{O}) \to 0.$$

It is well-known, see [F] page 41, that

$$\mathcal{E}xt^1(I_{\widetilde{W}} \otimes O(\sigma + (b-2)f), \mathcal{O}) = \mathcal{O}_{\widetilde{W}}$$

hence the exact sequence yields

$$0 \to \mathcal{O}(-\sigma - (b-2)f) \to \widetilde{F_2}^{\vee} \to I_{\widetilde{W}} \to 0.$$

Twisting by $L \otimes I_Z$ and taking cohomology, we see that the that α is surjective iff

$$H^1(I_Z \otimes I_{\widetilde{W}} \otimes L) = 0,$$

which is what (27) asserts. □

Lemma 2. *If W contains no two points in the same fiber, then the associated sheaf $\widetilde{F}_2$ is locally free.*

Proof. Consider the divisor Q of subschemes in $X^{[b]}$ containing two points in the same fiber. Let

$$\mathcal{D} \hookrightarrow \mathfrak{M}_2^b$$

be the codimension 1 locus of nonlocally free sheaves in the rank 2 moduli space. Lemma 4.41 of [Y] calculates

$$\mathcal{O}(\mathcal{D}) = \Theta_w \text{ on } \mathfrak{M}_2^b,$$

for the Mukai vector

$$w = (2, -\sigma - (b-2)f, (b-2)\omega).$$

Using now the formulas in [OG], page 27, we have

$$\Theta_w = \mathcal{O}(Q) = \mathcal{O}((b-1)f)^{[b]},$$

under the identification

$$X^{[b]} \dashrightarrow \mathfrak{M}_2^b.$$

Finally, (18) implies that

$$h^0(X^{[b]}, \mathcal{O}((b-1)f)^{[b]}) = \binom{h^0(\mathcal{O}((b-1)f))}{b} = 1$$

hence $\mathcal{D}$ and Q coincide as claimed. □

2.6. **The nongeneric locus.** In order to complete the proof of Theorem 4, we will need to take into account the overlaps of the theta divisor with R, Q and S.

First, as the divisors R, Q, S are invariant under ι, we write (29) equivalently as

$$\widetilde{\Theta}_{r,s} \setminus (\tau^\star R \cup \tau^\star Q \cup \tau^\star S) = \theta_{L,a,b} \setminus (\tau^\star R \cup \tau^\star Q \cup \tau^\star S), \tag{30}$$

where $\widetilde{\Theta}_{r,s}$ is the partial reflection of the divisor $\Theta_{r,s}$ obtained by acting with the involution ι on one of the factors.

We write

$$\widetilde{\Theta}_{r,s} = \Gamma \cup \Delta,$$

where Γ and Δ are divisors such that the intersection

$$\Delta \cap (\tau^\star Q \cup \tau^\star R \cup \tau^\star S)$$

is proper, and

$$\text{support } \Gamma \subset \tau^\star Q \cup \tau^\star R \cup \tau^\star S$$

Equation (30) shows in particular that Δ is a pullback divisor under τ,

$$\Delta = \tau^\star \Delta_0 \text{ for } \Delta_0 \subset X^{[a+b]}.$$

Since

$$\mathcal{O}(\widetilde{\Theta}_{r,s}) = L^{[a]} \boxtimes L^{[b]} = \tau^\star L^{[a+b]},$$

we have

$$\mathcal{O}(\Gamma) = \tau^\star (L^{[a+b]} \otimes \mathcal{O}(-\Delta_0)). \tag{31}$$

More strongly, we will show shortly that (31) implies that

Claim 1. Γ *as a divisor is a pullback under the morphism* τ,

$$\Gamma = \tau^\star \Gamma_0.$$

As a consequence,

$$\widetilde{\Theta}_{r,s} = \tau^\star (\Delta_0 \cup \Gamma_0).$$

Now

$$\theta_L = \{V \text{ such that } h^0(I_V \otimes L) \neq 0\}$$

is the only section of $L^{[a+b]}$ on $X^{[a+b]}$. Thus we must have that

$$\theta_L = \Delta_0 \cup \Gamma_0, \text{ and}$$

$$\widetilde{\Theta}_{r,s} = \tau^\star \theta_L = \{(I_Z, I_W) \text{ such that } h^0(I_Z \otimes I_W \otimes L) \neq 0\} = \theta_{L,a,b}.$$

This completes the proof of Theorem 4. Theorem 2 now follows via (23).

Proof of Claim 1. We will consider the three divisors Q, R and S over the the Hilbert schemes $X^{[a]}$, $X^{[b]}$ or $X^{[a+b]}$. All these divisors are irreducible. Let

$$\tau^\star Q = Q_1 \cup Q_2 \cup Q_3, \quad \tau^\star R = R_1 \cup R_2,$$

$$\tau^\star S = S_1 \cup S_2$$

be the irreducible components of the pullbacks on the product $X^{[a]} \times X^{[b]}$. Here

$$Q_1 = Q \times X^{[b]}, \quad Q_2 = X^{[a]} \times Q,$$

while Q_3 is the divisor of cycles $(I_Z, I_W) \in X^{[a]} \times X^{[b]}$ such that Z, W intersect the same elliptic fiber. In the same fashion, we have

$$R_1 = R \times X^{[b]}, R_2 = X^{[a]} \times R,$$

$$S_1 = S \times X^{[b]}, S_2 = X^{[a]} \times S.$$

Note first that $\widetilde{\Theta}_{r,s}$ does not contain the divisor Q_3. Indeed,

$$\mathcal{O}(Q) = \mathcal{O}((a+b-1)f)^{[a+b]} \text{ on } X^{[a+b]},$$

so

$$\tau^\star \mathcal{O}(Q) = \mathcal{O}((a+b-1)f)^{[a]} \boxtimes \mathcal{O}((a+b-1)f)^{[b]} \text{ on } X^{[a]} \times X^{[b]}.$$

As

$$\mathcal{O}(Q_1) = \mathcal{O}((a-1)f)^{[a]} \boxtimes \mathcal{O} \quad \text{and} \quad \mathcal{O}(Q_2) = \mathcal{O} \boxtimes \mathcal{O}((b-1)f)^{[b]},$$

it follows that

$$\mathcal{O}(Q_3) = \mathcal{O}(bf)_{(a)} \boxtimes \mathcal{O}(af)_{(b)}.$$

Assuming $\widetilde{\Theta}_{r,s}$ contained Q_3, we would have

$$H^0(X^{[a]} \times X^{[b]}, \mathcal{O}(\widetilde{\Theta}_{r,s} - Q_3)) \neq 0. \tag{32}$$

However, we will show that (32) is false. Indeed,

$$\mathcal{O}(\widetilde{\Theta}_{r,s} - Q_3) = L(-bf)^{[a]} \boxtimes L(-af)^{[b]}.$$

From (18), we have

$$h^0\left(L(-bf)^{[a]}\right) = \binom{h^0(L(-bf))}{a},$$

$$h^0\left(L(-af)^{[b]}\right) = \binom{h^0(L(-af))}{b}.$$

It suffices to explain that either

$$h^0(L(-bf)) = 0 \text{ or } h^0(L(-af)) = 0. \tag{33}$$

On the surface X, we generally have

$$h^0(X, \mathcal{O}(m\sigma + nf)) = \begin{cases} 0, \text{ if } m \geq 0, \ n < 0 \\ 2 + m(n-m), \text{ if } m > 0, \ n \geq 2m \end{cases}. \tag{34}$$

The first dimension count is immediate as $h^0(X, \mathcal{O}(m\sigma)) = 1$ for all $m \geq 0$, and the second holds as in that case $\mathcal{O}(m\sigma + nf)$ is big and nef, so has no higher cohomology. Now, recall that

$$L = \mathcal{O}\left((r+s)\sigma + \left(r + s + \frac{a+b-2}{r+s}\right) f\right) \text{ on } X.$$

The numerical constraint (24)

$$a + b > (r+s)^2 + 2$$

ensures that either $L(-af)$ or $L(-bf)$ has a negative number of fiber classes. This proves (33) using the dimension count (34).

Similarly, $\widetilde{\Theta}_{r,s}$ cannot contain both Q_1 and Q_2. Indeed, we calculate

$$\mathcal{O}(\widetilde{\Theta}_{r,s} - Q_1 - Q_2) = L((-a+1)f)_{(a)} \boxtimes L((-b+1)f)_{(b)}.$$

As before,

$$h^0(L((-a+1)f) = 0 \text{ or } h^0(L((-b+1)f) = 0.$$

Therefore, (18) implies that $\mathcal{O}(\widetilde{\Theta}_{r,s} - Q_1 - Q_2)$ has no sections.

Let us write

$$\Gamma = q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + r_1 R_1 + r_2 R_2 + s_1 S_1 + s_2 S_2.$$

The above argument shows that $q_3 = 0$ and that we can assume without loss of generality $q_2 = 0$. We calculate

$$\mathcal{O}(Q_1) = \mathcal{O}((a-1)f)^{[a]} \boxtimes \mathcal{O},$$

$$\mathcal{O}(R_1) = M^{-2} \boxtimes \mathcal{O}, \ \ \mathcal{O}(R_2) = \mathcal{O} \boxtimes M^{-2},$$

$$\mathcal{O}(S_1) = \mathcal{O}(\sigma)_{(a)} \boxtimes \mathcal{O}, \ \ \mathcal{O}(S_2) = \mathcal{O} \boxtimes \mathcal{O}(\sigma)_{(b)},$$

Consequently,

$$\mathcal{O}(\Gamma) = \left(\mathcal{O}\left(q_1(a-1)f + s_1\sigma\right)_{(a)} \otimes M^{q_1 - 2r_1}\right) \boxtimes \left(\mathcal{O}\left(s_2\sigma\right)_{(b)} \otimes M^{-2r_2}\right). \tag{35}$$

From (31) we know that this line bundle is a pullback under τ. This strongly constrains the coefficients in the expression (35). In fact, via the isomorphism

$$\text{Pic}(X^{[n]}) = \text{Pic}(X) \oplus \mathbb{Z}, \quad L_{(n)} \otimes M^r \mapsto (L, r),$$

the image

$$\tau^\star : \text{Pic}(X^{[a+b]}) \to \text{Pic}(X^{[a]}) \times \text{Pic}(X^{[b]})$$

corresponds to the diagonal embedding. Therefore, in (35), we must have

$$q_1 = 0, s_1 = s_2, q_1 - 2r_1 = -2r_2.$$

Hence

$$\Gamma = r_1(R_1 + R_2) + s_1(S_1 + S_2)$$

is a pullback of the divisor $\Delta_0 = r_1 R + s_1 S$. This establishes Claim 1.

Remark 1. The argument just given suffices to show that $\widetilde{\Theta}_{r,s} = \tau^\star \theta_L$. We note here however that $\widetilde{\Theta}_{r,s}$ intersects properly not only $\tau^\star Q$ but also $\tau^\star S$. Otherwise, we would have

$$H^0(X^{[a+b]}, L^{[a+b]} \otimes \mathcal{O}(-S)) \neq 0.$$

We calculate

$$L^{[a+b]} \otimes \mathcal{O}(-S) = L(-\sigma)^{[a+b]}.$$

From the dimension count (34) we have

$$h^0(L(-\sigma)) = a + b + 1 + \nu,$$

and therefore from (18),

$$h^0(L^{[a+b]} \otimes \mathcal{O}(-S)) = \binom{h^0(L(-\sigma))}{a+b} = \binom{a+b+1+\nu}{a+b} = 0.$$

2.7. **Arbitrary elliptic surfaces.** The case of arbitrary simply connected elliptic surfaces X with section can be approached by the same methods. In this section we prove Theorem 5. First, we write down O'Grady's construction in this more general setting, and then we reinterpret it via Fourier-Mukai transforms.

The holomorphic Euler characteristic of the fibration

$$\pi : X \to \mathbb{P}^1$$

will be denoted

$$\chi = \chi(\mathcal{O}) = 1 + h^2(\mathcal{O}_X) > 0.$$

We study normalized moduli spaces of sheaves $\mathfrak{M}_v$ such that

$$\chi(v) = 1 \implies c_1(v) = \sigma + \left(a - \frac{r(r-1)}{2}\chi\right) f,$$

where we write $2a$ for the dimension of $\mathfrak{M}_v$. A birational isomorphism

$$\Phi_v : X^{[a]} \dashrightarrow \mathfrak{M}_v$$

is constructed as follows. As in the case of $K3$ surfaces, we consider generic schemes Z of length a, satisfying the requirements (i)-(iv) of section 2.2. We set

$$E_1 = I_Z(\sigma + af).$$

Inductively, we construct nontrivial extensions

$$0 \to \mathcal{O} \to E_{r+1} \to E_r(-\chi f) \to 0. \tag{36}$$

Several statements are to be shown simultaneously during the induction step:

(a) $\mathrm{Ext}^0(E_r(-\chi f), \mathcal{O}) = 0$
(b) $\mathrm{Ext}^2(E_r(-\chi f), \mathcal{O}) = 0.$
(c) $\mathrm{Ext}^1(E_r(-\chi f), \mathcal{O}) \cong \mathbb{C}$. This shows that the extension (36) is unique.
(d) the restriction of E_r to the generic fiber is the Atiyah bundle W_r. This implies the stability of E_r with respect to suitable polarizations. For special fibers through $p \in Z$, the restriction splits as $\mathsf{W}_{r-1,p} \oplus \mathcal{O}_f(o - p)$.

Checking (a)-(d) for the base case $r = 1$ uses the requirements (i)-(iv) of section 2.2. We briefly explain the inductive step. The first vanishing in (a) follows by stability since for polarizations $H = \sigma + mf$ with $m >> 0$, we have

$$\frac{c_1(E_r(-\chi f)) \cdot H}{r} > 0.$$

Regarding (b), we consider the exact sequence induced by (36)

$$\mathrm{Ext}^2(E_r(-2\chi f), \mathcal{O}) \to \mathrm{Ext}^2(E_{r+1}(-\chi f), \mathcal{O}) \to \mathrm{Ext}^2(\mathcal{O}(-\chi f), \mathcal{O}) = 0.$$

Now (b) follows since the leftmost term also vanishes as one can see by considering the injection

$$\mathrm{Ext}^2(E_r(-2\chi f), \mathcal{O}) \hookrightarrow \mathrm{Ext}^2(E_r(-\chi f), \mathcal{O}) = 0.$$

Now, (a) and (b) imply (c) via a Riemann-Roch calculation. Finally, (d) is argued exactly as Lemma 1 above.

Now, we use (d) to calculate Fourier-Mukai transforms

$$\mathsf{S}_{X \to Y} : \mathbf{D}(X) \to \mathbf{D}(Y).$$

By the arguments of section 2.2, we obtain

(i) $\mathsf{S}_{X \to Y}(E_r^\vee) = I_Z(r\sigma + r\chi f)[-1]$
(ii) $\mathsf{S}_{X \to Y}(E_r) = \mathsf{C}_Z \otimes \mathcal{O}(-r\sigma - (r-1)\chi f)$.

Consider now two complementary moduli spaces $\mathfrak{M}_v$ and $\mathfrak{M}_w$. After twisting by fiber classes, we may assume v and w are normalized. We consider the Theta locus

$$\Theta = \{(E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w : h^0(E \otimes F \otimes \mathcal{O}(\nu f)) = 0\}$$

where

$$-\nu = \frac{a + b - \chi}{r + s} - (r + s - 1)\frac{\chi}{2} + 1 > \chi,$$

via the condition (ii) of Theorem 5. We set

$$L = \mathcal{O}_X\left((r+s)\sigma + ((r+s-1)\chi - \nu)f\right) \otimes K_X.$$

An easy calculation shows

$$h^0(L) = \chi(L) = a + b,$$

and that L has no higher cohomology. We therefore obtain a divisor

$$\theta_{L,a,b} \subset X^{[a]} \times X^{[b]},$$

and the associated twist

$$\tilde{\theta}_L = (1 \times i)^\star \theta_L = (i \times 1)^\star \theta_L$$

in the product of Hilbert schemes.

Repeating the argument for $K3$s, we obtain that under the birational map

$$\Phi_v \times \Phi_w : X^{[a]} \times X^{[b]} \dashrightarrow \mathfrak{M}_v \times \mathfrak{M}_w$$

we have

$$(\Phi_v \times \Phi_w)^\star \Theta \subset \tilde{\theta}_L, \tag{37}$$

at least along the nongeneric locus. This is enough to establish that Θ is a divisor.

Remark 2. Unfortunately, we cannot conclude equality in (37) since we are unable to estimate the codimension of the image of Φ_v and Φ_w in the two moduli spaces. We believe this codimension to be at least 2. Conjecture 1 would follow once this statement is established.

3. Generic strange duality

In this section we prove Theorem 1 by a deformation argument. We will keep the same notations as in the introduction.

Let (X, H) be a polarized $K3$ surface, and let

$$v = (r, H, \chi - r), \ \ w = (s, H, \chi' - s)$$

be the two orthogonal Mukai vectors with $\chi \leq 0, \chi' \leq 0$. Consider a deformation

$$\pi : (\mathcal{X}, \mathcal{L}) \to \Delta$$

of polarized $K3$ surfaces such that

(i) the $K3$ surface (X, H) appears as a generic fiber. In fact, we may assume that for $t \neq 0$, $\mathcal{L}_t$ is the unique ample generator of the Picard group of $\mathcal{X}_t$.

(ii) $\mathcal{X}_0$ is an elliptically fibered $K3$ surface with a section, such that $c_1(\mathcal{L}_0)$ is a numerical section.

For each $t \in \Delta$, we consider the Mukai vectors

$$v_t = (r, c_1(\mathcal{L}_t), \chi - r), \;\; w_t = (s, c_1(\mathcal{L}_t), \chi' - s).$$

Since intersection products are preserved by deformations, we have

$$\langle v_t^\vee, w_t \rangle = 0 \text{ for all } t \in \Delta.$$

We form two relative moduli spaces of $\mathcal{L}_t$-semistable sheaves

$$\mathfrak{M}[v] = \cup_{t\in\Delta} \mathfrak{M}_{v_t} \to \Delta, \;\; \mathfrak{M}[w] = \cup_{t\in\Delta} \mathfrak{M}_{w_t} \to \Delta.$$

The product

$$\mathfrak{M}[v] \times \mathfrak{M}[w] \to \Delta$$

carries a relative theta divisor $\Theta[v, w]$ obtained as the vanishing locus of a section of the relative Theta bundles

$$\Theta[w] \boxtimes \Theta[v] \to \mathfrak{M}[v] \times \mathfrak{M}[w].$$

Pushing forward to Δ via the projections π, we obtain the sheaves

$$\mathsf{V} = \pi_\star\left(\Theta[w]\right), \;\; \mathsf{W} = \pi_\star\left(\Theta[v]\right),$$

as well as a section D of $\mathsf{V} \otimes \mathsf{W}$.

We claim that V and W are vector bundles of equal rank. Let us consider first the sheaf V. Over the special fiber, there is a birational isomorphism

$$\mathfrak{M}_{v_0} \dashrightarrow \mathcal{X}_0^{[a]}$$

regular away from codimension 2, where a denotes half the dimension. The line bundle Θ_{w_0} corresponds to a line bundle of the form $L^{[a]}$ for some $L \to \mathcal{X}_0$ with $h^0(L) = a + b$. Hence,

$$h^0(\mathfrak{M}_{v_0}, \Theta_{w_0}) = h^0(\mathcal{X}_0^{[a]}, L^{[a]}) = \binom{h^0(L)}{a} = \binom{a+b}{a}.$$

Over the general fiber, the Lemma 3 below shows that

$$h^0(\mathfrak{M}_{v_t}, \Theta_{w_t}) = \chi(\mathfrak{M}_{v_t}, \Theta_{w_t}) = \binom{a+b}{a}.$$

The calculation of the Euler characteristic in the equation above can be found in O'Grady [OG2]. By Grauert's theorem, V is a vector bundle whose formation commutes with restriction to fibers. The same arguments apply to W.

Lemma 3. *Let H be an ample generator of the Picard of the $K3$ surface X. Assume that v and w are Mukai vectors such that*

(i) $\langle v^\vee, w \rangle = 0$,

(ii) $c_1(v) = c_1(w) = H$,

(iii) $\chi(v) \leq 0,\ \chi(w) \leq 0$.

The line bundle $\Theta_w \to \mathfrak{M}_v$ *is big and nef, hence it does not have higher cohomology.*

Proof. For a Mukai vector $v = (v_0, v_2, v_4)$, define

$$\lambda_v = (0, -v_0 H, H \cdot v_2)$$

and

$$\mu_v = (-H \cdot v_2, v_4 H, 0).$$

These vectors are perpendicular to v. It is shown by Jun Li that $\Theta_{-\lambda_v}$ is big and nef [Li1]; in fact, $\Theta_{-\lambda_v}$ defines a morphism from the Gieseker space to the Uhlenbeck space.

Using reflections along rigid sheaves, Yoshioka proves that $\Theta_{-\lambda_v - \mu_v}$ is also big and nef [Y], and that it determines a morphism

$$\pi : \mathfrak{M}_v \to \mathfrak{X},$$

where

$$\mathfrak{X} \subset \bigcup_{k \geq -\chi(v)} \mathfrak{M}_{v_k},$$

for the vectors

$$v_k = v + k\langle 1, 0, 1\rangle.$$

The explicit construction is as follows. Since $c_1(v) = H$, by stability it follows that

$$H^2(E) = 0 \implies h^0(E) - h^1(E) = \chi(v) \leq 0.$$

For each $k \geq -\chi(v)$, consider the Brill-Noether locus

$$\mathfrak{M}_k = \{E : h^1(E) = k\} \hookrightarrow \mathfrak{M}_v$$

and for $E \in \mathfrak{M}_k$ construct the universal extension

$$0 \to H^1(E) \otimes \mathcal{O}_X \to \widetilde{E} \to E \to 0.$$

Then, the assignment

$$\mathfrak{M}_v \ni E \mapsto \tilde{E} \in \mathfrak{X}$$

defines a birational map onto its image. In fact, the fibers over sheaves E in the Brill-Noether locus $\mathfrak{M}_k$ are Grassmannians $\mathbb{G}(k, 2k + \chi(v))$.

Now, under the assumptions of the Lemma, we have

$$\Theta_w^{H^2} = \Theta_{-\lambda_v}^{-\chi(w)} \otimes \Theta_{-\lambda_v - \mu_v}^{s}$$

which shows that Θ_w is big and nef as well. □

The section giving $\Theta[v, w]$ induces a morphism

$$\mathsf{D} : \mathsf{V}^\vee \to \mathsf{W}.$$

The lemma below suffices to show that D is an isomorphism over the generic fiber, thus proving Theorem 1

Lemma 4. *Over the central fiber, the duality morphism*

$$\mathsf{D}_0 : H^0(\mathfrak{M}_{v_0}, \Theta_{w_0})^\vee \to H^0(\mathfrak{M}_{w_0}, \Theta_{v_0}) \tag{38}$$

is an isomorphism.

Proof. A proof in rank 2 will be given first. The necessary modifications in higher rank are explained at the end.

Consider a suitable polarization H_+ with respect to both v and w, and write $\mathfrak{M}^+_{v_0}$ and $\mathfrak{M}^+_{w_0}$ for the moduli spaces of H_+-semistable sheaves over $\mathcal{X}_0$. Theorem 2 ensures that

$$\mathsf{D}^+_0 : H^0(\mathfrak{M}^+_{v_0}, \Theta_{w_0})^\vee \to H^0(\mathfrak{M}^+_{w_0}, \Theta_{v_0}) \tag{39}$$

is an isomorphism. In rank 2, it follows that (38) is an isomorphism by Lemma 5 below, applied to the pair (H_+, H_-) where

$$H_- = c_1(v) = c_1(w).$$

Requirement (ii) of the Lemma is satisfied for the polarization H_+ by [F]. The statement holds for the polarization H_-. Indeed, by [Y2], the generic fibers of the flat morphism

$$\mathfrak{M}[v] \to \Delta$$

are nonempty of the expected dimension

$$\langle v_t, v_t \rangle + 2 = \langle v_0, v_0 \rangle + 2.$$

The same must be true about the special fiber.

In higher rank, we need to replace Lemma 5 by arguments of Yoshioka presented in the Appendix. Yoshioka's estimates are stated on the stack, but the translation to the moduli schemes is straightforward. The key fact is that that lifting sections from the moduli scheme to the moduli stack is an isomorphism, by Proposition 8.4 in [BL].

To spell out the details, write

$$\mathbf{M}^{ss}[v] \to \Delta, \ \ \mathbf{M}^{ss}[w] \to \Delta$$

for the relative moduli stacks of $\mathcal{L}$-semistable sheaves, and consider the theta bundles

$$\mathbf{\Theta}_w \to \mathbf{M}^{ss}[v], \;\; \mathbf{\Theta}_v \to \mathbf{M}^{ss}[w].$$

Note the morphisms to the moduli schemes

$$p : \mathbf{M}^{ss}[v] \to \mathfrak{M}[v], \quad p : \mathbf{M}^{ss}[w] \to \mathfrak{M}[w]$$

which match the theta bundles accordingly

$$\mathbf{\Theta}_w = p^\star \Theta_w, \quad \mathbf{\Theta}_v = p^\star \Theta_v.$$

Now, the central fibers $\mathbf{M}^{ss}_{v_0}$ and $\mathbf{M}^{ss}_{w_0}$ consist in semistable sheaves with respect to a polarization $\mathcal{L}_0$ which may lie on a wall. Under the assumptions

$$\langle v, v\rangle \geq 2r(r^2 - r + 1), \;\; \langle w, w\rangle \geq 2s(s^2 - s + 1),$$

Corollary 1 of the Appendix shows that the central fibers are isomorphic away from codimension 2 to the moduli stacks of H_+-semistable sheaves $\mathbf{M}^+_{v_0}$ and $\mathbf{M}^+_{w_0}$, for suitable polarizations H_+. Lifting (39) to the stack, we obtain that

$$\mathsf{D}_0^+ : H^0(\mathbf{M}^+_{v_0}, \mathbf{\Theta}_{w_0})^\vee \to H^0(\mathbf{M}^+_{w_0}, \mathbf{\Theta}_{v_0})$$

is an isomorphism. In turn, this shows that

$$\mathsf{D}_0 : H^0(\mathbf{M}^{ss}_{v_0}, \mathbf{\Theta}_{w_0})^\vee \to H^0(\mathbf{M}^{ss}_{w_0}, \mathbf{\Theta}_{v_0})$$

is an isomorphism as well, which establishes (38) descending once again to the moduli scheme. □

Lemma 5. *Let $H_\pm$ be two ample polarizations on an elliptic $K3$ surface X, and let $\mathfrak{M}_\pm$ be the moduli spaces of $H_\pm$ Gieseker semistable sheaves with Mukai vector*

$$v = (2, c_1(v), \chi - 2).$$

We assume

(i) *$c_1(v)$ is an ample numerical section and $\chi(v) \leq 0$,*
(ii) *the moduli spaces $\mathfrak{M}_\pm$ are nonempty of the expected dimension $\langle v, v\rangle + 2$.*

Then, there is a common open subset

$$U \hookrightarrow \mathfrak{M}_\pm$$

whose complement has codimension at least 2 in the two moduli spaces, consisting of sheaves which are both $H_\pm$-Gieseker semistable.

Proof. The proof follows closely the arguments of [Q]. Consider a sheaf

$$\mathcal{F} \in \mathfrak{M}_+ \setminus \mathfrak{M}_-.$$

Since $\mathcal{F}$ is not H_- semistable, there is a destabilizing rank 1 sheaf $I_Z(D)$, where D is a divisor on X, and Z is a zero dimensional scheme of length z. We form the exact sequence

$$0 \to I_Z(D) \to \mathcal{F} \to I_W(E) \to 0, \tag{40}$$

Note that

$$D + E = c_1(v),$$

$$z + w = -\chi(v) + 4 + \frac{D^2 + E^2}{2}.$$

Because $I_Z(D)$ is H_--destabilizing for $\mathcal{F}$, we must have

$$(D - E) \cdot H_- \geq 0. \tag{41}$$

Similarly, since $\mathcal{F}$ is H_+ semistable, $\mathcal{F}$ is μ-semistable with respect to H_+, hence

$$(D - E) \cdot H_+ \leq 0. \tag{42}$$

In particular, the last two inequalities imply that neither $E - D$ nor $D - E$ can be effective, since otherwise $D = E$ contradicting that $c_1(v)$ is a numerical section.

We claim

$$\text{Ext}^0(I_Z(D), I_W(E)) = \text{Ext}^2(I_W(D), I_Z(E)) = 0.$$

Indeed, we have

$$\text{Ext}^0(I_Z(D), I_W(E)) \hookrightarrow \text{Ext}^0(I_Z(D), \mathcal{O}(E)) = \text{Ext}^2(\mathcal{O}(E), I_Z(D))$$

$$= H^2(I_Z(D - E)) = H^2(\mathcal{O}(D - E)) = H^0(\mathcal{O}(E - D)) = 0.$$

Similarly,

$$\text{Ext}^2(I_Z(D), I_W(E)) = \text{Ext}^0(I_W(E), I_Z(D)) = 0.$$

Therefore,

$$\text{Ext}^1(I_Z(D), I_W(E)) = -\chi(I_Z(D), I_W(E)) = -\frac{(D - E)^2}{2} - 2 + z + w.$$

We can now to count moduli for the extensions (40):

$$2z + 2w + \text{Ext}^1(I_Z(D), I_W(E)) - 1 = D^2 + E^2 + D \cdot E - 3\chi + 9.$$

Since $\mathfrak{M}_+$ has dimension

$$2 + \langle v, v \rangle = c_1(v)^2 - 4\chi + 10 = (D + E)^2 - 4\chi + 10,$$

the lemma follows once we establish that

$$D \cdot E \geq \chi + 1.$$

Write

$$c_1(v) = \sigma + af, \quad D = m\sigma + nf, \quad E = (1-m)\sigma + (a-n)f.$$

Similarly, let

$$H_{\pm} = \sigma + \lambda_{\pm} f$$

where $\lambda_{\pm} > 2$. Since

$$(D - E) \cdot H_{\pm} = (\lambda_{\pm} - 2)(2m - 1) + (2n - a),$$

inequalities (41), (42) give

$$(2m-1)(\lambda_+ - 2) \leq (a - 2n), \quad (2m-1)(\lambda_- - 2) \geq (a - 2n).$$

As a corollary, we see that $(2m-1)$ and $(a-2n)$ must have the same sign, i.e.

$$(2m-1)(a-2n) \geq 0 \implies n(1-m) + m(a-n) \geq \frac{a}{2} > 0.$$

Therefore,

$$D \cdot E = 2m(m-1) + n(1-m) + m(a-n) > 0.$$

This completes the proof. □

Remark 3. In the above argument, the condition (i) may be relaxed to

$$\frac{a}{2} \geq \chi + 1 \iff \langle v, v \rangle \geq 10.$$

APPENDIX: CHANGE OF POLARIZATION FOR MODULI SPACES OF HIGHER RANK SHEAVES OVER $K3$ SURFACES

BY KOTA YOSHIOKA

Let X be a $K3$ surface, and fix a Mukai vector

$$v := (r, \xi, a) \in H^*(X, \mathbb{Z})$$

with $r > 0$. For an ample divisor H on X, denote by $\mathbf{M}(v)$, $\mathbf{M}_H(v)^{ss}$, and $\mathbf{M}_H(v)^{\mu-ss}$ the stacks of sheaves, of Gieseker H-semistable sheaves, and of slope H-semistable sheaves respectively – all of type v.

Lemma 6. *If H is general with respect to v, that is, H does not lie on a wall with respect to v, then*

$$\dim \mathbf{M}_H(v)^{ss} = \begin{cases} \langle v^2\rangle + 1, & \langle v^2\rangle > 0 \\ \langle v^2\rangle + l, & \langle v^2\rangle = 0 \\ \langle v^2\rangle + l^2 = -l^2, & \langle v^2\rangle < 0 \end{cases}, \tag{43}$$

where $l = \gcd(r, \xi, a)$. In particular, $\dim \mathbf{M}_H(v)^{ss} \leq \langle v^2\rangle + r^2$.

Proof. If $\langle v^2\rangle \geq 0$, then the claims are Lemma 3.2 and 3.3 in [KY]. If $\langle v^2\rangle < 0$, then $\mathbf{M}_H(v)^{ss}$ consists of $E_0^{\oplus l}$, where E_0 is the unique member of $\mathbf{M}_H(v/l)^{ss}$. In this case, $\mathbf{M}_H(v)^{ss} = BGL(l)$, and $\dim \mathbf{M}_H(v)^{ss} = -\dim \mathrm{Aut}(E_0^{\oplus l}) = -l^2$. □

Let $\mathcal{F}_H(v_1, v_2, \ldots, v_s)$ be the stack of the Harder-Narashimhan filtrations

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E, \;\; E \in \mathbf{M}(v) \tag{44}$$

such that the quotients F_i/F_{i-1}, $1 \leq i \leq s$ are semistable with respect to H and

$$v(F_i/F_{i-1}) = v_i. \tag{45}$$

Then Lemma 5.3 in [KY] implies

$$\dim \mathcal{F}_H(v_1, v_2, \ldots, v_s) = \sum_{i=1}^{s} \dim \mathbf{M}_H(v_i)^{ss} + \sum_{i<j} \langle v_i, v_j\rangle. \tag{46}$$

Note that

$$\mathrm{Hom}(F_i/F_{i-1}, F_j/F_{j-1}) = 0 \;\text{ for }\; i < j,$$

as reduced Hilbert polynomials are strictly decreasing in the Harder-Narasimhan filtration.

Let H_1 be an ample divisor on X which belongs to a wall W with respect to v and H an ample divisor which belongs to an adjacent chamber. Then Gieseker H-semistable sheaves are H_1 slope-semistable

$$\mathbf{M}_H(v)^{ss} \hookrightarrow \mathbf{M}_{H_1}(v)^{\mu\text{-}ss}$$

We shall estimate the codimension of

$$\mathbf{M}_{H_1}(v)^{\mu\text{-}ss} \setminus \mathbf{M}_H(v)^{ss}.$$

Specifically, we shall prove

Proposition 3.

$$(\langle v^2\rangle + 1) - \dim(\mathbf{M}_{H_1}(v)^{\mu\text{-}ss} \setminus \mathbf{M}_H(v)^{ss}) \geq \frac{1}{r}\langle v^2\rangle/2 + r - r^2 + 1. \tag{47}$$

As a consequence, we have

Corollary 1. *Assume that*

$$\frac{1}{r}\langle v^2\rangle/2 + r - r^2 + 1 \geq 2.$$

Then $\mathbf{M}_H(v)^{ss}$ does not depend on the choice of ample line bundle H (generic or on a wall) up to codimension 1.

Proof. Let E be an H_1 slope-semistable sheaf, which is however not H-semistable. Consider its Harder-Narasimhan filtration relative to H,

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E.$$

All the subsheaves in the filtration are H-destabilizing for E. As E is H_1 slope-semistable, we must have equalities of slopes,

$$\mu_{H_1}(F_1) = \mu_{H_1}(F_2) = \ldots = \mu_{H_1}(E),$$

or in the notation of (45),

$$\frac{c_1(v_i)\cdot H_1}{\mathrm{rk}\ v_i} = \frac{c_1(v)\cdot H_1}{\mathrm{rk}\ v}, \quad 1 \leq i \leq s. \tag{48}$$

Thus

$$\mathbf{M}_{H_1}(v)^{\mu\text{-}ss} \setminus \mathbf{M}_H(v)^{ss} = \cup_{v_1,\ldots,v_s}\mathcal{F}_H(v_1, v_2, \ldots, v_s),$$

where (48) is satisfied. We shall estimate $\sum_{i<j}\langle v_i, v_j\rangle$. We set $v_i := (r_i, \xi_i, a_i)$. Since

$$\langle (v_i/r_i - v_j/r_j)^2\rangle = (\xi_i/r_i - \xi_j/r_j)^2, \tag{49}$$

we get

$$\langle v_i, v_j\rangle = \frac{r_j}{r_i}\langle v_i^2\rangle/2 + \frac{r_i}{r_j}\langle v_j^2\rangle/2 - \frac{(r_j\xi_i - r_i\xi_j)^2}{2r_ir_j}. \tag{50}$$

Then we have

$$\begin{aligned}\langle v^2\rangle/2 &= \sum_{i<j}\langle v_i, v_j\rangle + \sum_i \langle v_i^2\rangle/2 \\ &= \sum_i \frac{r}{r_i}\langle v_i^2\rangle/2 - \sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j}.\end{aligned} \tag{51}$$

Hence

$$\sum_i \frac{1}{r_i}\langle v_i^2\rangle/2 = \frac{1}{r}\langle v^2\rangle/2 + \frac{1}{r}\sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j} \tag{52}$$

and

$$\begin{aligned}\sum_{i<j}\langle v_i, v_j\rangle &= \sum_i \frac{r-r_i}{r_i}\langle v_i^2\rangle/2 - \sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j}\\ &= \sum_i \frac{r-r_i}{r_i}(\langle v_i^2\rangle/2 + r_i^2) - \sum_i (r-r_i)r_i - \sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j}\\ &\geq \sum_i \frac{1}{r_i}(\langle v_i^2\rangle/2 + r_i^2) - \sum_i (r-r_i)r_i - \sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j}\\ &= \sum_i \frac{1}{r_i}\langle v_i^2\rangle/2 + r - r^2 + \sum_i r_i^2 - \sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j}\\ &= \frac{1}{r}\langle v^2\rangle/2 + \frac{1}{r}\sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j} + r - r^2 + \sum_i r_i^2 - \sum_{i<j}\frac{(r_i\xi_j - r_j\xi_i)^2}{2r_ir_j}\\ &\geq \frac{1}{r}\langle v^2\rangle/2 + r - r^2 + \sum_i r_i^2,\end{aligned}$$

where we also used the Hodge index theorem and Bogomolov's inequality

$$\langle v_i^2\rangle + 2r_i^2 \geq 0.$$

Therefore

$$\begin{aligned}(\langle v^2\rangle + 1) - \dim \mathcal{F}_H(v_1, v_2, \ldots, v_s) &= \sum_{i<j}\langle v_i, v_j\rangle + 1 - \sum_i (\dim \mathbf{M}_H(v_i)^{ss} - \langle v_i^2\rangle)\\ &\geq \frac{1}{r}\langle v^2\rangle/2 + r - r^2 + 1,\end{aligned}$$

which implies the claim. □

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Department of Mathematics

University of Illinois at Chicago
E-mail address: alina@math.uic.edu

Department of Mathematics

University of California, San Diego
E-mail address: doprea@math.ucsd.edu

Department of Mathematics

Faculty of Science, Kobe University, Kobe, 657, Japan
E-mail address: yoshioka@math.kobe-u.ac.jp