Problem 1.

Find the critical points of the function

$$f(x, y) = 2x^3 - 3x^2 y - 12x^2 - 3y^2$$

and determine their type i.e. local min/local max/saddle point. Are there any global min/max?

Problem 2.

Determine the global max and min of the function

$$f(x, y) = x^2 - 2x + 2y^2 - 2y + 2xy$$

over the compact region

$$-1 \le x \le 1,\ \ 0 \le y \le 2.$$

Problem 3.

Using Lagrange multipliers, optimize the function

$$f(x, y) = x^2 + (y + 1)^2$$

subject to the constraint

$$2x^2 + (y - 1)^2 \le 18.$$

Problem 4.

Consider the function

$$w = e^{x^2 y}$$

where

$$x = u\sqrt{v},\ \ y = \frac{1}{uv^2}.$$

Using the chain rule, compute the derivatives

$$\frac{\partial w}{\partial u},\ \frac{\partial w}{\partial v}.$$

Problem 5.

(i) For what value of the parameter a, will the planes

$$ax + 3y - 4z = 2,\ \ x - ay + 2z = 5$$

be perpendicular?

(ii) Find a vector parallel to the line of intersection of the planes

$$x - y + 2z = 2,\ \ 3x - y + 2z = 1.$$

(iii) Find the plane through the origin parallel to

$$z = 4x - 3y + 8.$$

(iv) Find the angle between the vectors

$$\mathbf{v} = (1, -1, 2),\ \ \mathbf{w} = (1, 3, 0).$$

(v) A plane has equation

$$z = 5x - 2y + 7.$$

For what values of a is the vector

$$(a, 1, \frac{1}{2})$$

normal to the plane?

Problem 6.

(i) Compute the second degree Taylor polynomial of the function

$$f(x, y) = e^{x^2 - y}$$

around $(1, 1)$.

(ii) Compute the second degree Taylor polynomial of the function

$$f(x) = \sin(x^2)$$

around $x = \sqrt{\pi}$.

(iii) The second degree Taylor polynomial of a certain function $f(x, y)$ around $(0, 1)$ equals

$$1 - 4x^2 - 2(y - 1)^2 + 3x(y - 1).$$

Can the point $(0, 1)$ be a local minimum for f? How about a local maximum?

Problem 7.

(i) The temperature $T(x, y)$ in a long thin plane at the point (x, y) satisfies Laplace's equation

$$T_{xx} + T_{yy} = 0.$$

Does the function

$$T(x, y) = \ln(x^2 + y^2)$$

satisfy Laplace's equation?

(ii) For the function

$$f(x, y) = \sin(x^2 + y^2)\ln(x^4y^4 + 1)\tan(xy)$$

is it true that

$$f_{xyxyy} = f_{yyxyx}?$$

Problem 8.

Consider the function $f(x, y) = \frac{x^2}{y^4}$.

(i) Carefully draw the level curve passing through $(1, -1)$. On this graph, draw the gradient of the function at $(1, -1)$.

(ii) Compute the directional derivative of f at $(1, -1)$ in the direction $\mathbf{u} = \left(\frac{4}{5}, \frac{3}{5}\right)$. Use this calculation to estimate

$$f((1, -1) + .01\mathbf{u}).$$

(iii) Find the unit direction $\mathbf{v}$ of steepest descent for the function f at $(1, -1)$.

(iv) Find the two unit directions $\mathbf{w}$ for which the derivative $f_{\mathbf{w}} = 0$.

Problem 9.

Consider the function

$$f(x, y) = \sqrt{\ln(e^{2x}y^3)}.$$

(i) Write down the tangent plane to the graph of f at $(2, 1)$.
(ii) Find the approximate value of the number

$$\sqrt{\ln(e^{4.1}(1.02)^3)}.$$

Problem 10.

Suppose that

$$z = e^{3x+2y}, \quad y = \ln(3u - w), \quad x = u + 2v.$$

Calculate

$$\frac{\partial z}{\partial v}, \frac{\partial z}{\partial w}.$$

Problem 11.

(i) Find z such that

$$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots = 3.$$

(ii) Calculate the series

$$\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \ldots + \frac{2^{99}}{3^{100}}.$$

Problem 12.

The probability density function for the outcome x of a certain experiment is

$$p(x) = Ce^{-x}, \text{ for } x \geq 0.$$

(i) What is the value of the constant C?
(ii) What is the cumulative distribution function?
(iii) What is the median of the experiment?
(iv) What is the mean of the experiment?
(v) What is the probability that the outcome of the experiment is bigger than 1?

Problem 13.

Consider the function $f(x, y) = 5 - (x + 1)^2 - y^2$.

(i) Draw the cross section corresponding to $x = 1$.
(ii) Draw the contour diagram of f showing at least three levels.
(iii) Draw the graph of f.
(iv) What is the equation of the tangent plane to the graph of f at $(1, 0, 1)$?

Problem 14.

Find the point on the plane

$$2x + 3y + 4z = 29$$

that is closest to the origin. You may want to minimize the square of the distance to the origin.

Problem 15.

Find the critical points of the function $f(x, y) = 2x^3 + 6xy + 3y^2$ and describe their nature.