

GENERIC STRANGE DUALITY FOR $K3$ SURFACES

ALINA MARIAN AND DRAGOS OPREA

ABSTRACT. Strange duality is shown to hold over generic $K3$ surfaces in a large number of cases. The isomorphism for elliptic $K3$ surfaces is established first via Fourier-Mukai techniques. Applications to Brill-Noether theory for sheaves on $K3$ s are also obtained. The appendix written by Kota Yoshioka discusses the behavior of the moduli spaces under change of polarization, as needed in the argument.

CONTENTS

1. Introduction	1
1.1. The strange duality morphism	1
1.2. Results	3
1.3. Acknowledgments	7
2. The theta isomorphism for elliptic $K3$ surfaces	7
2.1. O’Grady’s construction	7
2.2. O’Grady’s moduli space via Fourier-Mukai	9
2.3. Line bundles and theta divisors over the Hilbert scheme of points	16
2.4. The strange duality setup	18
2.5. The theta divisor over the generic locus	19
2.6. The nongeneric locus	20
2.7. Arbitrary simply connected elliptic surfaces	24
3. Generic strange duality	26
Appendix: Change of polarization for moduli spaces of higher rank sheaves over $K3$ surfaces	30
References	33

1. INTRODUCTION

1.1. **The strange duality morphism.** We consider moduli spaces of sheaves over $K3$ surfaces, and the strange duality map on spaces of generalized theta functions associated to them.

To start, we recall the general geometric setting for strange duality phenomena. Let (X, H) be a smooth polarized complex projective surface. To give our exposition a uniform character, we assume that X is simply connected. Let v be a class in the topological K -theory $K_{\text{top}}(X)$ of the surface, and denote by \mathfrak{M}_v the moduli space of Gieseker H -semistable sheaves on X of topological type v .

The moduli space \mathfrak{M}_v carries natural line bundles which we now discuss. Consider the bilinear form on $K_{\text{top}}(X)$ given by

$$(1) \quad (v, w) = \chi(v \cdot w), \text{ for } v, w \in K_{\text{top}}(X),$$

where the product in K -theory is used. Let

$$v^\perp \subset K_{\text{top}}(X)$$

contain the K -classes orthogonal to v relative to this form. When \mathfrak{M}_v consists of stable sheaves only¹, there is a group homomorphism

$$\Theta : v^\perp \rightarrow \text{Pic } \mathfrak{M}_v, \quad w \mapsto \Theta_w,$$

studied among others in [LeP1], [Li2]. If \mathfrak{M}_v carries a universal sheaf

$$\mathcal{E} \rightarrow \mathfrak{M}_v \times X,$$

and w is the class of a vector bundle F , we have

$$\Theta_w = \det \mathbf{R}p_*(\mathcal{E} \otimes q^*F)^{-1}.$$

Here p and q are the two projection maps from $\mathfrak{M}_v \times X$. The theta line bundle is also defined in the absence of a universal sheaf, by descent from the Quot scheme [LeP1], [Li2].

We consider now two classes v and w in $K_{\text{top}}(X)$ satisfying

$$(v, w) = 0.$$

If the conditions

$$(2) \quad H^2(E \otimes F) = 0, \quad \text{Tor}^1(E, F) = \text{Tor}^2(E, F) = 0$$

hold in $\mathfrak{M}_v \times \mathfrak{M}_w$ away from codimension 2, the locus

$$(3) \quad \Theta = \{(E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w \text{ such that } H^0(E \otimes F) \neq 0\}$$

has expected codimension 1. The vanishings (2) are fundamental in the theory of theta divisors; their full significance is explained in [Sc]. In turn, the natural scheme structure of Θ is discussed in [D2]. Considering the associated line bundle, we have the splitting

$$\mathcal{O}(\Theta) = \Theta_w \boxtimes \Theta_v.$$

¹The homomorphism is defined in all generality from a more restricted domain [LeP1].

Therefore, Θ induces a map

$$D : H^0(\mathfrak{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_v).$$

The main query concerning this map is

Question 1. *When nonzero, is D an isomorphism?*

In the context of moduli of sheaves over surfaces, the question was asked and first studied by Le Potier [LeP2]. While an affirmative answer is not expected in this generality, the isomorphism was shown to hold for many pairs $(\mathfrak{M}_v, \mathfrak{M}_w)$ of moduli spaces of sheaves over either $K3$ or rational surfaces, cf. [A] [D1] [D2] [G] [OG2] [S] [Yu]. In all examples however, one of the moduli spaces involved has small dimension and the other consists of rank 2 sheaves. A survey of some of the known results is contained in [MO1]. In this paper, we establish the isomorphism on moduli spaces over generic $K3$ surfaces for a large class of topological types of the sheaves, allowing in particular for arbitrarily high ranks and dimensions. The precise statements are as follows.

1.2. Results. Let (X, H) be a polarized $K3$ surface. We use as customary the Mukai vector

$$v(E) = \text{ch}E\sqrt{tdX} \in H^*(X, \mathbb{Z})$$

to express the topological type of a sheaf E on X . We write

$$v = v_0 + v_2 + v_4$$

to distinguish cohomological degrees in v , and set

$$v^\vee = v_0 - v_2 + v_4.$$

Note also the Mukai pairing on cohomology:

$$\langle v, w \rangle = \int_S v_2 w_2 - v_0 w_4 - v_4 w_0.$$

In terms of the pairing (1), we have

$$(v, w) = -\langle v, w^\vee \rangle = -\langle v^\vee, w \rangle.$$

When it consists only of stable sheaves, the moduli space \mathfrak{M}_v is an irreducible holomorphic symplectic manifold whose dimension is simply expressed in terms of the Mukai pairing

$$\dim \mathfrak{M}_v = \langle v, v \rangle + 2.$$

We show

Theorem 1. *Assume (X, H) is a generic primitively polarized K3 surface of fixed degree. Consider any primitive Mukai vectors v and w of ranks $r \geq 2$ and $s \geq 3$, orthogonal with respect to the pairing (\cdot, \cdot) , and satisfying*

- (i) $c_1(v) = c_1(w) = H$,
- (ii) $\chi(v) \leq 0, \chi(w) \leq 0$,
- (iii) $\langle v, v \rangle \geq 2(r-1)(r^2+1), \langle w, w \rangle \geq 2(s-1)(s^2+1)$.

Then

$$D : H^0(\mathfrak{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

In particular, the theorem asserts that the locus Θ of equation (3) is a divisor. The genericity means that the statement holds on a nonempty open subset of the moduli space of polarized K3s of fixed degree, for all possible vectors v and w at once.

Similarly, in rank 2, we prove

Theorem 1A. *Assume (X, H) is a generic primitively polarized K3 surface of degree at least 8, and consider orthogonal primitive Mukai vectors v and w of rank 2 such that*

- (i) $c_1(v) = c_1(w) = H$,
- (ii) $\chi(v) \leq 0, \chi(w) \leq 0$.

Then

$$D : H^0(\mathfrak{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

The generic strange duality statements of Theorems 1 and 1A lead to a natural approach to establish the isomorphism for *all* K3 surfaces. This is explained in [MO2], which focuses on the global geometry of the Verlinde bundles over the moduli of polarized K3s. We note here that the methods of this paper can also be used to show strange duality over abelian surfaces. We will return to this setting in [MO3].

The theorems are obtained by deformation to moduli spaces over a smooth elliptic K3 surface X with a section. Better results are in fact available here. Let us assume that the fibers have at worst nodal singularities and that the Néron-Severi group is

$$\mathrm{NS}(X) = \mathbb{Z}\sigma + \mathbb{Z}f,$$

where σ and f are the classes of the section and of the fiber respectively. For fixed Mukai vectors, we consider stability with respect to polarizations $H = \sigma + mf$ suitable in the sense of [F]. We show

Theorem 2. *Let v and w be orthogonal Mukai vectors corresponding to sheaves of ranks r and s on X with $r, s \geq 2$. Assume further that*

- (i) $c_1(v) \cdot f = c_1(w) \cdot f = 1$,
- (ii) $\langle v, v \rangle + \langle w, w \rangle \geq 2(r + s)^2$.

Then the duality map

$$D : H^0(\mathfrak{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

Along the way we establish the following Brill-Noether result for sheaves on $K3$ elliptic surfaces.

Theorem 3. *Under the assumptions of Theorem 2, the locus Θ has codimension 1 in the product of moduli spaces $\mathfrak{M}_v \times \mathfrak{M}_w$. In particular, for a generic sheaf $E \in \mathfrak{M}_v$,*

$$\Theta_E = \{F \in \mathfrak{M}_w : h^0(E \otimes F) \neq 0\}$$

is a divisor in \mathfrak{M}_w .

The proofs use the fact that the moduli spaces \mathfrak{M}_v and \mathfrak{M}_w are birational to Hilbert schemes of points on X ,

$$(4) \quad \mathfrak{M}_v \dashrightarrow X^{[a]}, \quad \mathfrak{M}_w \dashrightarrow X^{[b]}, \quad \text{with } a = \frac{\langle v, v \rangle}{2} + 1, \quad b = \frac{\langle w, w \rangle}{2} + 1.$$

The birational maps (4) were described in [OG1] and were shown to be regular away from codimension 2. Theorem 2 is then a consequence of the explicit identification of the theta divisor (3) with a divisor in the product $X^{[a]} \times X^{[b]}$ known to induce strange duality. Specifically, for any line bundle L on X with $\chi(L) = a + b$ and no higher cohomology, one can consider the divisor associated to the locus

$$\theta_{L,a,b} = \{(I_Z, I_W) \text{ such that } h^0(I_Z \otimes I_W \otimes L) \neq 0\} \subset X^{[a]} \times X^{[b]}.$$

Furthermore, observe the involution on the elliptic surface X , given by fiberwise reflection across the origin of the fiber:

$$p \in f \mapsto -p \in f.$$

The involution is defined away from the codimension 2 locus of singular points of fibers of X . It induces an involution on any Hilbert scheme of points on X , defined outside a codimension 2 locus,

$$\iota : X^{[a]} \dashrightarrow X^{[a]}, \quad Z \mapsto \tilde{Z}.$$

Consider the pullback

$$\tilde{\theta}_{L,a,b} = (\iota \times 1)^* \theta_{L,a,b},$$

under the birational map

$$\iota \times 1 : X^{[a]} \times X^{[b]} \dashrightarrow X^{[a]} \times X^{[b]}, \quad (Z, W) \mapsto (\tilde{Z}, W).$$

It can be shown that

$$(\iota \times 1)^* \theta_{L,a,b} = (1 \times \iota)^* \theta_{L,a,b}.$$

Viewing Θ as a locus in $X^{[a]} \times X^{[b]}$ via (4), we prove

Theorem 4. *There exists a line bundle L on X with $\chi(L) = a + b$ and no higher cohomology, such that*

$$\Theta = \tilde{\theta}_{L,a,b} \text{ in the product } X^{[a]} \times X^{[b]}.$$

$\theta_{L,a,b}$ is known to give an isomorphism on the associated spaces of sections on $X^{[a]}$ and $X^{[b]}$, cf. [MO1]. Therefore so does $\tilde{\theta}_{L,a,b}$, yielding Theorem 2.

The identification of the two theta divisors of Theorem 4 is difficult even though the O'Grady birational isomorphism with the Hilbert scheme is explicit. To achieve it, in Propositions 1 and 2 of Section 2.2, we interpret the O'Grady construction by means of Fourier-Mukai transforms. We show there that the Fourier-Mukai transforms of *generic* O'Grady sheaves are two-term complexes in the derived category, derived dual to ideal sheaves. More importantly, a careful analysis is necessary to keep track of the special loci where the generic description may fail.

The same method gives results for arbitrary simply connected (not necessarily $K3$) elliptic surfaces

$$\pi : X \rightarrow \mathbb{P}^1$$

with a section and at worst nodal irreducible fibers. The dimension of the two complementary moduli spaces will be taken large enough compared to the constant

$$\Delta = \chi(X, \mathcal{O}_X) \cdot ((r+s)^2 + (r+s) + 2) - 2(r+s).$$

Assuming as before that the polarization is suitable, we have

Theorem 5. *Assume v and w are two orthogonal Mukai vectors of ranks $r, s \geq 3$, with*

- (i) $c_1(v) \cdot f = c_1(w) \cdot f = 1$,
- (ii) $\dim \mathfrak{M}_v + \dim \mathfrak{M}_w \geq \Delta$.

Then, Θ is a divisor in $\mathfrak{M}_v \times \mathfrak{M}_w$. Furthermore,

$$D : H^0(\mathfrak{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

The result in this new setting was initially established up to the statement that the birational isomorphism of \mathfrak{M}_v and \mathfrak{M}_w with Hilbert schemes of points holds away from codimension 2. Subsequently, M. Bernardara and G. Hein obtained a proof of this statement [BH], thus completing the last missing step and yielding the theorem above.

The article is structured as follows. The main part of the argument concerns the case of elliptic $K3$ surfaces and is presented in Section 2. The last part of Section 2 treats the case of arbitrary simply connected elliptic surfaces. Section 3 explains generic strange duality via a deformation argument. The appendix written by Kota Yoshioka contains a discussion of change of polarization for higher-rank moduli spaces of sheaves over $K3$ s.

1.3. Acknowledgments. We thank Marcello Bernardara, Kieran O’Grady, Georg Hein, Eyal Markman, Michael Thaddeus, and Kota Yoshioka for helpful conversations and correspondence. In particular, Kota Yoshioka suggested a simplification of the main calculation of Section 2.5. We thank the referees for carefully reading the paper and for useful suggestions. Finally, we gratefully acknowledge the support of the NSF and of the Sloan Foundation.

2. THE THETA ISOMORPHISM FOR ELLIPTIC $K3$ SURFACES

2.1. O’Grady’s construction. Keeping the notations of the introduction, we let $\pi : X \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface with a section σ , whose fibers are irreducible and have at worst nodal singularities. The Picard rank of X thus equals 2. We have

$$\sigma^2 = -2, f^2 = 0, \sigma \cdot f = 1.$$

We are concerned with sheaves on X with Mukai vector of type

$$v = r + (\sigma + kf) + p\omega,$$

for some $k, p \in \mathbb{Z}$, with ω being the class of a point in X . We consider a v -suitable polarization

$$H = \sigma + mf \text{ for } m \gg 0.$$

This means that H lies in a v -chamber of the ample cone of X adjacent to the class f of the fiber [OG1]. The moduli space \mathfrak{M}_v of H -semistable sheaves consists of slope-stable sheaves only, and the choice of H ensures that $E \in \mathfrak{M}_v$ is stable if and only if its restriction to a generic fiber is stable. The restriction to special fibers may be unstable, as we will show in Lemma 1 below.

As explained in Sections *I.4 – I.5* of [OG1], we can inductively build the moduli spaces \mathfrak{M}_v as follows. Note first that tensoring with $\mathcal{O}(f)$ gives an isomorphism

$$\mathfrak{M}_v \cong \mathfrak{M}_{\tilde{v}}, \text{ where } \tilde{v} = r + (\sigma + (k+r)f) + (p+1)\omega.$$

Such a twist raises the Euler characteristic by 1. We normalize the moduli space by requiring that $p = 1 - r$; when it has dimension $2a$ we refer to it as \mathfrak{M}_r^a . Points in \mathfrak{M}_r^a are rank r sheaves with Mukai vector

$$v_{r,a} = r + (\sigma + (a - r(r-1))f) + (1-r)\omega.$$

The normalization amounts to imposing that

$$\chi(E) = 1 \text{ for } E \in \mathfrak{M}_v.$$

In rank 1, note the isomorphism

$$X^{[a]} \cong \mathfrak{M}_1^a, \quad I_Z \mapsto I_Z(\sigma + af).$$

For any r , the generic point E_r of \mathfrak{M}_r^a has exactly one section

$$h^0(E_r) = 1,$$

as expected since the Euler characteristic is 1; see Proposition *I.3.3* in [OG1]. Moreover, $h^0(E_r(-f)) = 0$ generically, and

$$h^0(E_r(-2f)) = 0 \text{ for } E_r \text{ outside a codimension 2 locus in } \mathfrak{M}_r^a.$$

In addition, stability forces the vanishing $h^2(E_r(-2f)) = 0$ for all sheaves in \mathfrak{M}_r^a , so

$$(5) \quad h^1(E_r(-2f)) = -\chi(E_r(-2f)) = 1$$

outside a codimension 2 locus in \mathfrak{M}_r^a . These statements are contained in Claim *I.5.2* and Proposition *I.5.17* in [OG1]. There, an open subscheme $U_r^a \subset \mathfrak{M}_r^a$ is singled out, for which (5) holds. For sheaves E_r in U_r^a there is a unique nontrivial extension

$$(6) \quad 0 \rightarrow \mathcal{O} \rightarrow E_{r+1} \rightarrow E_r \otimes \mathcal{O}(-2f) \rightarrow 0.$$

The resulting middle term E_{r+1} is torsion-free, with Mukai vector $v_{r+1,a}$, and is stable unless E_r belongs to a divisor D_r in U_r^a , cf. Section *I.4* and Lemma *I.5.7* [OG1]. In the latter case, a stabilization procedure is required to ensure that the resulting rank $r+1$ sheaf also belongs to \mathfrak{M}_{r+1}^a . The assignment

$$E_r \mapsto E_{r+1}$$

identifies open subschemes

$$U_r^a \cong U_{r+1}^a,$$

giving rise to a birational map

$$(7) \quad \phi_r : \mathfrak{M}_r^a \dashrightarrow \mathfrak{M}_{r+1}^a,$$

and therefore a birational morphism away from codimension 2,

$$(8) \quad \Phi_r : X^{[a]} \cong \mathfrak{M}_1^a \dashrightarrow \mathfrak{M}_r^a.$$

It will not be necessary for us to dwell on the details of the semistable reduction process along the D_r s although this, together with the identification of the D_r s themselves as divisors on the Hilbert scheme $X^{[a]}$, constitutes the most difficult part of [OG1]. We record here however, for future use, that

$$(9) \quad D_1 = Q \cup S, \text{ and } D_r = S \text{ for } r \geq 2.$$

Here, Q is the divisor on $X^{[a]}$ consisting of ideals I_Z such that

$$h^0(I_Z((a-1)f)) \neq 0.$$

Equivalently, Q is the divisor of subschemes in $X^{[a]}$ with at least two points contained in the same elliptic fiber of X . Furthermore, S is the divisor of subschemes in $X^{[a]}$ which intersect the section σ of the elliptic fibration.

2.2. O'Grady's moduli space via Fourier-Mukai. We will reinterpret here the birational map

$$\Phi_r : X^{[a]} \dashrightarrow \mathfrak{M}_r^a$$

as a Fourier-Mukai transform. This will be crucial for the identification of the theta divisor and in particular for the proof of Theorem 4.

We let $Y \rightarrow \mathbb{P}^1$ denote the dual elliptic $K3$ surface *i.e.*, the relative moduli space of rank 1 degree 0 sheaves over the fibers of $\pi : X \rightarrow \mathbb{P}^1$. In fact, X and Y are canonically isomorphic. Writing

$$\mathcal{P} \rightarrow X \times_{\mathbb{P}^1} Y$$

for the universal sheaf, we consider the Fourier-Mukai transform

$$S_{X \rightarrow Y} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y),$$

with kernel \mathcal{P} , given by

$$(10) \quad S_{X \rightarrow Y}(x) = \mathbf{R}q_* \left(\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathbf{L}p^*x \right).$$

Here p and q are the two projections. We will normalize \mathcal{P} such that

$$S_{X \rightarrow Y}(\mathcal{O}) = \mathcal{O}_\sigma[-1].$$

In fact, we have

$$c_1(\mathcal{P}) = \Delta - p^*\sigma - q^*\sigma$$

where Δ is the diagonal in $X \times_{\mathbb{P}^1} Y$. In a similar fashion, we set

$$\mathcal{Q} = \mathcal{P}^\vee,$$

and use this as the kernel of the transform

$$\mathbb{T}_{Y \rightarrow X} : \mathbf{D}(Y) \rightarrow \mathbf{D}(X)$$

It was shown in [Br] that the functors $\mathbb{S}_{X \rightarrow Y}$ and $\mathbb{T}_{Y \rightarrow X}$ are equivalences of categories and that

$$(11) \quad \mathbb{S} \circ \mathbb{T} = \mathbf{1}_{\mathbf{D}(Y)}[-2], \quad \mathbb{T} \circ \mathbb{S} = \mathbf{1}_{\mathbf{D}(X)}[-2].$$

Fix a cycle $Z \in X^{[a]}$, and let E_r denote the sheaf in \mathfrak{M}_r^a corresponding to Z under the O'Grady isomorphism Φ_r . We will consider *generic* subschemes Z , in the sense that

- (i) Z consists of distinct points,
- (ii) no two points of Z lie in the same fiber,
- (iii) Z is disjoint from the section,
- (iv) Z does not contain any singular points of the fibers.

We determine the images of E_r and of its derived dual E_r^\vee under the functor (10). The answer is simpler for the dual, which is in fact *WIT*₁ relative to $\mathbb{S}_{X \rightarrow Y}$. Indeed, using the natural identification $Y \cong X$, we show

Proposition 1. *For generic Z , we have*

$$\mathbb{S}_{X \rightarrow Y}(E_r^\vee) = I_{\tilde{Z}}(r\sigma + 2rf)[-1].$$

Furthermore,

$$\mathbb{S}_{X \rightarrow Y}(E_r^\vee(nf)) = \mathbb{S}_{X \rightarrow Y}(E_r^\vee) \otimes \mathcal{O}(nf).$$

Here, we recall from the introduction that

$$\tilde{Z} = \iota^* Z$$

is the cycle obtained by taking the inverses of all points in Z in the group law of their corresponding fibers. This makes sense even for singular fibers using the group law of the regular locus.

The Fourier-Mukai transform of E_r is expressed in terms of derived duals of ideal sheaves. We have

Proposition 2. *For $r \geq 1$, we have*

$$\mathbb{S}_{X \rightarrow Y}(E_r) = I_Z^\vee \otimes \mathcal{O}(-r\sigma - 2(r-1)f).$$

The rest of this section is devoted to the proofs of Propositions 1 and 2. We study first how the generic sheaf E_r restricts to the fibers. Consider a fiber f of $\pi : X \rightarrow \mathbb{P}^1$ with origin $o = \sigma \cap f$, and let

$$W_r \rightarrow f$$

be the unique rank r stable bundle on f with determinant $\mathcal{O}_f(o)$. The W_r 's were constructed by Atiyah over smooth elliptic curves. His arguments extend verbatim to nodal genus 1 curves: we define W_r inductively as the unique nontrivial extension

$$(12) \quad 0 \rightarrow \mathcal{O} \rightarrow W_{r+1} \rightarrow W_r \rightarrow 0, \quad W_1 = \mathcal{O}_f(o).$$

Details can be found in several places in the literature, for instance in Section 3 of [BK]. Similarly, if p is any smooth point of the fiber f , we write

$$W_{r,p} \rightarrow f$$

for the Atiyah bundle over the fiber f of determinant $\mathcal{O}_f(p)$ such that

$$(13) \quad 0 \rightarrow \mathcal{O} \rightarrow W_{r+1,p} \rightarrow W_{r,p} \rightarrow 0.$$

The convention

$$W_{0,p} = \mathcal{O}_p$$

is used throughout.

For Z generic in the sense of (i) - (iv), we have the following

Lemma 1. (i) *If f is a fiber such that $Z \cap f = \emptyset$, then*

$$E_r|_f = W_r.$$

(ii) *If f is a fiber through $p \in Z$, then*

$$E_r|_f = W_{r-1,p} \oplus \mathcal{O}_f(o-p).$$

Proof. This is seen by induction starting with the case $r = 1$ when

$$E_1 = I_Z(\sigma + af).$$

The basic observation is that for $p \in X$ and I_p denoting its ideal sheaf in X , the restriction to the fiber f through p is

$$I_p|_f = \mathcal{O}_p \oplus \mathcal{O}_f(-p).$$

This gives the statement for E_1 . The inductive step from r to $r + 1$ follows from the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow E_{r+1} \rightarrow E_r(-2f) \rightarrow 0.$$

Its restriction to any fiber never splits as explained by Lemma *I.4.7* [OG1]. The restriction to a fiber avoiding Z must therefore coincide with the Atiyah bundle W_{r+1} , since the latter is the only nontrivial extension

$$0 \rightarrow \mathcal{O} \rightarrow W_{r+1} \rightarrow W_r \rightarrow 0.$$

The same argument holds for the fibers through points of Z , using that there is a unique nontrivial extension

$$0 \rightarrow \mathcal{O} \rightarrow W_{r,p} \oplus \mathcal{O}_f(o-p) \rightarrow W_{r-1,p} \oplus \mathcal{O}_f(o-p) \rightarrow 0.$$

□

Letting f be a smooth elliptic fiber, we record now the Fourier-Mukai transforms of the Atiyah bundles relative to the standard Poincaré kernel on $f \times f$. We use hatted notation for the transforms, and as before we let

$$\iota : f \rightarrow f$$

denote reflection about the origin of f . We have

$$(14) \quad \widehat{W}_r = \mathcal{O}_f(-r \cdot o),$$

$$(15) \quad \widehat{W}_{r,p} = \mathcal{O}_f(-(r+1) \cdot o + p).$$

By the results of [Muk], the last two equations imply

$$(16) \quad \widehat{W}_r^\vee = \mathcal{O}_f(r \cdot o)[-1],$$

$$(17) \quad \widehat{W}_{r,p}^\vee = \mathcal{O}_f((r+1) \cdot o - \iota^* p)[-1].$$

The first transform (14) is obtained inductively by applying the Fourier-Mukai functor to the defining sequence (12). The base case $r = 1$ is obvious. Similarly (15) can be derived using sequence (13). An alternate argument starts by noticing

$$W_{r,p} = W_r \otimes M$$

where

$$M^r = \mathcal{O}_f(p - o).$$

The line bundle M corresponds to a point $m \in f$. Then $rm = p$ holds in the group law of the fiber. Using the properties of the Fourier-Mukai transform [Muk], we obtain

$$\widehat{W}_{r,p} = t_m^* \widehat{W}_r = t_m^* \mathcal{O}_f(-r \cdot o) = \mathcal{O}_f(-r \cdot [-m]) = \mathcal{O}_f(-(r+1) \cdot o + p).$$

Equations (14), (15), (16), and (17) also hold for the singular nodal fibers; this is explained by Lemma 2.13, Definition 2.15, and Remark 2.17 in [BK]. Note that the transforms in [BK] are stated for the functor $\mathbb{T}_{Y \rightarrow X}$, but the results for the functor $\mathbb{S}_{X \rightarrow Y}$ follow via (11).

Proof of Proposition 1. We will first check that the isomorphism

$$(18) \quad \mathbf{S}_{X \rightarrow Y}(E_r^\vee)[1] = I_{\tilde{Z}}(r\sigma + 2rf)$$

holds fiberwise. Derived restriction to fibers commutes with Fourier-Mukai [Br], and Lemma 1 gives the restriction of E_r^\vee to each fiber. The Fourier-Mukai transform of the restriction to a general fiber is

$$\mathcal{O}_f(r \cdot o)[-1],$$

by (16). For a special fiber f containing a point $p \in Z$, equation (17) yields the transform

$$\mathcal{O}_f(r \cdot o - \iota^*p)[-1] \oplus \mathcal{O}_{\iota^*p}[-1].$$

The two formulas above give precisely the derived restriction of $I_{\tilde{Z}}(r\sigma + 2rf)[-1]$. We have therefore checked that the proposition holds on every fiber.

Since both sides of (18) are sheaves of rank 1, we complete the proof by checking equality of determinants. This is indeed sufficient. By Lemma 1 the restriction of E_r^\vee to each fiber contains no subbundles of positive slope. Therefore, Proposition 3.7 in [BH] guarantees that the Fourier-Mukai transform of E_r^\vee is torsion free. By the equality of determinants, the left hand side of (18) must take the form

$$\mathbf{S}_{X \rightarrow Y}(E_r^\vee)[1] = I_U(r\sigma + 2rf)$$

for some zero dimensional subscheme U . The restriction to fibers forces $U = \tilde{Z}$, as claimed.

Since X is simply connected, to match the determinants in (18) it is enough to prove that the first Chern classes of both sides agree. In general, let V be a rank r sheaf of Euler characteristic χ and

$$c_1(V) = l\sigma + mf.$$

Then, by Grothendieck-Riemann-Roch, we have

$$c_1(\mathbf{S}_{X \rightarrow Y}(V)) = q!(p^*\text{ch}(V) \cdot \text{td}(X \times_{\mathbb{P}^1} Y/Y) \cdot \text{ch}\mathcal{P})_{(2)}.$$

The Chern character of V is

$$\text{ch}(V) = r + (l\sigma + mf) + (\chi - 2r)\omega,$$

where ω is the class of a point. Moreover,

$$\text{td}(X \times_{\mathbb{P}^1} Y/Y) = p^*(1 - f + 2\omega).$$

Hence

$$\begin{aligned} c_1(\mathbf{S}_{X \rightarrow Y}(V)) &= rc_1(\mathbf{S}_{X \rightarrow Y}(\mathcal{O})) + (\chi - 2r - l)q!(p^*\omega) + q!(p^*(l\sigma + mf)c_1(\mathcal{P})) \\ &= -r\sigma + (\chi - 2r - l)f + 2lf \end{aligned}$$

$$= -r\sigma + (\chi - 2r + l)f.$$

For $V = E_r^\vee$ the Chern class calculation gives

$$c_1(\mathbf{S}_{X \rightarrow Y}(E_r^\vee)) = -r\sigma - 2rf,$$

which proves the first isomorphism. The calculation also shows that

$$c_1(\mathbf{S}_{X \rightarrow Y}(E_r^\vee(nf))) = c_1(\mathbf{S}_{X \rightarrow Y}(E_r^\vee)) - nf.$$

The claim about twisting by fibers follows by repeating the above argument for $E_r^\vee(nf)$ and comparing determinants. \square

Proof of Proposition 2. We consider the Fourier-Mukai functor

$$\mathbf{T}_{X \rightarrow Y} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$$

with kernel

$$\mathcal{Q} = \mathcal{P}^\vee.$$

By duality, for all $x \in \mathbf{D}(X)$, we have

$$\mathbf{S}_{X \rightarrow Y}(x)^\vee = \mathbf{T}_{X \rightarrow Y}(x^\vee \otimes \omega_{X/\mathbb{P}^1})[-1] = \mathbf{T}_{X \rightarrow Y}(x^\vee \otimes \mathcal{O}(2f))[-1].$$

Thus, the proposition follows once we establish that

$$\mathbf{T}_{X \rightarrow Y}(E_r^\vee(2f))[-1] = I_Z \otimes \mathcal{O}(r\sigma + 2(r-1)f).$$

The proof of this fact is similar to that of Proposition 1. First, the equality is checked fiberwise using Lemma 1. The Grothendieck-Riemann-Roch calculation completes the argument.

Remark 1. The derived dual of the ideal sheaf I_Z can be computed explicitly for generic schemes Z . We include this calculation for completeness, even though it is not necessary for the proofs of the main theorems.

First, O'Grady's construction gives rise to a rank 2 sheaf E_2 together with an exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow E_2 \rightarrow I_Z(\sigma + (a-2)f) \rightarrow 0.$$

Note that E_2 is locally free by Lemma 2 below. Setting

$$\mathbf{C}_Z = [E_2 \rightarrow \mathcal{O}(\sigma + (a-2)f)],$$

we claim

$$(19) \quad I_Z^\vee = \mathbf{C}_Z.$$

In particular, this implies that

$$(20) \quad \mathbf{S}_{X \rightarrow Y}(E_r) = \mathbf{C}_Z \otimes \mathcal{O}(-r\sigma - 2(r-1)f).$$

To prove (19), we dualize the sequence

$$0 \rightarrow \mathcal{O} \rightarrow E_2 \rightarrow I_Z \otimes \mathcal{O}(\sigma + (a-2)f) \rightarrow 0.$$

We obtain

$$0 \rightarrow \mathcal{O}(-\sigma - (a-2)f) \rightarrow E_2^\vee \rightarrow \mathcal{O} \rightarrow \mathcal{E}xt^1(I_Z \otimes \mathcal{O}(\sigma + (a-2)f), \mathcal{O}) \rightarrow 0.$$

It is well-known, see [F] page 41, that

$$\mathcal{E}xt^1(I_Z \otimes \mathcal{O}(\sigma + (a-2)f), \mathcal{O}) = \mathcal{O}_Z$$

hence the exact sequence yields

$$0 \rightarrow \mathcal{O}(-\sigma - (a-2)f) \rightarrow E_2^\vee \rightarrow I_Z \rightarrow 0.$$

Equation (19) follows from here. \square

Lemma 2. *If Z contains no two points in the same fiber, then the associated sheaf E_2 is locally free.*

Proof. Consider the divisor Q of subschemes in $X^{[a]}$ containing two points in the same fiber. Let

$$\mathcal{D} \hookrightarrow \mathfrak{M}_2^a$$

be the codimension 1 locus of nonlocally free sheaves in the rank 2 moduli space. Lemma 4.41 of [Y2] calculates

$$\mathcal{O}(\mathcal{D}) = \Theta_w \text{ on } \mathfrak{M}_2^a,$$

for the Mukai vector

$$w = (2, -\sigma - (a-2)f, (a-2)\omega).$$

Using now the formulas in [OG1], page 27, and (22) below, we have

$$\Theta_w = \mathcal{O}(Q)$$

under the identification

$$X^{[a]} \dashrightarrow \mathfrak{M}_2^a.$$

Finally, we will remark in (24) below that the line bundle $\mathcal{O}(Q) \rightarrow X^{[a]}$ has a unique section, hence \mathcal{D} and Q coincide as claimed. \square

2.3. Line bundles and theta divisors over the Hilbert scheme of points. The birational isomorphism (8) allows us to identify the Picard group of \mathfrak{M}_r^a with that of the Hilbert scheme $X^{[a]}$.

For any smooth projective surface X and any line bundle L on it, we indicate by $L_{(a)}$ the line bundle on $X^{[a]}$ induced from the symmetric line bundle $L^{\boxtimes a}$ on the product $X \times \dots \times X$. Letting p and q be the projections

$$p : X^{[a]} \times X \rightarrow X^{[a]}, \quad q : X^{[a]} \times X \rightarrow X,$$

and letting $\mathcal{O}_{\mathcal{Z}}$ denote the universal structure sheaf on $X^{[a]} \times X$, we further set

$$(21) \quad L^{[a]} = \det p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^*L).$$

It is well known that the line bundles $L_{(a)}$ for $L \in \text{Pic } X$ and $M = \mathcal{O}^{[a]}$ generate the Picard group of $X^{[a]}$, cf. [EGL], and that for any L on X ,

$$L^{[a]} = L_{(a)} \otimes M.$$

We have, for instance,

$$\mathcal{O}(S) = \mathcal{O}(\sigma)_{(a)},$$

and

$$(22) \quad \mathcal{O}(Q) = \mathcal{O}((a-1)f)^{[a]}.$$

We note for future use the formulas of [EGL],

$$(23) \quad h^0(X^{[a]}, L_{(a)}) = \binom{h^0(X, L) + a - 1}{a}, \quad h^0(X^{[a]}, L^{[a]}) = \binom{h^0(X, L)}{a}.$$

To illustrate, we compute

$$(24) \quad h^0(X^{[a]}, \mathcal{O}(Q)) = h^0(X^{[a]}, \mathcal{O}((a-1)f)^{[a]}) = \binom{h^0(\mathcal{O}((a-1)f))}{a} = 1.$$

Consider now two Hilbert schemes of points $X^{[a]}$ and $X^{[b]}$, and the rational morphism, defined away from codimension 2,

$$(25) \quad \tau : X^{[a]} \times X^{[b]} \dashrightarrow X^{[a+b]}, \quad (I_Z, I_W) \mapsto I_Z \otimes I_W.$$

Assume that L is a line bundle on X with no higher cohomology, and such that

$$\chi(L) = h^0(L) = a + b.$$

From (23), we note that

$$h^0(X^{[a+b]}, L^{[a+b]}) = \binom{h^0(X, L)}{a+b} = 1.$$

The unique section of $L^{[a+b]}$ vanishes on the locus

$$(26) \quad \theta_L = \{I_V \in X^{[a+b]}, \text{ such that } H^0(I_V \otimes L) \neq 0\},$$

whose pullback under τ is the divisor

$$\theta_{L,a,b} = \{(I_Z, I_W) \in X^{[a]} \times X^{[b]} \text{ such that } H^0(I_Z \otimes I_W \otimes L) \neq 0\}.$$

We furthermore have

$$(27) \quad \mathcal{O}(\theta_{L,a,b}) = \tau^* L^{[a+b]} = L^{[a]} \boxtimes L^{[b]} \text{ on } X^{[a]} \times X^{[b]}.$$

It was observed in [MO1] that $\theta_{L,a,b}$ induces an isomorphism

$$(28) \quad \mathbb{D} : H^0(X^{[a]}, L^{[a]})^\vee \rightarrow H^0(X^{[b]}, L^{[b]}).$$

The discussion of this subsection has been carried out for arbitrary smooth projective surfaces. We revert now to the original context of an elliptically fibered $K3$ with section and Picard rank 2, for which the line bundle L takes the form

$$L = \mathcal{O}(m\sigma + nf).$$

It will be important for our arguments to consider the following partial reflection of the divisor $\theta_{L,a,b}$:

$$\tilde{\theta}_{L,a,b} = \{(Z, W) \in X^{[a]} \times X^{[b]} \text{ such that } h^0(I_Z \otimes I_{\widetilde{W}} \otimes L) \neq 0\}.$$

As usual, the subschemes

$$\widetilde{Z} = \iota^* Z, \quad \widetilde{W} = \iota^* W$$

are obtained from the fiberwise reflection $\iota : X \dashrightarrow X$. There is a seeming asymmetry in the roles of Z and W in the definition of $\tilde{\theta}_{L,a,b}$, but in fact we also have

$$\tilde{\theta}_{L,a,b} = \{(Z, W) \in X^{[a]} \times X^{[b]} \text{ such that } h^0(I_{\widetilde{Z}} \otimes I_W \otimes L) \neq 0\}.$$

To explain this equality, note first that the line bundle L is invariant under ι

$$\iota^* L = L.$$

Hence, so are the tautological line bundles $L^{[a]}$, $L^{[b]}$ and $L^{[a+b]}$. On $X^{[a+b]}$, the divisor θ_L of (26) corresponds to the unique section of $L^{[a+b]}$, therefore must be invariant under ι as well,

$$\iota^* \theta_L = \theta_L.$$

The same is then true for the pullback

$$\theta_{L,a,b} = \tau^* \theta_L,$$

which implies that

$$h^0(I_Z \otimes I_{\widetilde{W}} \otimes L) = 0 \text{ if and only if } h^0(I_{\widetilde{Z}} \otimes I_W \otimes L) = 0.$$

The above discussion also shows that $\tilde{\theta}_{L,a,b}$ is a section of the line bundle $L^{[a]} \boxtimes L^{[b]}$ and that furthermore it induces an isomorphism

$$(29) \quad \widetilde{\mathbb{D}} : H^0(X^{[a]}, L^{[a]})^\vee \rightarrow H^0(X^{[b]}, L^{[b]}).$$

2.4. The strange duality setup. We now place ourselves in the setting of Theorems 2, 3 and 4 *i.e.*, we take X to be an elliptically fibered $K3$ surface with section, and consider two moduli spaces of sheaves \mathfrak{M}_v and \mathfrak{M}_w with orthogonal Mukai vectors satisfying

- (i) $\langle v, w^\vee \rangle = 0$,
- (ii) $c_1(v) \cdot f = c_1(w) \cdot f = 1$,
- (iii) $\langle v, v \rangle + \langle w, w \rangle \geq 2(r+s)^2$.

Equivalently, we consider two normalized moduli spaces \mathfrak{M}_r^a and \mathfrak{M}_s^b such that

$$(30) \quad r+s \mid a+b-2, \text{ and moreover } -\nu \stackrel{\text{def}}{=} \frac{a+b-2}{r+s} - (r+s-2) \geq 2.$$

We also assume that $r, s \geq 2$. The divisibility condition and the definition of ν are so as to ensure that

$$\chi(E_r \cdot F_s(\nu f)) = 0, \text{ for sheaves } E_r \in \mathfrak{M}_r^a, F_s \in \mathfrak{M}_s^b.$$

Furthermore, the stability condition implies that

$$H^2(E_r \otimes F_s(\nu f)) = 0.$$

The vanishing

$$\text{Tor}^1(E_r, F_s) = \text{Tor}^2(E_r, F_s) = 0$$

is satisfied when E_r or F_s are locally free, which occurs away from codimension 2 in the product space, cf. Proposition 0.5 in [Y1].

We write

$$\Theta_{r,s} = \{(E_r, F_s) \in \mathfrak{M}_r^a \times \mathfrak{M}_s^b \text{ such that } h^0(E_r \otimes F_s(\nu f)) \neq 0\}.$$

The condition defining $\Theta_{r,s}$ is divisorial, but it is not a priori clear that this locus actually has codimension 1. Nonetheless, using the explicit formulas in Section I.6 of [OG1], the line bundle $\mathcal{O}(\Theta_{r,s})$ on $\mathfrak{M}_r^a \times \mathfrak{M}_s^b$ can be expressed on the product $X^{[a]} \times X^{[b]}$ via the birational map

$$(\Phi_r, \Phi_s) : X^{[a]} \times X^{[b]} \dashrightarrow \mathfrak{M}_r^a \times \mathfrak{M}_s^b.$$

Letting

$$(31) \quad L = \mathcal{O}((r+s)\sigma + (2(r+s) - 2 - \nu)f) \text{ on } X,$$

it was shown in [MO1] that

$$(32) \quad \mathcal{O}(\Theta_{r,s}) = L^{[a]} \boxtimes L^{[b]}.$$

We will prove that

Theorem 4. $\Theta_{r,s} = \tilde{\theta}_{L,a,b}$ on $X^{[a]} \times X^{[b]}$.

2.5. The theta divisor over the generic locus. In this section and the one following it, we prove Theorems 2, 3 and 4.

We first identify the theta divisor $\Theta_{r,s}$ on the locus corresponding to generic Z and W . Our genericity assumptions were specified in (i)-(iv) of Section 2.2. On any Hilbert scheme of points of X , we consider then the following:

- (i) the divisor R of non-reduced subschemes,
- (ii) the divisor Q of subschemes intersecting a fiber in more than one point,
- (iii) the divisor S of subschemes which intersect the section.

Recall that along the divisors S and Q the extensions (6) have unstable middle terms needing to undergo semi-stable reduction.

We single out here only the nongeneric loci corresponding to divisors, as for our purposes we can ignore higher codimension phenomena. Thus we will disregard the loci corresponding to

- (iv) schemes whose supports contain singular points of fibers of X .

We work with the rational morphism

$$\tau : X^{[a]} \times X^{[b]} \dashrightarrow X^{[a+b]},$$

and we will pullback the divisors R , S and Q to the product of Hilbert schemes and of moduli spaces \mathfrak{M}_r^a and \mathfrak{M}_s^b . We set

$$\mathfrak{M} = \mathfrak{M}_r^a \times \mathfrak{M}_s^b \setminus (\tau^*R \cup \tau^*Q \cup \tau^*S).$$

For $(E_r, F_s) \in \mathfrak{M}$, we prove

$$(33) \quad H^0(E_r \otimes F_s(\nu f)) = H^1(I_{\tilde{Z}} \otimes I_W \otimes L)^\vee.$$

In particular, this shows

$$(34) \quad \Theta_{r,s} \setminus (\tau^*R \cup \tau^*Q \cup \tau^*S) = \tilde{\theta}_{L,a,b} \setminus (\tau^*R \cup \tau^*Q \cup \tau^*S).$$

To establish (33), we use the Fourier-Mukai functor $\mathbf{S}_{X \rightarrow Y}$ defined in (10), as well as Propositions 1 and 2. We calculate

$$\begin{aligned} H^0(E_r \otimes F_s(\nu f)) &= \mathrm{Hom}_{\mathbf{D}(X)}(E_r^\vee(-\nu f), F_s) \\ &= \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{S}_{X \rightarrow Y}(E_r^\vee(-\nu f)), \mathbf{S}_{X \rightarrow Y}(F_s)) \\ &= \mathrm{Hom}_{\mathbf{D}(Y)}(I_{\tilde{Z}}(r\sigma + 2rf - \nu f), I_W^\vee \otimes \mathcal{O}(-s\sigma - 2(s-1)f)[1]) \\ &= \mathrm{Ext}^1(I_{\tilde{Z}} \otimes L, I_W^\vee) \\ &= \mathrm{Ext}^1(I_W^\vee, I_{\tilde{Z}} \otimes L)^\vee \\ &= H^1(I_W \otimes I_{\tilde{Z}} \otimes L)^\vee. \end{aligned}$$

Note that this calculation establishes Theorem 3. \square

2.6. The nongeneric locus. In order to complete the proof of Theorem 4, we will need to analyze the overlaps of the theta divisor with R , Q and S .

First, as the divisors R, Q, S are invariant under ι , we write (34) equivalently as

$$(35) \quad \tilde{\Theta}_{r,s} \setminus (\tau^*R \cup \tau^*Q \cup \tau^*S) = \theta_{L,a,b} \setminus (\tau^*R \cup \tau^*Q \cup \tau^*S),$$

where $\tilde{\Theta}_{r,s}$ is the partial reflection of the divisor $\Theta_{r,s}$ obtained by acting with the involution ι on one of the factors.

We write

$$\tilde{\Theta}_{r,s} = \Gamma \cup \Delta,$$

where Γ and Δ are divisors such that the intersection

$$\Delta \cap (\tau^*Q \cup \tau^*R \cup \tau^*S)$$

is proper, and

$$\text{support } \Gamma \subset \tau^*Q \cup \tau^*R \cup \tau^*S.$$

Equation (35) shows in particular that Δ is a pullback divisor under τ ,

$$\Delta = \tau^*\Delta_0 \text{ for } \Delta_0 \subset X^{[a+b]}.$$

Since

$$\mathcal{O}(\tilde{\Theta}_{r,s}) = L^{[a]} \boxtimes L^{[b]} = \tau^*L^{[a+b]},$$

we have

$$(36) \quad \mathcal{O}(\Gamma) = \tau^*(L^{[a+b]} \otimes \mathcal{O}(-\Delta_0)).$$

More strongly, we will show shortly that (36) implies that

Claim 1. Γ as a divisor is a pullback under the morphism τ ,

$$\Gamma = \tau^*\Gamma_0.$$

As a consequence,

$$\tilde{\Theta}_{r,s} = \tau^*(\Delta_0 \cup \Gamma_0).$$

Now

$$\theta_L = \{V \text{ such that } h^0(I_V \otimes L) \neq 0\}$$

is the only section of $L^{[a+b]}$ on $X^{[a+b]}$. Thus we must have that

$$\theta_L = \Delta_0 \cup \Gamma_0, \text{ and}$$

$$\tilde{\Theta}_{r,s} = \tau^*\theta_L = \{(I_Z, I_W) \text{ such that } h^0(I_Z \otimes I_W \otimes L) \neq 0\} = \theta_{L,a,b}.$$

This completes the proof of Theorem 4. Theorem 2 follows as well via (29). \square

Proof of Claim 1. We will consider the three divisors Q , R and S over the Hilbert schemes $X^{[a]}$, $X^{[b]}$ or $X^{[a+b]}$. All these divisors are irreducible. Let

$$\begin{aligned}\tau^*Q &= Q_1 \cup Q_2 \cup Q_3, & \tau^*R &= R_1 \cup R_2, \\ \tau^*S &= S_1 \cup S_2\end{aligned}$$

be the irreducible components of the pullbacks on the product $X^{[a]} \times X^{[b]}$. Here

$$Q_1 = Q \times X^{[b]}, \quad Q_2 = X^{[a]} \times Q,$$

while Q_3 is the divisor of cycles $(I_Z, I_W) \in X^{[a]} \times X^{[b]}$ such that Z, W intersect the same elliptic fiber. In the same fashion, we have

$$\begin{aligned}R_1 &= R \times X^{[b]}, R_2 = X^{[a]} \times R, \\ S_1 &= S \times X^{[b]}, S_2 = X^{[a]} \times S.\end{aligned}$$

Note first that $\tilde{\Theta}_{r,s}$ does not contain the divisor Q_3 . Indeed,

$$\mathcal{O}(Q) = \mathcal{O}((a+b-1)f)^{[a+b]} \text{ on } X^{[a+b]},$$

so

$$\tau^*\mathcal{O}(Q) = \mathcal{O}((a+b-1)f)^{[a]} \boxtimes \mathcal{O}((a+b-1)f)^{[b]} \text{ on } X^{[a]} \times X^{[b]}.$$

As

$$\mathcal{O}(Q_1) = \mathcal{O}((a-1)f)^{[a]} \boxtimes \mathcal{O} \quad \text{and} \quad \mathcal{O}(Q_2) = \mathcal{O} \boxtimes \mathcal{O}((b-1)f)^{[b]},$$

it follows that

$$\mathcal{O}(Q_3) = \mathcal{O}(bf)_{(a)} \boxtimes \mathcal{O}(af)_{(b)}.$$

Assuming $\tilde{\Theta}_{r,s}$ contained Q_3 , we would have

$$(37) \quad H^0(X^{[a]} \times X^{[b]}, \mathcal{O}(\tilde{\Theta}_{r,s} - Q_3)) \neq 0.$$

However, we will show that (37) is false. Indeed,

$$\mathcal{O}(\tilde{\Theta}_{r,s} - Q_3) = L(-bf)^{[a]} \boxtimes L(-af)^{[b]}.$$

From (23), we have

$$\begin{aligned}h^0(L(-bf)^{[a]}) &= \binom{h^0(L(-bf))}{a}, \\ h^0(L(-af)^{[b]}) &= \binom{h^0(L(-af))}{b}.\end{aligned}$$

It suffices to explain that either

$$(38) \quad h^0(L(-bf)) = 0 \text{ or } h^0(L(-af)) = 0.$$

On the surface X , we generally have

$$(39) \quad h^0(X, \mathcal{O}(m\sigma + nf)) = \begin{cases} 0, & \text{if } m \geq 0, n < 0 \\ 2 + m(n-m), & \text{if } m > 0, n \geq 2m \end{cases}.$$

The first dimension count is immediate as

$$h^0(X, \mathcal{O}(m\sigma)) = 1$$

for all $m \geq 0$, and the second holds as in that case $\mathcal{O}(m\sigma + nf)$ is big and nef, so has no higher cohomology. Now, recall that

$$L = \mathcal{O} \left((r+s)\sigma + \left(r+s + \frac{a+b-2}{r+s} \right) f \right) \text{ on } X.$$

The numerical constraint (30)

$$a+b \geq (r+s)^2 + 2$$

ensures that either $L(-af)$ or $L(-bf)$ has a negative number of fiber classes. This proves (38) using the dimension count (39).

Similarly, $\tilde{\Theta}_{r,s}$ cannot contain both Q_1 and Q_2 . Indeed, we calculate

$$\mathcal{O}(\tilde{\Theta}_{r,s} - Q_1 - Q_2) = L((-a+1)f)_{(a)} \boxtimes L((-b+1)f)_{(b)}.$$

As in (38), unless

$$r = s = 2, a = b = 9,$$

we have

$$(40) \quad h^0(L((-a+1)f)) = 0 \text{ or } h^0(L((-b+1)f)) = 0,$$

therefore (23) implies that $\mathcal{O}(\tilde{\Theta}_{r,s} - Q_1 - Q_2)$ has no sections.

Let us write

$$\Gamma = q_1Q_1 + q_2Q_2 + q_3Q_3 + r_1R_1 + r_2R_2 + s_1S_1 + s_2S_2.$$

The above argument shows that $q_3 = 0$ and that we can assume without loss of generality $q_2 = 0$. We calculate

$$\begin{aligned} \mathcal{O}(Q_1) &= \mathcal{O}((a-1)f)^{[a]} \boxtimes \mathcal{O}, \\ \mathcal{O}(R_1) &= M^{-2} \boxtimes \mathcal{O}, \quad \mathcal{O}(R_2) = \mathcal{O} \boxtimes M^{-2}, \\ \mathcal{O}(S_1) &= \mathcal{O}(\sigma)_{(a)} \boxtimes \mathcal{O}, \quad \mathcal{O}(S_2) = \mathcal{O} \boxtimes \mathcal{O}(\sigma)_{(b)}, \end{aligned}$$

Consequently,

$$(41) \quad \mathcal{O}(\Gamma) = \left(\mathcal{O}(q_1(a-1)f + s_1\sigma)_{(a)} \otimes M^{q_1-2r_1} \right) \boxtimes \left(\mathcal{O}(s_2\sigma)_{(b)} \otimes M^{-2r_2} \right).$$

From (36) we know that this line bundle is a pullback under τ . This strongly constrains the coefficients in the expression (41). In fact, via the isomorphism

$$\text{Pic}(X^{[n]}) = \text{Pic}(X) \oplus \mathbb{Z}, \quad L_{(n)} \otimes M^r \mapsto (L, r),$$

the image

$$\tau^* : \text{Pic}(X^{[a+b]}) \rightarrow \text{Pic}(X^{[a]}) \times \text{Pic}(X^{[b]})$$

corresponds to the diagonal embedding. Therefore in (41) we must have

$$q_1 = 0, s_1 = s_2, q_1 - 2r_1 = -2r_2.$$

Hence

$$\Gamma = r_1(R_1 + R_2) + s_1(S_1 + S_2)$$

is a pullback of the divisor $\Gamma_0 = r_1R + s_1S$. This establishes Claim 1 unless

$$r = s = 2, a = b = 9.$$

In this latter case, the dimension calculated in (40) is

$$h^0(L((-a+1)f)) = h^0(L((-b+1)f)) = h^0(\mathcal{O}(4\sigma)) = 1,$$

and

$$\mathcal{O}(\tilde{\Theta}_{r,s} - Q_1 - Q_2) = \mathcal{O}(4\sigma)_{(a)} \boxtimes \mathcal{O}(4\sigma)_{(b)}$$

has a unique section supported on $S_1 \cup S_2$. Thus if Q_1, Q_2 are both contained in $\tilde{\Theta}_{r,s}$, then $\tilde{\Theta}_{r,s}$ is supported in $Q_1 \cup Q_2 \cup S_1 \cup S_2$. This implies via (35) that

$$\theta_L \setminus (R \cup Q \cup S) = \emptyset.$$

However, the following remark shows that this is not possible. \square

Remark 2. We note here that in fact θ_L on $X^{[a+b]}$ intersects both S and Q properly, therefore $\tilde{\Theta}_{r,s}$ intersects τ^*S and τ^*Q properly. Otherwise, we would have

$$H^0(X^{[a+b]}, L^{[a+b]} \otimes \mathcal{O}(-S)) \neq 0,$$

$$H^0(X^{[a+b]}, L^{[a+b]} \otimes \mathcal{O}(-Q)) \neq 0.$$

We calculate

$$L^{[a+b]} \otimes \mathcal{O}(-S) = L(-\sigma)^{[a+b]}.$$

From the dimension count (39) we have

$$h^0(L(-\sigma)) = a + b + 1 + \nu,$$

and therefore from (23),

$$h^0(L^{[a+b]} \otimes \mathcal{O}(-S)) = \binom{h^0(L(-\sigma))}{a+b} = \binom{a+b+1+\nu}{a+b} = 0.$$

Similarly,

$$L^{[a+b]} \otimes \mathcal{O}(-Q) = L((-a-b+1)f)_{(a+b)}$$

has no sections on $X^{[a+b]}$ since $L((-a-b+1)f)$ has no sections on X .

2.7. Arbitrary simply connected elliptic surfaces. Theorem 5 will be proved in this section. First, we write down O'Grady's construction for arbitrary simply connected elliptic surfaces with section, and then we reinterpret it via Fourier-Mukai transforms.

The holomorphic Euler characteristic of the fibration

$$\pi : X \rightarrow \mathbb{P}^1$$

will be denoted

$$\chi = \chi(\mathcal{O}) = 1 + h^2(\mathcal{O}_X) > 0.$$

We study normalized moduli spaces of sheaves \mathfrak{M}_v such that

$$\chi(v) = 1 \implies c_1(v) = \sigma + \left(a - \frac{r(r-1)}{2} \chi \right) f,$$

where we write $2a$ for the dimension of \mathfrak{M}_v . A birational isomorphism

$$\Phi_v : X^{[a]} \dashrightarrow \mathfrak{M}_v$$

is constructed as follows. As in the case of $K3$ surfaces, we consider generic schemes Z of length a , satisfying the requirements (i)-(iv) of section 2.2. We set

$$E_1 = I_Z(\sigma + af).$$

Inductively, we construct nontrivial extensions

$$(42) \quad 0 \rightarrow \mathcal{O} \rightarrow E_{r+1} \rightarrow E_r(-\chi f) \rightarrow 0.$$

Several statements are to be proved simultaneously during the induction step:

- (a) $\text{Ext}^0(E_r(-\chi f), \mathcal{O}) = 0$
- (b) $\text{Ext}^2(E_r(-\chi f), \mathcal{O}) = 0$.
- (c) $\text{Ext}^1(E_r(-\chi f), \mathcal{O}) \cong \mathbb{C}$. This shows that the extension (42) is unique.
- (d) the restriction of E_r to the generic fiber is the Atiyah bundle W_r . This implies the stability of E_r with respect to suitable polarizations. For special fibers through $p \in Z$, the restriction splits as $W_{r-1,p} \oplus \mathcal{O}_f(o-p)$.

Checking (a)-(d) for the base case $r = 1$ uses the requirements (i)-(iv) of section 2.2. We briefly explain the inductive step. The first vanishing in (a) follows by stability since for polarizations $H = \sigma + mf$ with $m \gg 0$, we have

$$\frac{c_1(E_r(-\chi f)) \cdot H}{r} > 0.$$

Regarding (b), we consider the exact sequence induced by (42)

$$\text{Ext}^2(E_r(-2\chi f), \mathcal{O}) \rightarrow \text{Ext}^2(E_{r+1}(-\chi f), \mathcal{O}) \rightarrow \text{Ext}^2(\mathcal{O}(-\chi f), \mathcal{O}) = 0.$$

Now (b) follows since the leftmost term also vanishes, as one can see by considering the injection

$$\mathrm{Ext}^2(E_r(-2\chi f), \mathcal{O}) \hookrightarrow \mathrm{Ext}^2(E_r(-\chi f), \mathcal{O}) = 0.$$

Now, (a) and (b) imply (c) via a Riemann-Roch calculation. Finally, (d) is argued exactly as Lemma 1 above.

We use (d) to calculate Fourier-Mukai transforms

$$\mathbf{S}_{X \rightarrow Y} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y).$$

By the arguments of section 2.2, we obtain

- (i) $\mathbf{S}_{X \rightarrow Y}(E_r^\vee) = I_{\tilde{Z}}(r\sigma + r\chi f)[-1]$
- (ii) $\mathbf{S}_{X \rightarrow Y}(E_r) = I_{\tilde{Z}}^\vee \otimes \mathcal{O}(-r\sigma - (r-1)\chi f)$.

Consider now two complementary moduli spaces \mathfrak{M}_v and \mathfrak{M}_w . After twisting by fiber classes, we may assume v and w are normalized. Consider the theta locus

$$\Theta = \{(E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w : h^0(E \otimes F \otimes \mathcal{O}(\nu f)) = 0\}$$

where

$$-\nu = \frac{a+b-\chi}{r+s} - (r+s-1)\frac{\chi}{2} + 1 \geq \chi,$$

by the condition (ii) of Theorem 5. We set

$$L = \mathcal{O}_X((r+s)\sigma + ((r+s-1)\chi - \nu)f) \otimes K_X.$$

An easy calculation shows

$$h^0(L) = \chi(L) = a + b,$$

and that L has no higher cohomology. We therefore obtain a divisor

$$\theta_{L,a,b} \subset X^{[a]} \times X^{[b]},$$

and the associated twist

$$\tilde{\theta}_L = (1 \times i)^* \theta_L = (i \times 1)^* \theta_L$$

in the product of Hilbert schemes.

Repeating the argument for elliptic $K3$ s, we obtain that under the birational map

$$\Phi_v \times \Phi_w : X^{[a]} \times X^{[b]} \dashrightarrow \mathfrak{M}_v \times \mathfrak{M}_w$$

we have

$$(43) \quad (\Phi_v \times \Phi_w)^* \Theta \subset \tilde{\theta}_L,$$

at least along the generic locus. This is enough to establish that Θ is a divisor.

As proved in Section 5 of [BH], Φ_v and Φ_w as well as their inverses are defined away from codimension 2 via Fourier-Mukai transforms. This allows us to conclude

equality in (43), first along the generic locus, and then over the entire moduli spaces. This establishes Theorem 5. \square

3. GENERIC STRANGE DUALITY

In this section we prove Theorem 1 by considering the strange duality map relatively over the moduli space of primitively polarized $K3$ surfaces of fixed degree. We first establish the isomorphism at points in the moduli space corresponding to elliptic $K3$ surfaces polarized by a numerical section. Then we show that the isomorphism holds generically.

Let (X_0, H_0) be a polarized elliptic $K3$ with section and Picard rank 2, and assume H_0 is a numerical section. Let

$$v_0 = (r, H_0, \chi - r), \quad w_0 = (r', H_0, \chi' - r)$$

be two orthogonal Mukai vectors, and consider the moduli spaces \mathfrak{M}_{v_0} and \mathfrak{M}_{w_0} of H_0 -semistable sheaves of type v_0 and w_0 .

Proposition 3. *The duality morphism*

$$(44) \quad D_0 : H^0(\mathfrak{M}_{v_0}, \Theta_{w_0})^\vee \rightarrow H^0(\mathfrak{M}_{w_0}, \Theta_{v_0})$$

is an isomorphism, if $(r, s) \neq (2, 2)$ and the dimension inequalities

$$(45) \quad \langle v_0, v_0 \rangle \geq 2(r-1)(r^2+1), \quad \langle w_0, w_0 \rangle \geq 2(s-1)(s^2+1)$$

hold.

When $r = s = 2$, the same conclusion is true provided $H_0^2 \geq 8$.

Proof. Consider a polarization H_+ suitable with respect to both v_0 and w_0 , and write $\mathfrak{M}_{v_0}^+$ and $\mathfrak{M}_{w_0}^+$ for the moduli spaces of H_+ -semistable sheaves over X_0 . Theorem 2 ensures that

$$(46) \quad D_0^+ : H^0(\mathfrak{M}_{v_0}^+, \Theta_{w_0})^\vee \rightarrow H^0(\mathfrak{M}_{w_0}^+, \Theta_{v_0})$$

is an isomorphism. Assumption (45) is stronger than what is needed to apply Theorem 2, provided $(r, s) \neq (2, 2)$. When $r = s = 2$, Theorem 2 also holds if $H_0^2 \geq 8$.

However, (45) allows us to apply Corollary 1 and Remark 3 of the Appendix. Thus the semistable moduli *stacks* do not depend on the choice of polarization away from codimension 2 loci. Therefore, spaces of sections of theta bundles on the moduli stacks are unaffected by the change of polarization. The translation to the moduli *schemes* is straightforward, as lifting sections from the moduli scheme to the moduli stack is an isomorphism, by Proposition 8.4 in [BL].

To spell out the details, write \mathbf{M}_{v_0} and \mathbf{M}_{w_0} for the moduli stacks of H_0 -semistable sheaves, and consider the theta bundles

$$\Theta_{w_0} \rightarrow \mathbf{M}_{w_0}, \quad \Theta_{v_0} \rightarrow \mathbf{M}_{v_0}.$$

Note the morphisms to the moduli schemes

$$p : \mathbf{M}_{v_0} \rightarrow \mathfrak{M}_{v_0}, \quad p : \mathbf{M}_{w_0} \rightarrow \mathfrak{M}_{w_0}$$

which match the theta bundles accordingly

$$\Theta_{w_0} = p^* \Theta_{w_0}, \quad \Theta_{v_0} = p^* \Theta_{v_0}.$$

Lifting (46) to the stacks $\mathbf{M}_{v_0}^+$ and $\mathbf{M}_{w_0}^+$ of H_+ - semistable sheaves, we obtain that

$$D_0^+ : H^0(\mathbf{M}_{v_0}^+, \Theta_{w_0})^\vee \rightarrow H^0(\mathbf{M}_{w_0}^+, \Theta_{v_0})$$

is an isomorphism. In turn, by the polarization invariance of spaces of theta sections, we derive that

$$D_0 : H^0(\mathbf{M}_{v_0}, \Theta_{w_0})^\vee \rightarrow H^0(\mathbf{M}_{w_0}, \Theta_{v_0})$$

is an isomorphism as well. (44) is established descending once again to the moduli scheme. This concludes the proof of Proposition 3. \square

We next show that the higher cohomology of the theta bundles vanishes for at least one point in the moduli space of polarized $K3$ surfaces.

Proposition 4. *Let (X, H) be a polarized $K3$ with H an ample generator of $\text{Pic}(X)$. Assume that v and w are Mukai vectors such that*

- (i) $\langle v^\vee, w \rangle = 0$,
- (ii) $c_1(v) = c_1(w) = H$,
- (iii) $\chi(v) \leq 0$, $\chi(w) \leq 0$.

The line bundle $\Theta_w \rightarrow \mathfrak{M}_v$ is big and nef, hence it does not have higher cohomology.

Proof. For a Mukai vector $v = (v_0, v_2, v_4)$, define

$$\lambda_v = (0, -v_0 H, H \cdot v_2)$$

and

$$\mu_v = (-H \cdot v_2, v_4 H, 0).$$

These vectors are perpendicular to v . It was shown by Jun Li that $\Theta_{-\lambda_v}$ is big and nef [Li1]; in fact, $\Theta_{-\lambda_v}$ defines a morphism from the Gieseker to the Uhlenbeck compactification.

Using reflections along rigid sheaves, Yoshioka proved that $\Theta_{-\lambda_v - \mu_v}$ is also big and nef [Y2], and that it determines a morphism

$$\pi : \mathfrak{M}_v \rightarrow \mathfrak{X}.$$

The image \mathfrak{X} is contained in a union of moduli spaces

$$\mathfrak{X} \subset \bigcup_{k \geq -\chi(v)} \mathfrak{M}_{v_k},$$

where the Mukai vectors v_k are of the form

$$v_k = v + k \langle 1, 0, 1 \rangle.$$

The explicit construction of π is as follows. Since $c_1(v) = H$, by stability it follows that

$$H^2(E) = 0 \implies h^0(E) - h^1(E) = \chi(v) \leq 0.$$

For each $k \geq -\chi(v)$, consider the Brill-Noether locus

$$\mathfrak{M}_k = \{E : h^1(E) = k\} \hookrightarrow \mathfrak{M}_v$$

and for $E \in \mathfrak{M}_k$ construct the universal extension

$$0 \rightarrow H^1(E) \otimes \mathcal{O}_X \rightarrow \tilde{E} \rightarrow E \rightarrow 0.$$

Then, the assignment

$$\mathfrak{M}_v \ni E \mapsto \tilde{E} \in \mathfrak{X}$$

defines a birational map onto its image. In fact, the fibers of π through sheaves E in the Brill-Noether locus \mathfrak{M}_k are Grassmannians $\mathbb{G}(k, 2k + \chi(v))$.

Now, under the assumptions of the lemma, we have

$$\Theta_w^{H^2} = \Theta_{-\lambda_v}^{-\chi(w)} \otimes \Theta_{-\lambda_v - \mu_v}^s$$

which shows that Θ_w is big and nef as well. \square

At this point, we are ready to prove Theorems 1 and 1A. We consider the moduli stack \mathcal{K} of primitively polarized $K3$ surfaces (X, H) of fixed degree, and the universal family

$$(\mathcal{X}, \mathcal{H}) \rightarrow \mathcal{K}.$$

Fix integers $\chi \leq 0, \chi' \leq 0$, and ranks $r, s \geq 2$. For each $t \in \mathcal{K}$ representing a polarized $K3$ surface $(\mathcal{X}_t, \mathcal{H}_t)$, consider the Mukai vectors

$$v_t = (r, c_1(\mathcal{H}_t), \chi - r), \quad w_t = (s, c_1(\mathcal{H}_t), \chi' - s).$$

We assume that the Mukai vectors are fiberwise orthogonal

$$\langle v_t^\vee, w_t \rangle = 0 \text{ for all } t \in \mathcal{K}.$$

We form the two relative moduli spaces of \mathcal{H}_t -semistable sheaves of type v_t and w_t

$$\mathfrak{M}[v] \rightarrow \mathcal{K}, \quad \mathfrak{M}[w] \rightarrow \mathcal{K}.$$

The product

$$\mathfrak{M}[v] \times_{\mathcal{K}} \mathfrak{M}[w] \rightarrow \mathcal{K}$$

carries a relative theta divisor $\Theta[v, w]$ obtained as the vanishing locus of a section of the relative theta bundle

$$\Theta[w] \boxtimes \Theta[v] \rightarrow \mathfrak{M}[v] \times_{\mathcal{K}} \mathfrak{M}[w].$$

Pushing forward to \mathcal{K} via the natural projections π , we obtain the sheaves

$$\mathbf{V} = \pi_{\star}(\Theta[w]), \quad \mathbf{W} = \pi_{\star}(\Theta[v]),$$

as well as a section \mathbf{D} of $\mathbf{V} \otimes \mathbf{W}$.

Semicontinuity and Proposition 4 ensure that the theta bundles carry no higher cohomology over a nonempty open substack $\mathcal{K}' \hookrightarrow \mathcal{K}$. Thus the sheaf \mathbf{V} is locally free over \mathcal{K}' , with rank

$$h^0(\mathfrak{M}_{v_t}, \Theta_{w_t}) = \chi(\mathfrak{M}_{v_t}, \Theta_{w_t}) = \binom{a+b}{a},$$

where a and b are half the dimensions of the moduli spaces \mathfrak{M}_{v_t} and \mathfrak{M}_{w_t} . The calculation of the Euler characteristic in the equation above can be found in [OG2]. Similarly, \mathbf{W} is locally free over \mathcal{K}' , also of rank $\binom{a+b}{a}$.

We let \mathcal{K}° be the maximal open substack corresponding to surfaces $(\mathcal{X}_t, \mathcal{H}_t)$ with $t \in \mathcal{K}$ such that

$$h^0(\mathfrak{M}_{v_t}, \Theta_{w_t}) = h^0(\mathfrak{M}_{w_t}, \Theta_{v_t}) = \binom{a+b}{a}.$$

By Grauert's theorem, \mathbf{V} and \mathbf{W} are locally free of rank

$$\text{rank } \mathbf{V} = \text{rank } \mathbf{W} = \binom{a+b}{a}$$

over \mathcal{K}° . Furthermore, the fibers of \mathbf{V} and \mathbf{W} over $t \in \mathcal{K}^\circ$ are calculated by the spaces of global sections $H^0(\mathfrak{M}_{v_t}, \Theta_{w_t})$ and $H^0(\mathfrak{M}_{w_t}, \Theta_{v_t})$. Thus over \mathcal{K}° we obtain a morphism of vector bundles

$$\mathbf{D} : \mathbf{V}^\vee \rightarrow \mathbf{W}$$

which packages the fiberwise strange duality maps.

We claim \mathcal{K}° contains the locus of elliptic $K3$ surfaces of Picard rank 2 polarized by numerical sections. Indeed, for any such surface (X_0, H_0) , there is a birational isomorphism

$$\mathfrak{M}_{v_0} \dashrightarrow X_0^{[a]}$$

regular away from codimension 2, cf. Section 2 and the Appendix. The line bundle Θ_{w_0} corresponds to a line bundle of the form $L^{[a]}$, for some

$$L \rightarrow X_0 \text{ with } h^0(L) = a + b.$$

Hence, by (23),

$$h^0(\mathfrak{M}_{v_0}, \Theta_{w_0}) = h^0(X_0^{[a]}, L^{[a]}) = \binom{h^0(L)}{a} = \binom{a+b}{a}.$$

The argument involving W is similar.

By Proposition 3, D is an isomorphism over the locus of polarized elliptic $K3$ surfaces (X_0, H_0) of Picard rank 2, with H_0 a numerical section. The locus where D drops rank is therefore a proper closed substack of \mathcal{K}° . The strange duality map D is an isomorphism for all surfaces in the nonempty complement of this closed substack in \mathcal{K}° .

Finally, as the number of choices for (v, w) satisfying the assumptions (i)-(iii) of the Theorem is finite in each fixed degree, taking intersections of the above open substacks, we construct surfaces for which D is an isomorphism for all vectors v and w at once. \square

APPENDIX: CHANGE OF POLARIZATION FOR MODULI SPACES OF HIGHER RANK SHEAVES OVER $K3$ SURFACES

By Kota Yoshioka

Let X be a $K3$ surface, and fix a Mukai vector

$$v := (r, \xi, a) \in H^*(X, \mathbb{Z})$$

with $r > 0$. For an ample divisor H on X , denote by $\mathbf{M}(v)$, $\mathbf{M}_H(v)^{ss}$, and $\mathbf{M}_H(v)^{\mu-ss}$ the stacks of sheaves, of Gieseker H -semistable sheaves, and of slope H -semistable sheaves respectively – all of type v .

Lemma 3. *If H is general with respect to v , that is, H does not lie on a wall with respect to v , then*

$$(47) \quad \dim \mathbf{M}_H(v)^{ss} = \begin{cases} \langle v, v \rangle + 1, & \langle v, v \rangle > 0 \\ \langle v, v \rangle + l, & \langle v, v \rangle = 0, \\ \langle v, v \rangle + l^2 = -l^2, & \langle v, v \rangle < 0 \end{cases}$$

where $l = \gcd(r, \xi, a)$. In particular,

$$\dim \mathbf{M}_H(v)^{ss} \leq \langle v, v \rangle + r^2.$$

Proof. If $\langle v, v \rangle \geq 0$, then the claims are Lemma 3.2 and 3.3 in [KY]. If $\langle v, v \rangle < 0$, then $\mathbf{M}_H(v)^{ss}$ consists of $E_0^{\oplus l}$, where E_0 is the unique member of $\mathbf{M}_H(v/l)^{ss}$. In this case, $\mathbf{M}_H(v)^{ss} = BGL(l)$, and $\dim \mathbf{M}_H(v)^{ss} = -\dim \text{Aut}(E_0^{\oplus l}) = -l^2$. \square

Let $\mathcal{F}_H(v_1, v_2, \dots, v_s)$ be the stack of the Harder-Narasimhan filtrations

$$(48) \quad 0 \subset F_1 \subset F_2 \subset \dots \subset F_s = E, \quad E \in \mathbf{M}(v)$$

such that the quotients F_i/F_{i-1} , $1 \leq i \leq s$ are semistable with respect to H and

$$(49) \quad v(F_i/F_{i-1}) = v_i.$$

Then Lemma 5.3 in [KY] implies

$$(50) \quad \dim \mathcal{F}_H(v_1, v_2, \dots, v_s) = \sum_{i=1}^s \dim \mathbf{M}_H(v_i)^{ss} + \sum_{i < j} \langle v_i, v_j \rangle.$$

Note that

$$\text{Hom}(F_i/F_{i-1}, F_j/F_{j-1}) = 0 \quad \text{for } i < j,$$

as reduced Hilbert polynomials are strictly decreasing in the Harder-Narasimhan filtration.

Let H_1 be an ample divisor on X which belongs to a wall W with respect to v and H an ample divisor which belongs to an adjacent chamber. Then Gieseker H -semistable sheaves are H_1 slope-semistable

$$\mathbf{M}_H(v)^{ss} \hookrightarrow \mathbf{M}_{H_1}(v)^{\mu-ss}$$

We shall estimate the codimension of

$$\mathbf{M}_{H_1}(v)^{\mu-ss} \setminus \mathbf{M}_H(v)^{ss}.$$

Specifically, we shall prove

Proposition 5.

$$(51) \quad (\langle v, v \rangle + 1) - \dim(\mathbf{M}_{H_1}(v)^{\mu-ss} \setminus \mathbf{M}_H(v)^{ss}) \geq \frac{1}{r} \langle v, v \rangle / 2 + r - r^2 + 1.$$

As a consequence, we have

Corollary 1. *Assume that*

$$\frac{1}{r} \langle v, v \rangle / 2 + r - r^2 + 1 \geq 2.$$

Then $\mathbf{M}_H(v)^{ss}$ is independent on the choice of ample line bundle H (generic or on a wall) away from codimension 2.

Proof. Let E be an H_1 slope-semistable sheaf, which is however not H -semistable. Consider its Harder-Narasimhan filtration relative to H ,

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E.$$

All the subsheaves in the filtration are H -destabilizing for E . As E is H_1 slope-semistable, we must have equalities of slopes,

$$\mu_{H_1}(F_1) = \mu_{H_1}(F_2) = \cdots = \mu_{H_1}(E),$$

or in the notation of (49),

$$(52) \quad \frac{c_1(v_i) \cdot H_1}{\text{rk } v_i} = \frac{c_1(v) \cdot H_1}{\text{rk } v}, \quad 1 \leq i \leq s.$$

Thus

$$\mathbf{M}_{H_1}(v)^{\mu\text{-}ss} \setminus \mathbf{M}_H(v)^{ss} = \cup_{v_1, \dots, v_s} \mathcal{F}_H(v_1, v_2, \dots, v_s),$$

where (52) is satisfied. We shall estimate

$$\sum_{i < j} \langle v_i, v_j \rangle.$$

We set $v_i := (r_i, \xi_i, a_i)$. Since

$$(53) \quad \langle (v_i/r_i - v_j/r_j)^2 \rangle = (\xi_i/r_i - \xi_j/r_j)^2,$$

we get

$$(54) \quad \langle v_i, v_j \rangle = \frac{r_j}{r_i} \langle v_i, v_i \rangle / 2 + \frac{r_i}{r_j} \langle v_j, v_j \rangle / 2 - \frac{(r_j \xi_i - r_i \xi_j)^2}{2r_i r_j}.$$

Then we have

$$(55) \quad \begin{aligned} \langle v, v \rangle / 2 &= \sum_{i < j} \langle v_i, v_j \rangle + \sum_i \langle v_i, v_i \rangle / 2 \\ &= \sum_i \frac{r}{r_i} \langle v_i, v_i \rangle / 2 - \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j}. \end{aligned}$$

Hence

$$(56) \quad \sum_i \frac{1}{r_i} \langle v_i, v_i \rangle / 2 = \frac{1}{r} \langle v, v \rangle / 2 + \frac{1}{r} \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j}$$

and

(57)

$$\begin{aligned}
 \sum_{i < j} \langle v_i, v_j \rangle &= \sum_i \frac{r - r_i}{r_i} \langle v_i, v_i \rangle / 2 - \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j} \\
 &= \sum_i \frac{r - r_i}{r_i} (\langle v_i, v_i \rangle / 2 + r_i^2) - \sum_i (r - r_i) r_i - \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j} \\
 &\geq \sum_i \frac{1}{r_i} (\langle v_i, v_i \rangle / 2 + r_i^2) - \sum_i (r - r_i) r_i - \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j} \\
 &= \sum_i \frac{1}{r_i} \langle v_i, v_i \rangle / 2 + r - r^2 + \sum_i r_i^2 - \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j} \\
 &= \frac{1}{r} \langle v, v \rangle / 2 + \frac{1}{r} \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j} + r - r^2 + \sum_i r_i^2 - \sum_{i < j} \frac{(r_i \xi_j - r_j \xi_i)^2}{2r_i r_j} \\
 &\geq \frac{1}{r} \langle v, v \rangle / 2 + r - r^2 + \sum_i r_i^2,
 \end{aligned}$$

where we also used the Hodge index theorem and Bogomolov's inequality

$$\langle v_i, v_i \rangle + 2r_i^2 \geq 0.$$

Therefore

$$\begin{aligned}
 (\langle v, v \rangle + 1) - \dim \mathcal{F}_H(v_1, v_2, \dots, v_s) &= \sum_{i < j} \langle v_i, v_j \rangle + 1 - \sum_i (\dim \mathbf{M}_H(v_i)^{ss} - \langle v_i, v_i \rangle) \\
 &\geq \frac{1}{r} \langle v, v \rangle / 2 + r - r^2 + 1,
 \end{aligned}$$

which implies the claim. \square

Remark 3. When $c_1(v)$ is primitive, the estimate (57) is strict. Indeed, in this case, equality cannot occur in the Hodge index theorem. Therefore, the assumption of Corollary 1 may be relaxed to

$$\langle v, v \rangle \geq 2(r - 1)(r^2 + 1).$$

REFERENCES

- [A] T. Abe, *Deformations of rank 2 quasi-bundles and some strange dualities for rational surfaces*, Duke Math. J., 155 (2010), 577–620.
- [BH] M. Bernardara, G. Hein, *The Fourier-Mukai algorithm for elliptic surfaces*, arXiv:1002.4986.
- [Br] T. Bridgeland, *Fourier-Mukai transform over elliptic $K3$ surfaces*, J. Reine Angew. Math. 498 (1998), 115–133.
- [BK] I. Burban, B. Kreuzler, *Fourier-Mukai transforms and semi-stable sheaves on nodal Weierstrass cubics*, J. reine angew. Math. 584 (2005), 45–82.

- [BL] A. Beauville, Y. Laszlo, *Conformal blocks and generalized theta functions*, Commun. Math. Phys. 164, 385-419 (1994).
- [D1] G. Dănilă, *Sections du fibré déterminant sur l'espace de modules des faisceaux semi-stables de rang 2 sur le plan projectif*, Ann. Inst. Fourier (Grenoble) 50 (2000), no. 5, 1323–1374.
- [D2] G. Dănilă, *Résultats sur la conjecture de dualité étrange sur le plan projectif*, Bull. Soc. Math. France 130 (2002), no. 1, 1–33.
- [EGL] G. Ellingsrud, L. Göttsche, M. Lehn, *On the cobordism class of the Hilbert scheme of a surface*, J. Algebraic Geom. 10 (2001), 81-100.
- [F] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer, New York, 1998.
- [G] L. Göttsche, IAS and Oberwolfach lectures, 2008.
- [LeP1] J. Le Potier, *Fibré déterminant et courbes de saut sur les surfaces algébriques*, Complex Projective Geometry (Trieste, 1989/Bergen, 1989), 213-240, London Math. Soc. Lecture Notes Ser., 179, Cambridge Univ. Press, Cambridge, 1992.
- [LeP2] J. Le Potier, *Dualité étrange sur le plan projectif*, Exposé donné à Luminy en décembre 1996.
- [Li1] J. Li, *Algebraic geometric interpretation of Donaldson's polynomial invariants*, J. Differential Geom. 37 (1993), no. 2, 417–466.
- [Li2] J. Li, *Picard groups of the moduli spaces of vector bundles over algebraic surfaces*, Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), 129-146, Lecture notes in Pure and Appl. Math., 179, Dekker, New York, 1996.
- [MO1] A. Marian, D. Oprea, *A tour of theta dualities on moduli spaces of sheaves*, 175–202, Contemporary Mathematics, 465, American Mathematical Society, Providence, Rhode Island (2008).
- [MO2] A. Marian, D. Oprea, *The Verlinde sheaves and strange duality for K3 surfaces*, in preparation.
- [MO3] A. Marian, D. Oprea, *On the strange duality conjecture for abelian surfaces*, [arXiv:1207.5307](https://arxiv.org/abs/1207.5307).
- [Muk] S. Mukai, *Duality between $D(X)$ and $D(\tilde{X})$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153-175.
- [OG1] K. O'Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface*, J. Algebraic Geom. 6 (1997), no. 4, 599–644.
- [OG2] K. O'Grady, *Involutions and linear systems on holomorphic symplectic manifolds*, Geom. Funct. Anal. 15 (2005), no. 6, 1223–1274.
- [KY] K. Kurihara, K. Yoshioka, *Holomorphic vector bundles on non-algebraic tori of dimension 2*, Manuscripta Math. 126 (2008), no. 2, 143–166.
- [S] J. Sawon, *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces*, J. Algebraic Geom. 16 (2007), no. 3, 477–497.
- [Sc] L. Scala, *Dualité étrange de Le Potier et cohomologie du schéma de Hilbert ponctuel d'une surface*, Gaz. Math., no 112 (2007), 53-65.
- [Y1] K. Yoshioka, *Irreducibility of moduli spaces of vector bundles on K3 surfaces*, [arXiv:math/9907001](https://arxiv.org/abs/math/9907001).
- [Y2] K. Yoshioka, *Brill-Noether problem for sheaves on K3 surfaces*, Proceedings of the Workshop “Algebraic Geometry and Integrable Systems related to String Theory,” Surikaisekikenkyusho Kokyuroku No. 1232 (2001), 109–124.
- [Y3] K. Yoshioka, *Twisted stability and Fourier-Mukai transform. I.*, Compositio Math. 138 (2003), no. 3, 261–288.
- [Yu] Y. Yuan, *Determinant line bundles on Moduli spaces of pure sheaves on rational surfaces and Strange Duality*, [arXiv:1005.3201](https://arxiv.org/abs/1005.3201).

DEPARTMENT OF MATHEMATICS

NORTHEASTERN UNIVERSITY

E-mail address: a.marian@neu.edu

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CALIFORNIA, SAN DIEGO

E-mail address: doprea@math.ucsd.edu

DEPARTMENT OF MATHEMATICS

FACULTY OF SCIENCE, KOBE UNIVERSITY, KOBE, 657, JAPAN

E-mail address: yoshioka@math.kobe-u.ac.jp