IRREDUCIBILITY AND DIMENSION

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1. Irreducibility

Let us assume that k is an algebraically closed field. We will study affine algebraic sets in a bit more detail. To begin with, we will break them into smaller pieces which cannot be broken further, the irreducible algebraic sets.

Example 1. Let X be the union of the two coordinate axes in $\mathbb{A}^2$. Then $X = \mathcal{Z}(xy)$. It is clear that we can write

$$X = X_1 \cup X_2$$

where X_1 is the x-axis given by $X_1 = \mathcal{Z}(y)$ and X_2 is the y-axis given by $X_2 = \mathcal{Z}(x)$. Both X_1 and X_2 are affine algebraic sets, which cannot be further decomposed into smaller affine algebraic sets. The set X is said to be reducible, and X_1 and X_2 are its irreducible components.

This example motivates the following:

Definition 1. *An affine algebraic set is reducible if $X = X_1 \cup X_2$ for two proper affine algebraic subsets X_1 and X_2.*

Remark 1.
- (i) It turns out that this definition can be generalized to arbitrary topological spaces (it is not specific to the Zariski topology). If X is any topological space, we say that X is reducible if $X = X_1 \cup X_2$ for two proper closed subsets of X. If X is not reducible, we will call X *irreducible*. Note that we do not require X_1 and X_2 to be disjoint.
- (ii) Under the additional requirement that X_1 and X_2 are disjoint, X is said to be *disconnected.* It is clear that a disconnected set it reducible.

Example 2. The affine line $\mathbb{A}^1$ is irreducible in the Zariski topology. This can be seen as follows. The only proper affine algebraic sets of A^1 are finite sets. It is clear that $\mathbb{A}^1$ cannot be written as union of two proper algebraic sets.

Nonetheless, when $k = \mathbb{C}$, the affine line $\mathbb{A}^1$ is reducible in the usual topology. To see this, set

$$X_1 = \{z \in \mathbb{C} : |z| \geq 1\},\ X_2 = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Both these sets are closed in the usual topology, and they cover $\mathbb{A}^1$. Note however that $\mathbb{A}^1$ is connected.

Example 3. Irreducible topological spaces X are strange. For instance, if U and V are nonempty open subsets of X, then $U \cap V \neq \emptyset$. Indeed, otherwise we would be able to cover X by the closed sets $X \setminus U, X \setminus V$, violating irreducibility.

Similarly, one can show that nonempty open sets U of irreducible spaces are dense, in the sense that their closure $\overline{U} = X$. Otherwise $\overline{U}$ and $X \setminus U$ would again cover X.

Definition 2. *An irreducible affine algebraic set is called an affine variety.*

2. Prime ideals

We have seen that affine algebraic sets are in $1-1$ correspondence with radical ideals. What kind of ideals do affine algebraic sets correspond to?

To answer this question, it is helpful to reconsider the first example of the previous section. The reason we could write X as union of two proper algebraic sets is that the equation defining X split into a product of two other equations. To conceptualize this idea, we introduce the following

Definition 3. *Let A be a commutative ring. An ideal $\mathfrak{p} \subset A$ is prime provided the following condition is satisfied:*

$$\text{for any } a, b \in A \text{ such that } ab \in \mathfrak{p} \implies a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

Equivalently, we require that $A/\mathfrak{p}$ is an integral domain.

We will show:

Theorem 1. *An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible iff $\mathcal{I}(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$*

Proof: We will prove that X is reducible if and only if $I(X)$ is not prime.

First, if $X = X_1 \cup X_2$, with $X_1, X_2 \neq X$ proper closed subsets, there exist polynomials $f \in I(X_1) \setminus I(X)$ and $g \in I(X_2) \setminus I(X)$. Since f vanishes on X_1 and g vanishes on X_2, the product fg vanishes on $X_1 \cup X_2 = X$. Therefore $fg \in I(X)$, while $f, g \notin I(X)$, showing that $I(X)$ is not prime.

Conversely, pick two polynomials $f, g \notin I(X)$ with $fg \in I(X)$. Define X_1 as the zero set of the ideal $I(X) + (f)$, and similarly let X_2 be the zero set of the ideal $I(X) + (g)$. Clearly X_1, X_2 are proper closed subsets of X. Moreover $X_1 \cup X_2 = X$. Indeed, if $x \in X$, then $fg(x) = 0$, so either $f(x) = 0$ or $g(x) = 0$. Therefore $x \in X_1$ or $x \in X_2$. This completes the proof. □

Example 4. The ideal $\mathfrak{p} = (f) \subset k[X_1, \ldots, X_n]$ is prime iff f is an irreducible polynomial. The associated affine variety is called an irreducible (affine) hypersurface.

Example 5. The hypersurface $Y^2 - X^3 = 0$ in $\mathbb{A}^2$ is irreducible since the polynomial $Y^2 - X^3$ is irreducible. A hypersurface in $\mathbb{A}^2$ is called an affine curve.

Example 6. The ideal $\mathfrak{a} = (X^2Y - Y^2)$ is not prime since the generating polynomial is not irreducible. In fact,

$$Y \cdot (X^2 - Y) \in \mathfrak{a}$$

but $Y \notin \mathfrak{a}$ and $X^2 - Y \notin \mathfrak{a}$. Geometrically $\mathcal{Z}(\mathfrak{a})$ consists in the union of the X-axis $Y = 0$ and the parabola $Y = X^2$.

Example 7. If $\mathfrak{p} \subset k[X]$ is a proper ideal, then $\mathfrak{p}$ is principal. Thus $\mathfrak{p} = (f)$, where f is irreducible. Thus f must be a linear polynomial, hence

$$\mathfrak{p} = (X - a).$$

Therefore, $Z(\mathfrak{p})$ is a point. This corresponds to the obvious geometric fact that the only proper irreducible subsets of $\mathbb{A}^1$ are points.

Example 8. Let us generalize the previous example to several variables. If $a_1, \ldots, a_n \in k$, then

$$\mathfrak{m} = (X_1 - a_1, \ldots, X_n - a_n)$$

is a prime ideal in $k[X_1, \ldots, X_n]$ since the quotient $k[X_1, \ldots, X_n]/\mathfrak{m}$ is a field. Indeed, the morphism

$$k[X_1, \ldots, X_n]/\mathfrak{m} \to k, \; f \to f(a_1, \ldots, a_n)$$

is an isomorphism. This motivates the following:

Definition 4. *An ideal $\mathfrak{m} \neq (1)$ in a commutative ring A is said to be maximal if the quotient $A/\mathfrak{m}$ is a field.*

Remarks:

(i) Maximal ideals are maximal elements with respect to inclusions. Indeed, assume $\mathfrak{m}$ is maximal, and $\mathfrak{a} \neq (1)$ is an ideal in A such that $\mathfrak{m} \subset \mathfrak{a}$. If $\mathfrak{m} \neq \mathfrak{a}$, the field $A/\mathfrak{m}$ would contain a nontrivial proper ideal, the image $\mathfrak{a}$ under the projection $A \to A/\mathfrak{m}$. This is a contradiction since fields don't have proper ideals other than (0).

(ii) By definition, maximal ideals are prime.

(iii) We have shown that $(X_1 - a_1, \ldots, X_n - a_n)$ is a maximal ideal in $k[X_1, \ldots, X_n]$. These are the only maximal ideals in $k[X_1, \ldots, X_n]$. Indeed, if $\mathfrak{a}$ is another maximal ideal, then by Hillbert Nullstellensatz we know that $\mathcal{Z}(\mathfrak{a})$ is nonempty, say containing the point $(a_1, \ldots, a_n)$. But then

$$a \subset (X_1 - a_1, \ldots, X_n - a_n)$$

and by maximality, the inclusion is an equality.

Finally, note that maximal ideals in $k[X_1, \ldots, X_n]$ correspond to the smallest algebraic sets, namely points.

Example 9. Prime ideals are radical. Indeed if

$$f^r \in \mathfrak{p} \implies f \in \mathfrak{p}, \text{ hence } \sqrt{\mathfrak{p}} = \mathfrak{p}.$$

Example 10. We have established the following correspondences

$$\{\text{ affine algebraic sets in } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } k[X_1, \ldots, X_n]\}$$

$$\{\text{ affine varieties in } \mathbb{A}^n\} \leftrightarrow \{\text{prime ideals in } k[X_1, \ldots, X_n]\}$$

$$\{\text{ points in } \mathbb{A}^n\} \leftrightarrow \{\text{maximal ideals in } k[X_1, \ldots, X_n]\}.$$

Each line is contained in the line above e.g. maximal ideals are prime, and prime ideals are radical.

Example 11. Let us step back and look at Example 7 again. We have seen that the only proper irreducible subsets of $\mathbb{A}^1$ are single points. Conversely, all proper prime ideals in $k[X]$ are of the form $(X - a)$, hence they are maximal.

We would like to investigate the same problem in $\mathbb{A}^2$? Said it differently, what are the prime ideals of $k[X, Y]$? Let $\mathfrak{p}$ be such an ideal, and assume $\mathfrak{p} \neq (0), (1)$. We claim that

(i) $\mathfrak{p}$ is principal generated by one irreducible polynomial f or

(ii) $\mathfrak{p}$ is maximal, $\mathfrak{p} = (X - a, Y - b)$ for some $a, b \in k$.

To see this, pick $F \in \mathfrak{p}$, and factorize F into product of irreducibles $F = f_1 \dots f_r \in \mathfrak{p}$. Then $f_i \in \mathfrak{p}$ for some i. Thus $\mathfrak{p}$ contains one irreducible polynomial f. If $\mathfrak{p} \neq (f)$, we prove that $\mathfrak{p}$ is maximal. Pick an element $G \in \mathfrak{p} \setminus (f)$. We can factor G into irreducibles $g_1 \dots g_s$ where by assumption $g_i \neq f$. Thus, the prime ideal $\mathfrak{p}$ must contain a second irreducible polynomial $g \neq f$. Now,

$$(f, g) \subset \mathfrak{p} \implies \mathcal{Z}(\mathfrak{p}) \subset \mathcal{Z}(f, g).$$

Lemma 1. *Two distinct irreducible polynomials f and g in $k[X, Y]$ have only finitely many common roots.*

Therefore $\mathcal{Z}(f, g)$ is a finite set of points. Since $\mathcal{Z}(\mathfrak{p})$ is irreducible, $\mathcal{Z}(\mathfrak{p})$ is a point (a, b) . But we have seen in a previous example that this forces $\mathfrak{p}$ to be the maximal ideal $(X - a, Y - b)$.

Geometrically, this means that the affine subvarieties of $\mathbb{A}^2$ are points, irreducible affine curves, and $\mathbb{A}^2$.

Remark: The situation is more complicated in $\mathbb{A}^3$, where we encounter more irreducible sets. In order of "dimension," these are: single points, irreducible affine curves, irreducible affine surfaces (or hypersurfaces), $\mathbb{A}^3$.

3. Irreducible components

In this section, we will show that any affine algebraic set can be broken uniquely into irreducible pieces. We have already seen this phenomenon in example 1. This turns out to be a universal property of Noetherian topological spaces (such as affine algebraic sets in the Zariski topology).

Theorem 2. *Let X be a Noetherian topological space. Then X can be written as union of irreducible closed subsets*

$$X = \bigcup_i X_i,$$

such that $X_i \not\subset X_j$ for $i \neq j$. The decomposition is unique up to reordering of the X_i's.

Proof: To prove existence, argue by contradiction assuming that X cannot be written as union of irreducible components. In particular, X is reducible, hence $X = X_1 \cap X_1'$. Moreover, the statement of the Theorem must be false for at least one of these two subsets, say X_1. Hence we can break X_1 into a union $X_2 \cap X_2'$. Continuing this construction, one arrives at an infinite chain

$$X \supsetneq X_1 \supsetneq X_2 \dots \supsetneq X_n \supsetneq \dots$$

of closed subsets, which is a contradiction as X is Noetherian.

Definition 5. *The X_i in the above decomposition are called the irreducible components of X.*

Remark 2. A Noetherian topological space has finitely many irreducible components. Indeed, assume that this is false. Let $\mathcal{C}$ be the collection of all closed sets of X which have infinitely many irreducible components. By assumption $\mathcal{C}$ Let C be the minimal set in $\mathcal{C}$. Clearly, C is not irreducible, hence it should split as $C_1 \cup C_2$, with C_1, C_2 closed. Now, all irreducible components of C are the irreducible components of C_1 and irreducible components of C_2. Hence, one of the two sets C_1 and C_2 must have inifnitely many components. This contradicts minimality of C in the collection $\mathcal{C}$.

Remark 3. Algebraically, we have proved that any radical ideal $\mathfrak{a}$ can be written as intersection of prime ideals

$$\mathfrak{a} = \bigcap_i \mathfrak{p}_i$$

in a unique way. This holds true for any Noetherian ring A, essentially by the same argument as above translated to the algebraic category.

For instance $A = \mathbb{Z}$ is Noetherian. A radical ideal must have the form $\mathfrak{a} = (a)$ where $a = p_1 \ldots p_r$ must be product of distinct primes. Then

$$\mathfrak{a} = \bigcap_i (p_i).$$

Example 12. Consider the affine algebraic variety $Z = \mathcal{Z}(Y^3 - XY)$. It is clear that Z consists in the union of the axis $Y = 0$ and the parabola $Y^2 - X = 0$. These are the irreducible components of Z. The parabola is irreducible since the ideal $(Y^2 - X)$ is prime, as the polynomial $Y^2 - X$ is irreducible.

Example 13. Let S be the affine algebraic set given by the vanishing of

$$XZ - Y^2 = X^3 - YZ = 0.$$

Let us determine the irreducible components of S. First, if $X = 0$ then $Y = 0$, so the Z-axis is contained in S. Otherwise, the first equation implies

$$Z = Y^2/X,$$

while the second becomes

$$0 = X^3 - YZ = X^3 - Y^3/X \implies X = \left(\frac{Y}{X}\right)^3.$$

Setting

$$t = \frac{Y}{X},$$

we obtain

$$X = t^3, Y = t^4, Z = t^5.$$

Let C be the curve given by the points (t^3, t^4, t^5). This curve is the image of the polynomial map

$$f : \mathbb{A}^1 \to \mathbb{A}^3, t \to (t^3, t^4, t^5).$$

The following lemma will show that C is irreducible. This will show that S has tw o irreducible components, the Z-axis and the curve C.

It remains to explain that C is closed. We exhibit equations which describe C namely

$$C = \mathcal{Z}(X^4 - Y^3, X^5 - Z^3, Y^5 - Z^4).$$

Indeed, if (X, Y, Z) satisfies these three equations, we claim

$$X = t^3, Y = t^4, Z = t^5.$$

This can be seen setting $t = Y/X$, and solving for the variables. Note that we need 3 equations to define the set C, while only two equations are used to describe the set S.

Lemma 2. *If $f : \mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map and X is an irreducible affine set, then $f(X)$ is also irreducible.*

Remark: Note that we are not claiming that $f(X)$ is an affine algebraic set. In fact, we saw this is false.

Proof: Write $f(X) = Z_1 \cup Z_2$ where Z_1, Z_2 are proper closed subsets of $f(X)$. Then

$$Z_i = Y_i \cap f(X)$$

for some algebraic subsets Y_i of $\mathbb{A}^m$. Since f is a polynomial map, $f^{-1}(Y_i)$ is an algebraic set in $\mathbb{A}^n$, so

$$X_i = f^{-1}(Y_i) \cap X$$

is a closed subset of X. Moreover

$$X = X_1 \cup X_2,$$

so $X_1 = X$ or $X_2 = X$. This means

$$X \subset f^{-1}(Y_i) \implies f(X) \subset Y_i \implies Z_i = X.$$

But this contradicts the irreducibility of X.

Remark: It is false that the inverse image of an irreducible subset under a polynomial map is irreducible. For instance consider

$$f : \mathbb{A}^2 \to \mathbb{A}^1, (x, y) \to xy.$$

The preimage of 0 is the union of the two coordinate axes.

4. Dimension

In this section we will define the dimension of an irreducible Noetherian topological space, in particular of affine varieties. The underlying idea of the definition that follows is that every additional equation should cut down the dimension by 1. More precisely:

Definition 6. *An irreducible Noetherian topological space is said to be of dimension n if there is a descending chain of closed irreducible subsets*

$$X = X_0 \supsetneq X_1 \supsetneq \ldots \supsetneq X_n \neq \emptyset,$$

and any other chain has length at most or equal to n.

In this definition, we should think of the X_i's as having dimension i. This definition of dimension is short and intuitive, but we will see that it is difficult to apply for actual computations.

Remark 4. If X is any Noetherian topological space, not necessarily irreducible, the dimension of X is defined to be the supremum of the dimensions of its irreducible components.

Example 14. It is easy to see that the dimension of the affine line $\mathbb{A}^1$ is one, essentially because the only irreducible algebraic subsets of $\mathbb{A}^1$ are points.

Example 15. The dimension of $\mathbb{A}^2$ equals 2. This is because the only proper closed subsets of $\mathbb{A}^2$ are either points or have the form $\mathcal{Z}(f)$, for f irreducible. It is clear that any descending chain of length 3 in $\mathbb{A}^2$ would have to contain an inclusion $\mathcal{Z}(f) \supset \mathcal{Z}(g)$. But this implies that $f|g$ which is impossible for f and g distinct.

Example 16. In general, the dimension of $\mathbb{A}^n$ is at least n because of the descending chain

$$\mathbb{A}^n \supsetneq \mathbb{A}^{n-1} \supsetneq \ldots \supsetneq \mathbb{A}^1.$$

It can be shown that in fact the dimension of $\mathbb{A}^n$ is n, but we will have to develop more tools to prove this fact. In fact, it is hard to perform explicit computations using the above definition of dimension.

Example 17. The dimension of the algebraic set in example 13 is 1. This follows once we show that

$$C = f(\mathbb{A}^1) = \{(t^3, t^4, t^5) : t \in \mathbb{A}^1\}$$

is an algebraic set of dimension 1.

Indeed, assuming that

$$C \supsetneq X_1 \supsetneq X_2 \neq \emptyset,$$

where X_1, X_2 is irreducible, we have

$$\mathbb{A}^1 \supsetneq f^{-1}(X_1) \supsetneq f^{-1}(X_2) \neq \emptyset.$$

Thus, $f^{-1}(X_i)$ is a proper closed subset of $\mathbb{A}^1$, hence it should be finitely many points. Then the irreducible set $X_i \subset f(f^{-1}(X_i))$ is one point. But then $X_1 = X_2$ which is not allowed.

Example 18. It is clear that if $Y \subset X$, we have $\dim Y \leq \dim X$. In particular, any affine algebraic set $X \subset \mathbb{A}^n$ has finite dimension $\leq n$.

Example 19. There are examples of irreducible Noetherian topological spaces X of infinite dimension. Indeed, let $X = \mathbb{Z}$ and give X the topology whose closed sets are

$$C_i = \{1, 2, \ldots, i\}.$$

Clearly, X is Noetherian and irreducible. However, there are chains of arbitrary length m, namely

$$X \supsetneq C_m \subsetneq C_{m-1} \subsetneq \ldots \subsetneq C_0.$$

Example 20. Our intuition says that if U is dense in an irreducible Noetherian topological space X then

$$\dim U = \dim X.$$

This is false as the following example shows. Take $X = \{a, b\}$, and give X the topology whose closed sets are $\emptyset, \{a\}, X$. Clearly, X is irreducible of dimension 1, while $U = \{b\}$ is a dense open set of dimension 0.

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