TAUTOLOGICAL CLASSES ON THE MODULI SPACES OF STABLE MAPS TO $\mathbb{P}^r$ VIA TORUS ACTIONS

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Abstract. We present a localization proof of the fact that the cohomology of the moduli spaces of genus zero stable maps to projective spaces is entirely tautological. In addition, we obtain a description of a Białynicki-Birula decomposition of the stack of stable maps in the context of Gathmann's moduli spaces of maps with prescribed contact order to a fixed hyperplane. We show that the decomposition is filterable by exhibiting an explicit ordering of the fixed loci.

In our previous paper [O], we introduced the tautological rings of the genus zero moduli spaces of stable maps to homogeneous varieties X. We showed that in the case of SL flags, all rational cohomology classes on the stable map spaces are tautological using methods from Hodge theory. The goal here is to indicate how such a proof can be obtained using localization techniques in the case when X is a projective space.

To set the stage, we recall the definition of the tautological rings. The moduli stacks $\overline{\mathcal{M}}_{0,S}(\mathbb{P}^r, d)$ parametrize S-pointed genus 0 degree d stable maps to $\mathbb{P}^r$. We use the notation $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ when the labelling set is $S = \{1, 2, \ldots, n\}$. These moduli spaces are connected by a system of natural morphisms, which we enumerate below:

- forgetful morphisms $\pi : \overline{\mathcal{M}}_{0,S}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,T}(\mathbb{P}^r, d)$ defined whenever $T \subset S$,
- evaluation morphisms to the target space $ev_i : \overline{\mathcal{M}}_{0,S}(\mathbb{P}^r, d) \to \mathbb{P}^r$ for all $i \in S$,
- gluing morphisms which produce maps with nodal domains

$$gl : \overline{\mathcal{M}}_{0,S_1\cup\{\bullet\}}(\mathbb{P}^r, d_1) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,\{\star\}\cup S_2}(\mathbb{P}^r, d_2) \to \overline{\mathcal{M}}_{0,S_1\cup S_2}(\mathbb{P}^r, d_1 + d_2).$$

We note that the gluing maps induce morphisms

$$A^\star(\overline{\mathcal{M}}_{0,S_1\cup\{\bullet\}}(\mathbb{P}^r, d_1)) \otimes A^\star(\overline{\mathcal{M}}_{0,S_2\cup\{\star\}}(\mathbb{P}^r, d_2)) \to A^\star(\overline{\mathcal{M}}_{0,S_1\cup S_2}(\mathbb{P}^r, d_1 + d_2))$$

given by first pulling back to the fibered product $\overline{\mathcal{M}}_{0,S_1\cup\{\bullet\}}(\mathbb{P}^r, d_1) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,\{\star\}\cup S_2}(\mathbb{P}^r, d_2)$ via the two projections p, q, and then pushing forward via gl:

$$\alpha \otimes \beta \mapsto gl_\star(p^\star\alpha \cdot q^\star\beta).$$

The pullacks are well-defined since the projections p, q are smooth [KP].

Definition 1. The genus 0 tautological rings $R^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$ are the smallest system of subrings of $A^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$ such that:

- the system is closed under pushforwards by the natural morphisms,
- all monomials in the evaluation classes $ev_i^\star \alpha$ for $\alpha \in A^\star(X)$ are in the system.

The localization theorem in [EG] can be used to show that the *localized equivariant Chow rings* of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are tautological. Our goal is to prove the following stronger result, analogous to the main theorem in [O], by making use of the torus action:

Theorem. *All rational Chow classes on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are tautological.*

Localization is often used in Gromov-Witten theory. However, our approach is novel in two ways. First, we make use of a non-generic torus action on $\mathbb{P}^r$ which fixes one point $p = [1 : 0 : \ldots : 0]$ and a hyperplane $H = \{z_0 = 0\}$:

$$t \cdot [z_0 : z_1 : \ldots : z_r] = [z_0 : tz_1 : \ldots : tz_r].$$

Secondly, we completely determine the Białynicki-Birula *plus* decomposition of the *stack* of stable maps which describes the flow of maps under this action. In addition, we show that the decomposition is *filterable*. This is the algebraic analogue of the Morse stratification, whose cells can be ordered by the levels of the critical sets. As a consequence, we build up the stack of stable maps by adding cells in a *well determined order*, leading to a proof of the above theorem. A filterable decomposition also gives a way of computing the Poincaré polynomials of the moduli spaces of stable maps from those of the fixed loci. In low codimensions this method works quite well as we will demonstrate in a future paper.

Note that the Białynicki-Birula decomposition has not been established for *general* smooth Deligne-Mumford *stacks* with a torus action. However, in our case, we succeed to explicitly write it down in the context of Gathmann's stacks [G1]. These stacks, whose definition will be reviewed in Section 1.2, compactify the locus of marked maps with fixed contact orders $\alpha_1, \ldots, \alpha_n$ with the hyperplane H.

We now explain the main result. Decorated graphs Γ will be used to bookkeep the fixed loci, henceforth denoted $\mathcal{F}_\Gamma$. Specifically,

- the vertices of Γ correspond to components or points of the domain of an invariant morphism which are mapped entirely to p or H. The vertices carry carry degree labels and also carry legs corresponding to each of the markings
- the edges, also decorated by degrees, correspond to the remaining components.

We repackage the datum of a decorated graph Γ into an explicit fibered product $\overline{\mathcal{Y}}_\Gamma$ of Kontsevich and Gathmann spaces in equation (11). The theorem stated above follows from the following stronger result:

Theorem 1. *The stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ can be decomposed into disjoint locally closed substacks $\mathcal{F}_\Gamma^+$ (the "plus" cells of maps "flowing" into $\mathcal{F}_\Gamma$) such that:*

(1) *The fixed loci $\mathcal{F}_\Gamma$ are substacks of $\mathcal{F}_\Gamma^+$. There are projection morphisms $\mathcal{F}_\Gamma^+ \to \mathcal{F}_\Gamma$. On the level of coarse moduli schemes, we obtain the plus Białynicki-Birula decomposition of the coarse moduli scheme of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.*

(2) *The decomposition is filterable. That is, there is an* **explicit** *partial ordering of the graphs Γ such that*

$$\overline{\mathcal{F}_\Gamma^+} \subset \bigcup_{\Gamma' \geq \Gamma} \mathcal{F}_{\Gamma'}^+.$$

(3) *The closures of $\mathcal{F}_\Gamma^+$ are images of the fibered products $\overline{\mathcal{Y}}_\Gamma$ of Kontsevich and Gathmann spaces under tautological morphisms.*

(4) *The codimension of $\mathcal{F}_\Gamma^+$ can be explicitly computed from the graph Γ. If $\mathfrak{u}$ is the number of H-labeled vertices of degree 0, with no legs, and $\mathfrak{s}$ is the number of H-labeled vertices which have positive degree or total valency at least 3, then the codimension is $d + \mathfrak{s} - \mathfrak{u}$.*

(5) *The rational cohomology and rational Chow groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are isomorphic.*

(6) *(There exists a collection of substacks ξ which span the rational Chow groups of $\mathcal{F}_\Gamma$ and) there exist closed substacks $\overline{\xi^+}$ supported in $\overline{\mathcal{F}_\Gamma^+}$ (compactifying the locus of maps flowing into ξ), which span the rational Chow groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. The stacks $\overline{\xi^+}$ are images of fibered products of Gathmann spaces and substacks of the Kontsevich spaces to H.*

(7) *The cycles constructed in (6) are tautological.*

This paper is organized as follows. The first section contains prelimiary observations about localization on the moduli spaces of stable maps and about the Gathmann stacks. In the second section we discuss the Białynicki-Birula cells on a general smooth Deligne Mumford stack with an equivariant atlas. There, we establish the "homology basis theorem" under a general filterability assumption. In the third section we identify the decomposition explicitly for the stacks $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ and show its filterability. Finally, the last section proves the main results.

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Conventions. All schemes and stacks are defined over $\mathbb{C}$. All stacks considered here are Deligne-Mumford. $\mathbb{T}$ stands for the one dimensional torus, which henceforth will be identified with $\mathbb{C}^\star$ via a fixed isomorphism. For schemes/stacks X with a $\mathbb{T}$-action, $X^{\mathbb{T}}$ denotes the fixed locus.

1. Preliminaries

In this section we collect several useful facts about the fixed loci of the torus action on the moduli spaces of stable maps. We also discuss the Gathmann compactification of the stack of maps with prescribed contact orders to a fixed hyperplane.

1.1. **Localization on the moduli spaces of stable maps.** The main theme of this paper is a description of the flow of stable maps under the torus action on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. This flow is obtained by translation of maps under the torus action on the target $\mathbb{P}^r$. Traditionally, actions with isolated fixed points have been used. As it will become clear in the next sections, for our purposes it is better to consider the following action which in homogeneous coordinates is given by:

$$t \cdot [z_0 : z_1 : \ldots : z_r] = [z_0 : tz_1 : \ldots : tz_r], \text{ for } t \in \mathbb{C}^\star. \tag{1}$$

There are two fixed sets: one of them is the isolated point $p = [1 : 0 : \ldots : 0]$ and the other one is the hyperplane H given by the equation $z_0 = 0$. We observe that

$$\text{if } z \in \mathbb{P}^r - H \text{ then } \lim_{t\to 0} t \cdot z = p. \tag{2}$$

The fixed stable maps $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^r$ are obtained as follows. The image of f is an invariant curve in $\mathbb{P}^r$, while the images of the marked points, contracted components, nodes and ramification points are $\mathbb{C}^\star$ invariant i.e. they map to p or to H. The non-contracted components are either entirely contained in H, or otherwise they map to invariant curves in $\mathbb{P}^r$ joining p to a point q_H in H. The restriction of the map f to these latter components is totally ramified over p and q_H. This requirement determines the map uniquely. To each fixed stable map we associate a tree Γ as follows.

- The vertices of the tree correspond to the connected components of the set $f^{-1}(p) \cup f^{-1}(H)$. These vertices come with labels p and H such that adjacent vertices have distinct labels. Moreover, the vertices labeled H also come with degree labels, corresponding to the degree of the stable map on the component mapped to H (which is 0 if these components are isolated points).
- The edges correspond to the non-contracted components which are not contained in H. These edges are decorated with degrees.
- Γ has n numbered legs coming from the marked points.

We introduce the following notation for the graph Γ.

- By definition, the flags incident to a vertex are all incoming legs and half edges.

- Typically, v *stands for a vertex labeled* p and we let $n(v)$ be its total valency (total number of incident flags).
- Typically, w *stands for a vertex labeled* H and we let $n(w)$ be its total valency. The corresponding degree is d_w.
- The set of vertices is denoted $V(\Gamma)$. We write V and W for the number of vertices labeled p and H respectively.
- The set of edges is denoted $E(\Gamma)$, and the degree of the edge e is d_e. We write E for the total number of edges.
- For each vertex v, we write α_v for the ordered collection of degrees of the incoming flags. We agree that the degrees of the legs are 0. We use the notation $d_v = |\alpha_v|$, for the sum of the entries in α_v.
- A vertex w labeled H of degree $d_w = 0$ is called *unstable* if $n(w) \leq 2$ and *very unstable* if $n(w) = 1$.

The unstable vertices correspond to zero dimensional components of $f^{-1}(H)$, and have the following interpretation:

- the very unstable vertices come from unmarked smooth points of the domain mapping to H;
- the unstable vertices with one leg come from marked points of the domain mapping to H;
- the unstable vertices with two incoming edges come from nodes of the domain mapping to H.

The vertices w labeled H of positive degree or with $n(w) \geq 3$ are *stable.* Let $\mathfrak{s}$ be the number of stable vertices labeled H, and $\mathfrak{u}$ be the number of very unstable vertices.

The fixed locus corresponding to the decorated graph Γ will be denoted by $\mathcal{F}_\Gamma$. It can be described as the image of a finite morphism

$$\zeta_\Gamma : \prod_{v \text{ labeled } p} \overline{\mathcal{M}}_{0,n(v)} \times \prod_{w \text{ labeled } H} \overline{\mathcal{M}}_{0,n(w)}(H, d_w) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d). \tag{3}$$

To get the fixed locus we need to factor out the action of a finite group A_Γ of automorphisms, which is determined by the exact sequence below whose last term is the automorphism group of the decorated graph Γ:

$$1 \to \prod_{e \in E(\Gamma)} \mathbb{Z}/d_e\mathbb{Z} \to A_\Gamma \to \text{Aut } (\Gamma) \to 1.$$

The map ζ_Γ can be described as follows.

- Pick a collection of $n(v)$-marked genus 0 curves C_v for each vertex v labeled p.

- Similarly, pick a collection of stable maps f_w to H of degree d_w with $n(w)$ markings on the domain C_w, one for each vertex w labeled H.
- When necessary, we interpret C_v or C_w as points.

A $\mathbb{C}^\star$ fixed stable map f with n markings to $\mathbb{P}^r$ is obtained as follows.

- The components C_v will be mapped to p. The components C_w will be mapped to H with degree d_w under the map f_w.
- We join any two curves C_v and C_w by a rational curve C_e whenever there is an edge e of the graph Γ joining v and w. We map C_e to $\mathbb{P}^r$ with degree d_e such that the map is totally ramified over the special points.
- Finally, the marked points correspond to the legs of the graph Γ.

1.2. **Gathmann's moduli spaces.** Gathmann's moduli spaces are an important ingredient of our localization proof. We briefly discuss them below, referring the reader to [G1] for the results quoted in this section.

We let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a n tuple of non-negative integers. We will usually assume that:

$$|\alpha| = \sum \alpha_i = d.$$

Even without this assumption, the substack $\overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ parametrizes stable maps $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^r$ such that

- $f(x_i) \in H$ for all i such that $\alpha_i > 0$.
- $f^\star H - \sum_i \alpha_i x_i$ is effective.

Gathmann shows that this is an irreducible, reduced, proper substack of expected codimension $|\alpha| = \sum_i \alpha_i$ of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

We will show later that these stacks define tautological cycles on the moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. To this end we will make use of the recursive structure of Gathmann's stacks obtained from equations (4) and (5) below. We explain what happens if we increase the multiplicities. We let e_j be the elementary n-tuple with 1 in the jth position and 0 otherwise. Then, we have the following relation in $A_\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$:

$$[\overline{\mathcal{M}}_{\alpha+e_j}^H(\mathbb{P}^r, d)] = -(\alpha_j \psi_j + ev_j^\star H) \cdot [\overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)] + [\mathcal{D}_{\alpha,j}(\mathbb{P}^r, d)]. \tag{4}$$

The correction terms $\mathcal{D}_{\alpha,j}(\mathbb{P}^r, d)$ come from the boundary of the Gathmann stacks. These boundary terms account for the stable maps f with one "internal" component C_0 mapped to H with some degree d_0 and with some multiplicity conditions α^0 at the marked points of f lying on C_0. Moreover, we require that the point x_j lie on C_0. There are r (unions of) components attached to the internal component at r points. For all $1 \leq i \leq r$, the map has degree d_i on the component C_i and sends the intersection point with the internal component C_0 to H with multiplicity m^i. In addition, there are multiplicity conditions α^i at the marked points of f lying on C_i. We require that the d_i's sum up to d and that the α^i's form a partition of the n-tuple α.

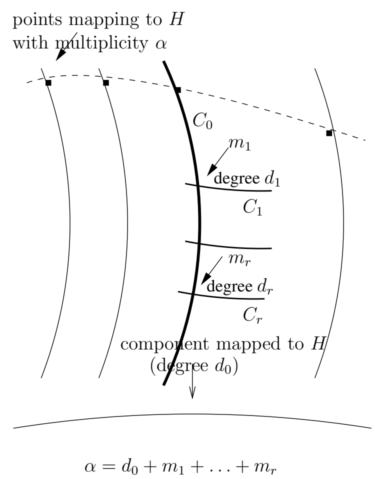

FIGURE 1. A map in the boundary of the Gathmann compactification.

The boundary terms we just described are fibered products of lower dimensional Kontsevich and Gathmann stacks:

$$\overline{\mathcal{M}}_{0,r+\ell(\alpha^0)}(H, d_0) \times_{H^r} \prod_{i=1}^{r} \overline{\mathcal{M}}^H_{\alpha^i \cup m^i}(\mathbb{P}^r, d_i),$$

with ℓ denoting the length of tuples. Gathmann shows that these stacks are irreducible of codimension 1 in $\overline{\mathcal{M}}^H_{\alpha+e_j}(\mathbb{P}^r, d)$. Their multiplicities are found from the equation:

$$[\mathcal{D}_{\alpha,j}(\mathbb{P}^r, d)] = \sum \frac{m^1 \ldots m^r}{r!} \left[\overline{\mathcal{M}}_{0,r+\ell(\alpha^0)}(H, d_0) \times_{H^r} \prod_{i=1}^{r} \overline{\mathcal{M}}^H_{\alpha^i \cup m^i}(\mathbb{P}^r, d_i) \right]. \tag{5}$$

Jun Li extended Gathmann's construction to arbitrary genera and targets [Li]. Li's stack $\mathfrak{M}^H_{0,\alpha}(\mathbb{P}^r, d)$ parametrizes genus 0 relative stable morphisms with contact order α to the pair $(\mathbb{P}^r, H)$. Morphisms to level k degenerations $\mathbb{P}^r[k]$ of the target $\mathbb{P}^r$ need to be considered. The collapsing maps $\mathbb{P}^r[k] \to \mathbb{P}^r$ induce a natural morphism

$$\mathfrak{M}^H_{0,\alpha}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$$

whose image is the Gathmann stack $\overline{\mathcal{M}}^H_{\alpha}(\mathbb{P}^r, d)$ [G2]. Morphisms in the boundary of the Gathmann spaces which have components contained entirely in H arise as collapsed images of morphisms to higher level degenerations of the target in Li's construction. For example, the map in Figure 1 corresponds to the relative stable morphism to the level 1 degeneration of $\mathbb{P}^r$. The "internal" component C_0 mapped to H is the collapsed image of the relative morphism restricted to the first level. The components $C_1, \ldots, C_r$ come from the initial level.

2. The decomposition on smooth stacks with a torus action

In this section we will construct the Białynicki-Birula cells of a smooth Deligne-Mumford stack with a torus actions under the additional assumption that there exists an equivariant affine étale atlas.[1] The existence of such an atlas should be a general fact, which we do not prove here since in the case of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ it can be constructed explicitly by hand. Finally, in Lemma 3 we prove a "homology basis theorem" for such stacks.

2.1. **The equivariant étale affine atlas.** In this subsection we will construct an equivariant affine atlas for the moduli stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Fix an arbitrary $\mathbb{T} = \mathbb{C}^\star$-action on $\mathbb{P}^r$ inducing an action by translation on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

Lemma 1. *Possibly after lifting the action, there exists a smooth étale affine $\mathbb{C}^\star$-equivariant surjective atlas $S \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.*

Proof. As a first step, we will find for any invariant stable map f, an equivariant étale atlas $S_f \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. The construction in [FP] shows that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is a global quotient $[J/PGL(W)]$, thus giving a smooth surjective morphism $\pi : J \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Here, J is a quasiprojective scheme, which in fact can be explicitly constructed as a locally closed subscheme of a product of Hilbert schemes on $\mathbb{P}(W) \times \mathbb{P}^r$ for some vector space W. Moreover, since π is smooth and $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is smooth, J should also be smooth. It is clear that the $\mathbb{T}$-action on the second factor equips J with a $\mathbb{T}$-action such that the morphism $\pi : J \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ equivariant. Moreover, from the explicit construction, it follows that $\pi^{\mathbb{T}} : J^{\mathbb{T}} \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)^{\mathbb{T}}$ is surjective.

For any invariant f, there exists a $\mathbb{T}$-invariant point $j_f \in J$ whose image is f. It follows from [S] that there exists an equivariant affine neighborhood J_f of $j = j_f$ in J. The map on tangent spaces $d\pi : T_j J_f \to T_f \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is equivariantly surjective. We can pick an equivariant subspace $V_f \hookrightarrow T_j J_f$ which maps isomorphically to $T_f \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. By Theorem 2.1 in [B1], we can construct a smooth equivariant affine subvariety S_f of J_f containing j such that $T_j S_f = V_f$. The map $\pi_f : S_f \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is étale at j. Replacing S_f by an equivariant open subset, we may assume π_f is étale everywhere. Shrinking futher, we can assume S_f is equivariant smooth affine [S].

[1] *Addendum:* Some of the arguments below also use the projectivity of the coarse moduli scheme.

We consider the case of non-invariant maps f. We let $\alpha : \mathbb{C}^\star \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ be the equivariant *nonconstant* translation morphism

$$\mathbb{C}^\star \ni t \to f^t \in \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d).$$

Proposition 6 in [FP] or Lemma 5 below show that, after possibly a base-change $\mathbb{C}^\star \to \mathbb{C}^\star$, we can extend this morphism across 0. The image of $0 \in \mathbb{C}$ under α is a $\mathbb{T}$-invariant map F so we can utilize the atlas S_F constructed above. We claim that the image of the atlas $\pi_F : S_F \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ contains f. First, the tangent vector of the family α at 0 gives a non-zero eigenvector for the $\mathbb{T}$-action on $T_F\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. This lifts via $d\pi_F$ to a non-zero eigenvector v in T_jS_F. Using [B1] again, we find a smooth curve C in S_F, invariant by the $\mathbb{T}$-action and passing through $j = j_F$, such that T_jC is spanned by v. It follows that, locally, the image of C under π_F is contained in the image of $\alpha \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. The $\mathbb{T}$-equivariance shows that the map f is contained in the image of $\pi_F(C)$.

We obtained equivariant smooth affine atlases $S_f \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ whose images cover $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Only finitelly many of them are necessary to cover $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$, and their disjoint union gives an affine smooth étale surjective atlas $S \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

Corollary 1. *Let X be a convex smooth projective variety with a $\mathbb{T}$-action. There exists an equivariant smooth étale affine surjective atlas $S \to \overline{\mathcal{M}}_{0,n}(X, \beta)$ as in Lemma 1.*

Proof. We embedd $i : X \hookrightarrow \mathbb{P}^r$ equivariantly, and base change the atlas S constructed in the lemma under the closed immersion $i : \overline{\mathcal{M}}_{0,n}(X, \beta) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Convexity of X is used to conclude that since $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is smooth, the étale atlas S is also smooth.

2.2. **The Białynicki-Birula cells.** In this section we construct the Białynicki-Birula cells for a smooth Deligne-Mumford stack $\mathcal{M}$ with a $\mathbb{T} = \mathbb{C}^\star$ action which admits an atlas as in Lemma 1. We first establish:

Lemma 2. *Let $f : X \to Y$ be an equivariant étale surjective morphism of smooth schemes (stacks) with $\mathbb{T}$ actions. Let Z be any component of the fixed locus of Y. Then $f^{-1}(Z)$ is union of components of $X^{\mathbb{T}}$ all mapping onto Z.*

Proof. It suffices to show that the $\mathbb{T}$-action on $f^{-1}(Z)$ is trivial. The $\mathbb{T}$-orbits in $f^{-1}(Z)$ need to be contracted by f since Z has a trivial action. Since the differential $df : TX \to TY$ is an isomorphism, it follows that all orbits are 0 dimensional. They must be trivial since they are also reduced and irreducible.

An affine fibration is a flat morphism $p : X \to Y$ which is étale locally trivial. Note that the transition functions need not be linear so p is not necessarily a vector bundle.

Proposition 1. *Let $\mathcal{M}$ be any smooth Deligne Mumford stack with a $\mathbb{T} = \mathbb{C}^\star$-action and a $\mathbb{C}^\star$-equivariant affine étale surjective atlas $\pi : S \to \mathcal{M}$. Let $\mathcal{F}$ be the fixed substack and $\mathcal{F}_i$ be its components. Then $\mathcal{M}$ can be covered by disjoint substacks $\mathcal{F}_i^+$ which are affine fibrations over $\mathcal{F}_i$.*

Proof. Let $R = S \times_{\mathcal{M}} S$. There are two étale surjective morphisms $s, t : R \to S$ which together define a morphism $j : R \to S \times S$. It is clear that R has a torus action such that s, t are both equivariant. Moreover, since $\mathcal{M}$ is Deligne-Mumford, j is quasi-finite, hence a composition of an open immersion and an affine morphism. Since S is affine, it follows that R is quasi-affine. As s is étale, we obtain that R is also smooth.

If $F = S \times_{\mathcal{M}} \mathcal{F}$ then $F \hookrightarrow S$ is a closed immersion. Since $S \to \mathcal{M}$ is étale and equivariant, by the above remark, F coincides with $S^{\mathbb{T}}$. Similarly $s^{-1}(F)$ and $t^{-1}(F)$ coincide with $R^{\mathbb{T}}$. Fixing i, we let $F_i = S \times_{\mathcal{M}} \mathcal{F}_i$. Then F_i is union of components F_{ij} of $S^{\mathbb{T}}$. Similarly, $s^{-1}(F_i) = t^{-1}(F_i)$ is a union of components R_{ik} of $R^{\mathbb{T}}$. We will construct the subtack $\mathcal{F}_i^+$ of $\mathcal{M}$ and the affine fibration $\alpha_i : \mathcal{F}_i^+ \to \mathcal{F}_i$ on the atlas S. We will make use of the results of [B1], where a *plus* decomposition for quasi-affine schemes with a torus action was constructed. For each component F_{ij}, we consider its *plus* scheme F_{ij}^+; similarly for the R_{ik}'s we look at the cells R_{ik}^+. We claim that[2]

$$s^{-1}(\cup_j F_{ij}^+) = t^{-1}(\cup_j F_{ij}^+) = \cup_k R_{ik}^+$$

and we let $\mathcal{F}_i^+$ be the stack which $F_i^+ = \cup_j F_{ij}^+ \hookrightarrow S$ defines in $\mathcal{M}$. It suffices to show that if R_{ik} is mapped to F_{ij} under s, then R_{ik}^+ is a component of $s^{-1}(F_{ij}^+)$. Let $r \in R_{ik}$. Then, by the fact that s is étale and the construction in [B1] we have the following equality of tangent spaces:

$$T_r s^{-1}(F_{ij}^+) = ds^{-1}\left(T_{s(r)} F_{ij}^+\right) = ds^{-1}\left((T_{s(r)} S)^{\geq 0}\right) = (T_r R)^{\geq 0} = T_r R_{ik}^+.$$

Here $V^{\geq 0}$ denotes the subspace of the equivariant vector space V where the $\mathbb{C}^\star$-action has positive weights. The uniqueness result in Corollary to Theorem 2.2 in [B1] finishes the proof. Note that the argument here shows that the codimension of $\mathcal{F}_i^+$ in $\mathcal{M}$ is given by the number of negative weights on the tangent bundle $T_r\mathcal{M}$ at a $\mathbb{C}^\star$-fixed point r.

[2] *Correction:* In e-mail correspondence (September, 2019), Alper, Hall and Rydth pointed out that the inclusion $s^{-1}(\cup F_{ij}^+) \subset \cup R_{ik}^+$ can fail, and thus so does the equality $s^{-1}(F_{ij}^+) = t^{-1}(F_{ij}^+)$. In our particular case, it is possible to patch the argument by defining the plus cells on the stack from the projective coarse moduli scheme and proving the affine fibration structure in each chart by the reasoning here. Note that Alper, Hall and Rydth establish a general result in *A Luna étale slice theorem for algebraic stacks*, Annals of Mathematics, 191 (2020), 675 – 738. Following Drinfeld, *On algebraic spaces with an action of* $\mathbb{G}_m$, the plus cells are defined as components of $\underline{\mathrm{Hom}}_{\mathbb{C}^\star}(\mathbb{A}^1, \mathcal{M})$. Arguments on the coarse moduli scheme show that the cells thus defined are locally closed, and similarly the étale equivariant affine charts are used to establish the affine fibration structure.

To check that $\mathcal{F}_i^+ \to \mathcal{F}_i$ is an affine fibration, we start with the observation that $F_{ij}^+ \to F_{ij}$ are affine fibrations. We also need to check that the pullback *fibrations* under s and t are isomorphic:

$$s^\star \left(\oplus_j F_{ij}^+\right) \simeq t^\star \left(\oplus_j F_{ij}^+\right) \simeq \oplus_k R_{ik}^+.$$

The argument is identical to the one above, except that one needs to invoke the Corollary of Proposition 3.1 in [B1] to identify the fibration structure. Similarly, one checks the cocycle condition on triple overlaps.

Finally, we need to check that the trivializing open sets U for $F_{ij}^+ \to F_{ij}$ descend to the stack $\mathcal{M}$ i.e. we need to check that we can pick U such that $s^{-1}(U) = t^{-1}(U)$. First, one runs the argument of Lemma 2.2.3 in [AV]; we may assume that replacing U by an etale open, the groupoid $R \rightrightarrows S$ is (étale locally) given by $U \times \Gamma \rightrightarrows U$, where Γ is a finite group acting on U. In this case, our claim is obvious.

2.3. **The homology basis theorem.** In this subsection we will establish the "homology basis theorem" (Lemma 3) extending a result which is well known for smooth projective schemes [C]. The proof does not contain any new ingredients, but we include it below, for completeness. We agree on the following conventions. The Chow groups we use are those defined by Vistoli in [V], while the cohomology theory is defined for example in [Be].

Let us consider a smooth Deligne Mumford stack $\mathcal{M}$ with a torus action whose fixed loci $\mathcal{F}_i$ are indexed by $i \in I$, with I finite, and whose Białynicki-Birula cells $\mathcal{F}_i^+$ were defined above. We furthermore assume that the decompositon is filterable. That is, there is a partial (reflexive, transitive and anti-symmetric) ordering of the indices such that:

(a) We have $\overline{\mathcal{F}_i^+} \subset \cup_{j \geq i} \mathcal{F}_j$.

(b) There is a unique minimal index $\mathfrak{m} \in I$, i.e. an index such that if $i \leq \mathfrak{m}$ for some $i \in I$, then $i = \mathfrak{m}$.

Filterability of the decomposition was shown in [B2] for projective schemes. For the stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$, filterability follows from the similar statement on the coarse moduli scheme. However, to prove the tautology of the Chow classes, we need the *stronger filterability condition* (c), which we will demonstrate in the next section.

(c) There is a family Ξ of substacks supported on the fixed loci such that:
   - The cycles in Ξ span the rational Chow groups of the fixed loci.
   - For all $\xi \in \Xi$ supported on $\mathcal{F}_i$, there is a *plus* substack $\xi^+ = p_i^{-1}(\xi)$ (flowing into ξ) supported on $\mathcal{F}_i^+$; here $p_i : \mathcal{F}_i^+ \to \mathcal{F}_i$ is the projection. We assume

ξ^+ is contained in a closed substack $\widehat{\xi^+}$ supported on $\overline{\mathcal{F}_i^+}$ (usually, but not necessarily, its closure) with the proprety:

$$\widehat{\xi^+} \setminus \xi^+ \subset \bigcup_{j>i} \mathcal{F}_j^+.$$

Lemma 3. *Assume that $\mathcal{M}$ is a smooth Deligne Mumford stack which satisfies the assumptions (a) and (b) above.*

(i) *The Betti numbers $h^m(\mathcal{M})$ of $\mathcal{M}$ can be computed as:*

$$h^m(\mathcal{M}) = \sum_i h^{m-2n_i^-}(\mathcal{F}_i).$$

Here n_i^- is the codimension of $\mathcal{F}_i^+$ which equals the number of negative weights on the tangent bundle of $\mathcal{M}$ at a fixed point in $\mathcal{F}_i$.

(ii) *If the rational Chow rings and the rational cohomology of each fixed stack $\mathcal{F}_i$ are isomorphic, then the same is true about $\mathcal{M}$.*

(iii) *Additionally, if assumption (c) is satisfied, the cycles $\widehat{\xi^+}$ for $\xi \in \Xi$ span the rational Chow groups of $\mathcal{M}$.*

Proof. Thanks to item (b), we can define an integer valued function $L(i)$ as the length of the shortest ascending path from $\mathfrak{m}$ to i. Specifically, let $I_0 = \{\mathfrak{m}\}$, and define the sets I_k inductively, letting I_{k+1} be the set of minimal elements in $I \setminus \cup_{\ell \leq k} I_\ell$. We define $L(i) = k$ for $i \in I_k$. Observe that $i < j$ implies $L(i) < L(j)$. Because of (a), we have that

$$\mathcal{Z}_k = \bigcup_{L(i)>k} \mathcal{F}_i^+ = \bigcup_{L(i)>k} \overline{\mathcal{F}_i^+}$$

is a closed substack of $\mathcal{M}$. Letting $\mathcal{U}_k$ denote its complement, we conclude that

$$\mathcal{U}_{k-1} \hookrightarrow \mathcal{U}_k \text{ and } \mathcal{U}_k \setminus \mathcal{U}_{k-1} \text{ is union of cells } \bigcup_{L(i)=k} \mathcal{F}_i^+.$$

The Gysin sequence associated to the pair $(\mathcal{U}_k, \mathcal{U}_{k-1})$ is:

$$\ldots \to \bigoplus_{L(i)=k} H^{m-2n_i^-}(\mathcal{F}_i^+) = \bigoplus_{L(i)=k} H^{m-2n_i^-}(\mathcal{F}_i) \to H^m(\mathcal{U}_k) \to H^m(\mathcal{U}_{k-1}) \to \ldots.$$

One immitates the usual argument for smooth schemes in [AB], [Ki] to prove that the long exact sequence splits if the fixed loci have no odd cohomology as assumed in (ii). Even without the assumption (ii), the same sequence splits in equivariant cohomology, and item (i) follows by evaluating dimensions.

To prove (ii), we compare all short exact Gysin sequences to the Chow exact sequences (for m even) and use the five lemma:

$$
\begin{tikzcd}
& \bigoplus_{L(i)=k} A^{m/2-n_i^-}(\mathcal{F}_i) \arrow[r] \arrow[d] & A^{m/2}(\mathcal{U}_k) \arrow[r] \arrow[d] & A^{m/2}(\mathcal{U}_{k-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{L(i)=k} H^{m-2n_i^-}(\mathcal{F}_i) \arrow[r] & H^m(\mathcal{U}_k) \arrow[r] & H^m(\mathcal{U}_{k-1}) \arrow[r] & 0.
\end{tikzcd}
\tag{6}
$$

The first exact sequence exploits the surjectivity of the pullback $p_i^\star : A_\star(\mathcal{F}_i) \to A_\star(\mathcal{F}_i^+)$ which follows by the local triviality of the affine fibration $p_i : \mathcal{F}_i^+ \to \mathcal{F}_i$. In fact, this is true only under the assumption that the fibers of p_i are affine [Gi]. For the second exact sequence, homotopy invariance in cohomology is used. This can be proved by the methods of [Be], Lemma 37 via a spectral sequence argument.

Finally, for (iii), we use (6) to prove inductively that

$$\text{the cycles } \widehat{\xi^+} \cap \mathcal{U}_k \text{ for } \xi \in \Xi \text{ supported on } \mathcal{F}_i \text{ with } L(i) \leq k \text{ span } A^\star(\mathcal{U}_k).$$

Condition (c) is used to prove that the image in $\mathcal{U}_k$ of a cycle ξ supported on a fixed locus $\mathcal{F}_i$ with $L(i) = k$ is among the claimed generators:

$$\xi^+ = \widehat{\xi^+} \cap \mathcal{U}_k.$$

3. The Białynicki-Birula decomposition on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$

In this section, we explicitly identify the Białynicki-Birula decomposition on the stack of stable maps $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. We start by analyzing the $\mathbb{C}^\star$ flow of individual stable maps. We will relate the decomposition to Gathmann's stacks in the next subsection. Finally, we will prove the filterability condition (c) needed to apply Lemma 3.

3.1. **The flow of individual maps.** We study the $\mathbb{C}^\star$ flow of individual maps. To fix the notation, we let $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^r$ be a degree d stable map to $\mathbb{P}^r$. We look at the sequence of translated maps:

$$f^t : (C, x_1, \ldots, x_n) \to \mathbb{P}^r, \quad f^t(z) = tf(z).$$

By the "compactness theorem," this sequence will have a stable limit. We want to understand this limit $F = \lim_{t\to 0} f^t$.

To construct F explicitly we need to lift the torus action $t \to t^D$ where $D = d!$. Henceforth, we will work with the lifted action:

$$t \cdot [z_0 : z_1 : \ldots : z_r] = [z_0 : t^D z_1 : \ldots : t^D z_r].$$

We seek to construct a family of stable maps $G : \mathcal{X} \to \mathbb{P}^r$ over $\mathbb{C}$, whose fiber over $t \neq 0$ is f^t and whose central fiber F will be explicitly described below.

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[rr, bend left, "F"] & \mathcal{X} \arrow[r, "G"] \arrow[d, "\pi"] & \mathbb{P}^r \\ 0 \arrow[r] \arrow[u, bend left, "x_i"] & \mathbb{C} \arrow[u, bend left, "x_i"] & \end{tikzcd} \tag{7}$$

First we assume that the domain C is an irreducible curve. In case the image of f lies entirely in H, the family (7) is trivial and $F = f$.

Otherwise, f intersects the hyperplane H at isolated points, some of them possibly being among the marked points. We make a further simplifying assumption: we may assume that all points in $f^{-1}(H)$ are marked points of the domain. If this is not the case, we mark the remaining points in $f^{-1}(H)$ thus obtaining a new stable map $\overline{f}$ living in a moduli space with more markings $\overline{\mathcal{M}}_{0,n+k}(\mathbb{P}^r, d)$. We will have constructed a family (7) whose central fiber is $\overline{F} = \lim_{t\to 0} \overline{f}^t$. A new family having f^t as the t-fiber is obtained by forgetting the markings. We use a multiple of the line bundle

$$\omega_\pi(\sum_i x_i) \otimes G^\star \mathcal{O}_{\mathbb{P}^r}(3)$$

to contract the unstable components of the central fiber. Thus, we obtain the limit F from $\overline{F}$ by forgetting the markings we added and stabilizing.

Henceforth we assume that all points in $f^{-1}(H)$ are among the markings of f, and f is not a map to H. Let $s_1, \ldots, s_k$ be the markings which map to H, say with multiplicities $n_1, \ldots, n_k$ such that $\sum n_i = d$. We let $t_1, \ldots, t_l$ be the rest of the markings. We let $q_i = f(s_i)$. The following lemma will be of crucial importance to us. The method of proof is an explicit stable reduction, and it is similar to that of Proposition 2 in [KP].

Lemma 4. *Let F be the following stable map with reducible domain:*

- *The domain has one component C_0 of degree 0 mapped to p. This component contains markings $T_1, \ldots, T_l$. The curve C_0, with its markings and nodes, is isomorphic to the original marked curve C.*
- *Additionally, there are k components $C_1, \ldots, C_k$ attached to the degree 0 component. The restriction of F to C_i has degree n_i, its image is the line joining p to $q_i = f(s_i)$ and the map is totally ramified over p and q_i.*
- *Moreover, if we let $S_i = F^{-1}(q_i)$, then $S_1, \ldots, S_k, T_1, \ldots, T_l$ are the marked points of the domain of F.*

Then, the stabilization of F is the limit $\lim_{t\to 0} f_t$.

Proof. It suffices to exhibit a family as in (7). We let $f^0, \ldots, f^r$ be the homogeneous components of the map f. We let C be the domain curve with coordinates $[z : w]$. The

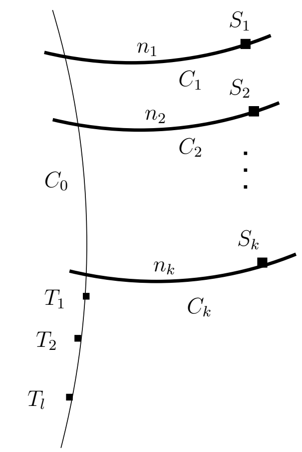

FIGURE 2. The limit in Lemma 4.

assumption about the contact orders of f with H shows that f^0 vanishes at $s_1, \dots, s_k$ of orders $n_1, \dots, n_k$ with $\sum_i n_i = d$.

There is a well defined map $G_0 : \mathbb{C}^\star \times C \to \mathbb{P}^r$ given by

$$(t, [z : w]) \mapsto \left[f^0(z : w) : t^D f^1(z : w) : \dots : t^D f^r(z : w) \right].$$

The projection map $\pi : \mathbb{C}^\star \times C \to \mathbb{C}^\star$ has constant sections $s_1, \dots, s_k, t_1, \dots, t_l$. It is clear that G_0 can be extended to a map

$$G_0 : \mathbb{C} \times C \setminus \bigcup_i (\{0\} \times \{s_i\}) \to \mathbb{P}^r.$$

A suitable sequence of blowups of $\mathbb{C} \times C$ at the points $\{0\} \times \{s_i\}$ will give a family of stable maps fibered over $\mathbb{C}$, $G : \mathcal{X} \to \mathbb{P}^r$ as in (7).

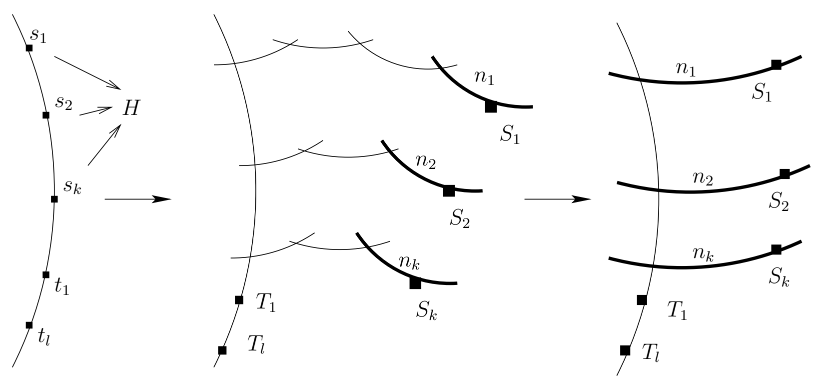

FIGURE 3. Obtaining the stable limit.

It is useful to understand these blowups individually. It suffices to work locally in quasi-affine patches U_i near s_i and then glue. An affine change of coordinates will ensure

$s_i = 0$. For $n = n_i$, we write $f^0 = z^n h$. We may assume that on U_i, h does not vanish and that $f_1, \ldots, f_r$ do not all vanish. Let $D = n \cdot e$. We will perform e blowups to resolve the map $G_0 : \mathbb{C} \times U_i \setminus \{(0, [0:1])\} \to \mathbb{P}^r$:

$$(t, [z:w]) \to [z^n h(z:w) : t^{ne} f_1(z:w) : \ldots : t^{ne} f_r(z:w)] .$$

The blowup at $(0, [0:1])$ gives a map:

$$G_1 : \mathcal{X}_1 \dashrightarrow \mathbb{P}^r .$$

In coordinates,

$$\mathcal{X}_1 = \{(t, [z:w], [A_1, B_1]) \text{ such that } A_1 z = t B_1 w\}$$

and

$$G_1 = \left[B_1^n \, h(tB_1 : A_1) : t^{ne-n} f_1(tB_1 : A_1) : \ldots : t^{ne-n} f_r(tB_1 : A_1)\right] .$$

The map is still undefined at $t = B_1 = 0$ so we will need to blow up again. After the kth blowup, we will have obtained a map:

$$G_k : \mathcal{X}_k \dashrightarrow \mathbb{P}^r ,$$

which in coordinates becomes:

$$\mathcal{X}_k = \{(t, [z:w], [A_1, B_1], \ldots, [A_k : B_k]) \mid A_1 z = t B_1 w, \ A_{i+1} B_i = t A_i B_{i+1}, \ 1 \leq i \leq k-1\}$$

$$G_k = \left[B_k^n \, h(t^k B_k : A_k) : t^{ne-nk} f_1(t^k B_k : A_k) : \ldots : t^{ne-nk} f_r(t^k B_k : A_k)\right] .$$

After the e^{th} blow up we obtain a well defined map. This map is constant on the first $e - 1$ exceptional divisors (hence they are unstable). On the e^{th} exceptional divisor the map is given by:

$$G_e = [B_e^n : A_e^n f_1(0:1) : \ldots : A_e^n f_r(0:1)] .$$

There, the map is totally ramified over two points in its image. It is easy to check that the sections $s_1, \ldots, s_k, t_1, \ldots, t_l$ extend over $t = 0$ as claimed in the lemma.

We obtain a family $G : \mathcal{X} \to \mathbb{P}^r$ of maps parametrized by $\mathbb{C}$ as in (7). The profile of the central fiber is the middle shape in Figure 3. There are unstable components coming from the exceptional divisors which need to be contracted successively to obtain the final limit we claimed. This completes the proof.

We consider the case when the domain curve is not irreducible. Assume that the stable map f is obtained by gluing maps f_1 and f_2 with fewer irreducible components at markings $\star$ and $\bullet$ on their domains with $f_1(\star) = f_2(\bullet)$. Inductively, we will have constructed families (7) of stable maps over $\mathbb{C}$ whose fibers over $t \neq 0$ are f_1^t and f_2^t. We glue the two families together at the sections $\star$ and $\bullet$ thus obtaining a family whose fiber over t

is f^t. The argument above proves that the limit for reducible maps can be obtained by taking the limits of each irreducible component and gluing the limits together along the corresponding sections.

Example. Figure 4 shows the limit in the case of a node x mapping to H with contact orders a_1 and a_2 on the two components C_1 and C_2 not contained in H. The node is replaced by two tails of degrees a_1 and a_2 joined at a node. The tails are joined to the rest of the domain $C_1 \cup C_2$ at nodes mapping to p.

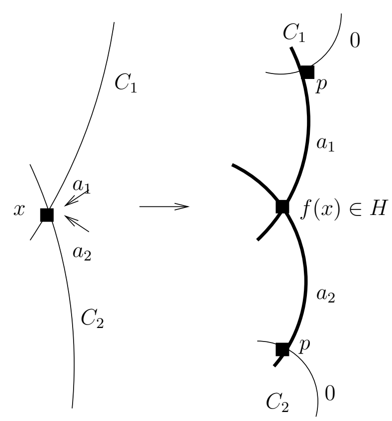

FIGURE 4. The limit of the $\mathbb{C}^\star$ flow when a node maps to H.

We obtain the following algorithm for computing the limit F.

(i) We consider each irreducible component of the domain individually. We mark the nodes on each such component.

(ii) We leave unaltered the irreducible components mapping to H.

(iii) The components which are not contained in H are replaced in the limit by reducible maps. The reducible maps have one back-bone component mapped to p. This component contains all markings which are not mapped to H. Moreover, rational tails are glued to the back-bone component at the points which map to H according to the item below. The markings which map to H are replaced by markings on the rational tails.

(iv) In the limit, each isolated point x of the domain curve which maps to $f(x) \in H$ with multiplicity n is replaced by a rational curve glued at a node to the rest of the domain. The node is mapped to p. The image under F of the rational tail is a curve in $\mathbb{P}^r$ joining p to $f(x) \in H$. The map F is totally ramified over these two points with order n. If the point x happens to be a section, we mark the point $F^{-1}(f(x))$ as a section of F.

(v) The map F is obtained by gluing all maps in (ii) and (iii) along the markings we added in (i) and then stabilizing.

Lemma 5. *For each stable map f, there is a family of stable maps (7) over $\mathbb{C}$, whose fiber over t is the translated map f^t and whose central fiber F can be obtained by the algorithm above.*

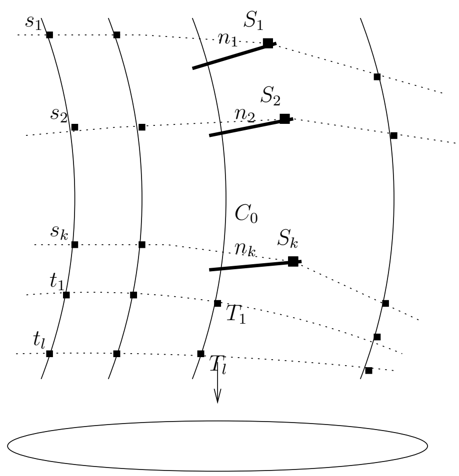

FIGURE 5. A family as in Lemma 5

3.2. **Relation to the Gathmann stacks.** We proceed to identify the Białynicki-Birula cells of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Recall from Section 1.1 that the fixed loci for the torus action on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are indexed by decorated graphs Γ. We will identify the closed stacks $\overline{\mathcal{F}_\Gamma^+}$ in terms of images of fibered products of Kontsevich and Gathmann stacks under the tautological morphisms.

We will need the following versions of Gathmann's construction.

(i) The substacks $\widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$ parametrize maps with the additional condition that the components of f intersect H properly. Thus, if we assume $|\alpha| = d$, the nodes are not mapped to H. The stack $\mathcal{M}_\alpha^H(\mathbb{P}^r, d)$ of maps with irreducible domains not mapping to H is an open substack of $\widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$, so that

$$\mathcal{M}_\alpha^H(\mathbb{P}^r, d) \hookrightarrow \widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d).$$

(ii-1) For each map f in $\widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$, the dual graph Δ of the domain is obtained as follows.

- Vertices labelled by degrees correspond to the irreducible components of f. Vertices of degree 0 satisfy the usual stability condition.
- The edges correspond to the nodes of f.
- Numbered legs correspond to the markings. The multiplicities α are distributed among the legs of Δ.
- We write α_v for the *ordered* collection of multiplicities of the legs incoming to the vertex v to which we adjoin 0's for all incoming edges (corresponding to the fact that the nodes of a map in $\widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$ cannot be sent to H). The assignment of the multiplicities to the incoming flags is part of the datum of α_v.
- The degree d_v of the vertex v is computed from the multiplicities: $|\alpha_v| = d_v$.

For each graph Δ as above, we consider the stratum in $\widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$ of maps whose dual graph is precisely Δ. This is the image of the fibered product $\mathcal{M}^{\Delta,H}_\alpha(\mathbb{P}^r, d)$ of open Gathamann spaces under the gluing maps:

$$\mathcal{M}^{\Delta,H}_\alpha(\mathbb{P}^r, d) = \left(\prod_{v\in V(\Delta)} \mathcal{M}^H_{\alpha_v}(\mathbb{P}^r, d_v)\right)^{E(\Gamma)} \to \widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d).$$

The fibered product is computed along the evaluation maps on the corresponding moduli spaces, as determined by the edges of Δ.

(ii-2) One also defines the stack $\widetilde{\mathcal{M}}^{\Delta,H}_\alpha(\mathbb{P}^r, d)$ by taking the analogous fiber product. Its image in $\widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$ are the maps intersecting H properly with domain type "at least" Δ. We write $\overline{\mathcal{M}}^\Delta_{0,n}$ for the closure of the stratum of marked stable curves whose dual graph is the graph underlying Δ (forgetting the multiplicity labels). It follows that:

$$\widetilde{\mathcal{M}}^{\Delta,H}_\alpha(\mathbb{P}^r, d) = \overline{\mathcal{M}}^\Delta_{0,n} \times_{\overline{\mathcal{M}}_{0,n}} \widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d). \tag{8}$$

(ii-3) The similar fibered product of closed Gathmann spaces is defined as:

$$\overline{\mathcal{M}}^{\Delta,H}_\alpha(\mathbb{P}^r, d) = \left(\prod_{v\in V(\Delta)} \overline{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v)\right)^{E(\Gamma)}. \tag{9}$$

We will see later that $\overline{\mathcal{M}}^{\Delta,H}_\alpha(\mathbb{P}^r, d)$ is truly a compactification of $\mathcal{M}^{\Delta,H}_\alpha(\mathbb{P}^r, d)$.

(iii) We will need to deal with unmarked smooth points of the domain mapping to H. This requires manipulations of a stack obtained from Gathmann's via the forgetful morphisms. We fix a collection of non-negative integers $\beta = (\beta_1, \dots, \beta_n)$ and a collection of positive integers $\delta = (\delta_1, \dots, \delta_m)$, satisfying the requirement $d = |\beta| + |\delta|$. The write $\mathcal{M}^H_{\beta,\delta}(\mathbb{P}^r, d)$ for the image of the open Gathmann stack via the forgetful morphism:

$$\mathcal{M}^{H}_{\beta\cup\delta}(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_{n+m}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d).$$

The open stack $\mathcal{M}^{H}_{\beta,\delta}(\mathbb{P}^r, d)$ parametrizes stable maps $f : (C, x_1 \ldots, x_n) \to \mathbb{P}^r$ with irreducible domains such that:

$$f^{\star}H = \sum \beta_i x_i + \sum \delta_j y_j,$$

for some distinct unmarked points $y_j \in C$. As before we can construct the companion intermediate and closed stacks denoted $\widetilde{\mathcal{M}}^{H}_{\beta,\delta}(\mathbb{P}^r, d)$ and $\overline{\mathcal{M}}^{H}_{\beta,\delta}(\mathbb{P}^r, d)$.

With these preliminaries under our belt, we repackage the datum carried by each of the graphs Γ indexing a fixed locus, into a fibered product $\mathcal{X}_{\Gamma}$ of Kontsevich and Gathmann spaces. Precisely, we define

$$\mathcal{X}_{\Gamma} = \left(\prod_{v} \mathcal{M}^{H}_{\beta_v,\delta_v}(\mathbb{P}^r, d_v) \times_H \prod_{w} \overline{\mathcal{M}}_{0,n(w)}(H, d_w) \right)^{E(\Gamma)}.$$

The set α_v defined in Section 1.1 is partitioned into two parts $\beta_v \cup \delta_v$. δ_v collects the degrees of the $\mathfrak{u}$ *terminal* edges whose endpoint labeled w is very unstable. The degree of the vertex v is computed from the the degrees of the incident edges:

$$d_v = \sum_{e} d_e = |\alpha_v| = |\beta_v| + |\delta_v|.$$

The fibered product above is obtained as usual along the evaluation maps on the moduli spaces, as determined by the edges of Γ.

A general point of $\mathcal{X}_{\Gamma}$ is obtained as follows.

- For each vertex w marked by (H, d_w) we construct a stable map f_w with $n(w)$ markings and domain curve C_w. The map f_w to H has degree d_w. For the *unstable* vertices w this construction is interpreted as points mapping to H.
- For each vertex v labeled p, we construct a stable map f_v with smooth domain C_v and contact orders $\beta_v \cup \delta_v$ with H.
- We join the domain curves C_v and C_w at a node each time there is an edge incident to both v and w. Edges e which contain unstable w's should be interpreted as giving a special point on C_v. This special point should be a node of the map if w has two incoming edges, or a marking if w has one incoming edge and an attached leg, or an unmarked point mapping to H when w is very unstable.
- For each v labeled p, the map f_v has degree $d_v = \sum_e d_e = |\beta_v| + |\delta_v|$ on the component C_v, the sum being taken over all edges incident to v. Moreover, such an edge e corresponds to a special point on C_v which maps to H and we require that the contact order of the map with H at that point be d_e.

It is clear that by Lemma 5, the limit of the flow of the above map has dual graph Γ.

To describe the Białynicki-Birula cell, we will carry out our discussion so that it only involves the stacks in (i) and (ii). To this end, we mark all the smooth domain points mapping to H. Combinatorially, this corresponds to modifying the very unstable vertices in Γ. We let γ be the graph obtained from Γ by adding legs at each of the terminal very unstable vertices w. Then $\mathcal{X}_\Gamma$ is the image of the fibered product

$$(\mathcal{X}_\gamma =) \, \mathcal{Y}_\Gamma = \left(\prod_v \mathcal{M}^H_{\alpha_v}(\mathbb{P}^r, d_v) \times_H \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w) \right)^{E(\Gamma)}$$

under the morphism:

$$\overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$$

which forgets the markings corresponding to the $\mathfrak{u}$ newly added legs of $\gamma \to \Gamma$. We analogously define the companion stacks $\widetilde{\mathcal{Y}}_\Gamma$ and $\overline{\mathcal{Y}}_\Gamma$ (and their images $\widetilde{\mathcal{X}}_\Gamma$ and $\overline{\mathcal{X}}_\Gamma$):

$$\widetilde{\mathcal{Y}}_\Gamma = \left(\prod_v \widetilde{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v) \times_H \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w) \right)^{E(\Gamma)} \tag{10}$$

$$\overline{\mathcal{Y}}_\Gamma = \left(\prod_v \overline{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v) \times_H \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w) \right)^{E(\Gamma)}. \tag{11}$$

There is a morphism $\overline{\mathcal{Y}}_\Gamma \to \overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ obtained as compositions of:

- gluing morphisms,
- forgetful morphisms,
- inclusions of Kontsevich stacks $\overline{\mathcal{M}}_{0,m}(H, d) \hookrightarrow \overline{\mathcal{M}}_{0,m}(\mathbb{P}^r, d)$,
- inclusions of Gathmann stacks $\overline{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_{0,m}(\mathbb{P}^r, d)$.

We obtained an open immersion whose image is dense as we will show in Lemma 8 :

$$\mathcal{Y}_\Gamma \hookrightarrow \widetilde{\mathcal{Y}}_\Gamma \hookrightarrow \overline{\mathcal{Y}}_\Gamma.$$

Lemma 6. *The stack $\widetilde{\mathcal{Y}}_\Gamma$ is a smooth locally closed substack of $\overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d)$ of codimension $d + \mathfrak{s} - \mathfrak{u}$.*

Proof. We observe that for any collection of weights α, the evaluation morphism:

$$ev_1 : \widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d) \to H \tag{12}$$

is smooth. First, the source is smooth. This is proved in [G1] for $\mathcal{M}^H_\alpha(\mathbb{P}^r, d)$. To pass to the nodal locus, an argument identical to that of lemma 10 in [FP] is required. As a consequence, there is a non-empty open set in the base over which the morphism is smooth. The group $PGL(H)$ acts on $\mathbb{P}^r$ in the usual fashion, and thus also on the domain and target of (12). As the morphism is $PGL(H)$-equivariant and the action on the base is transitive, the claim follows.

To prove the lemma, we follow an idea of [KP]. We will induct on the number of vertices of the tree Γ, the case of one vertex being clear. We will look at the terminal vertices of Γ with only one incident edge.

Pick a terminal stable vertex $\mathfrak{w}$ labelled $(H, d_{\mathfrak{w}})$, if it exists. A new graph Γ' is obtained by relabelling $\mathfrak{w}$ by $(H, 0)$ and removing all its legs. The morphism

$$\widetilde{\mathcal{Y}}_{\Gamma} \to \widetilde{\mathcal{Y}}_{\Gamma'}$$

is smooth as it is obtained by base change from the smooth morphism

$$ev : \overline{\mathcal{M}}_{0,n(\mathfrak{w})}(H, d_{\mathfrak{w}}) \to H.$$

We can now assume all terminal vertices are either labelled p, or labelled H but unstable. Removing all terminal H-labeled vertices from Γ, we obtain a new tree whose terminal vertices are all labeled p. Pick a termial vertex $\mathfrak{v}$ in the new tree. It is connected to (at most) one vertex $\mathfrak{w}$. $\mathfrak{v}$ was connected to the terminal vertices $\mathfrak{w}_1, \ldots, \mathfrak{w}_k$ in the old tree Γ. A new graph Γ' is obtained from Γ by removing the subtree spanned by the vertices $\mathfrak{v}, \mathfrak{w}_1, \ldots, \mathfrak{w}_k$ and replacing it with a leg attached at $\mathfrak{w}$. The same argument as before applies. We base change $\widetilde{\mathcal{Y}}_{\Gamma'}$ by the smooth morphism (12). Here α is the collection $\alpha_{\mathfrak{v}}$ of degrees of the edges incoming to $\mathfrak{v}$ and the evaluation is taken along the marking corresponding to the flag joining $\mathfrak{v}$ and $\mathfrak{w}$.

To compute the dimension of $\widetilde{\mathcal{Y}}_{\Gamma}$, we look at the contribution of each vertex w labeled (H, d_w), of each vertex v labeled p, and we subtract the contribution of contact multiplicities. Assuming all H labeled vertices are stable, we obtain the following formula for the dimension of $\widetilde{\mathcal{Y}}_{\Gamma}$:

$$\begin{aligned}
& \sum_{w} (rd_w + (r-1) + n(w) - 3) + \sum_{v} ((r+1)d_v + r + n(v) - 3 - |\alpha_v|) - \sum_{e} (r-1) \\
= \quad & r\left(\sum_{v} d_v + \sum_{w} d_w\right) + \left(\sum_{v} n(v) + \sum_{w} n(w)\right) + ((r-3)V + (r-4)W - (r-1)E) + \\
+ \quad & \left(\sum_{v} d_v - \sum_{v} |\alpha_v|\right) = rd + (n + 2E) + (-2E - W + r - 3) + 0.
\end{aligned}$$

In the above formula we used the fact that $V + W = E + 1$. Thus the codimension of $\widetilde{\mathcal{Y}}_{\Gamma}$ in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ equals $d + W$. The formula needs to be appended accordingly for the unstable vertices. The final answer for the codimension of $\widetilde{\mathcal{Y}}_{\Gamma}$ in $\overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d)$ becomes $d + \mathfrak{s} - \mathfrak{u}$.

Lemma 7. *There are $d + \mathfrak{s} - \mathfrak{u}$ negative weights on the normal bundle of $\mathcal{F}_{\Gamma}$.*

Proof. The arguments used to prove this lemma are well known (see for example [GP] for a similar computation). In the computation below, we will repeatedly use the fact that the tangent space $T_x\mathbb{P}^r$ has $\mathbb{C}^\star$ weights $D, \ldots, D$ for $x = p$ and weights $0, \ldots, 0, -D$ if $x \in H$.

Recall the description of the stable maps in $\mathcal{F}_\Gamma$ which was given in the discussion in the first section following equation (3). We let $(f, C, x_1, \ldots, x_n)$ be a generic stable map in $\mathcal{F}_\Gamma$ such that C_v and C_w are irreducible. We will compute the weights on the normal bundle at this generic point. These are the non-zero weights of the term $\mathcal{T}_f$ of the following exact sequence:

$$0 \to Ext^0(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to H^0(C, f^\star T\mathbb{P}^r) \to \mathcal{T}_f \to Ext^1(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to 0. \tag{13}$$

We will count the negative weights on the first, second and fourth term above.

The first term gives the infinitesimal deformations of the marked domain. Contributions come from deformation of the domains of type C_e. An explicit computation shows that the deformation space of such rational components with two special points, which need to be fixed by the deformation, is one dimensional with trivial weight. There is one exception in case the special points are not marked or nodes. This exceptional case corresponds to very unstable vertices. We obtain one negative weight for each such vertex, a total number of $\mathfrak{u}$, as one checks by a local coordinate computation.

Similarly, the fourth term corresponds to deformations of the marked domains. We are interested in the smoothings of nodes x lying on two components D_1 and D_2. The deformation space is $T_xD_1 \otimes T_xD_2$. The nodes lying on C_e and C_v give positive contributions. We obtain negative weights for nodes joining components C_e and C_w for stable w, and also for nodes lying on two compoents C_{e_1} and C_{e_2}, which correspond to unstable w's with two incoming edges. The number of such weights equals F. Here F is the number of flags (legs or half-edges) whose initial vertex is labeled w and which is either stable or unstable with two incoming edges.

The weights on the second term will be computed from the exact sequence

$$0 \to H^0(C, f^\star T\mathbb{P}^r) \to \bigoplus_v H^0(C_v, f_v^\star T\mathbb{P}^r) \bigoplus_w H^0(C_w, f_w^\star T\mathbb{P}^r) \bigoplus_e H^0(C_e, f_e^\star T\mathbb{P}^r) \to$$
$$\to \bigoplus_{\mathfrak{f}_v} T_{\mathfrak{f}_v}\mathbb{P}^r \bigoplus_{\mathfrak{f}_w} T_{\mathfrak{f}_w}\mathbb{P}^r \to 0.$$

Here $\mathfrak{f}_v$, $\mathfrak{f}_w$ are the flags of Γ labeled by their initial vertices v and w. They correspond to nodes of the domain mapping to p and H, hence the terms $T_{\mathfrak{f}_v}\mathbb{P}^r$ and $T_{\mathfrak{f}_w}\mathbb{P}^r$ in the exact sequence above. The third term of the exact sequence above receives one negative contribution for each of the flags $\mathfrak{f}_w$. We obtain the following contributions to the negative weights of $H^0(C, f^\star T\mathbb{P}^r)$ coming from the middle term. There are no negative contribution to $H^0(C_v, f_v^\star T\mathbb{P}^r) = T_p\mathbb{P}^r$. The Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathbb{C}^{r+1} \to T\mathbb{P}^r \to 0$$

and arguments similar to [GP] can be used to deal with the remaining two middle terms. Stable vertices labeled w will contribute $d_w + 1$ negative weights on $H^0(C_w, f_w^\star T\mathbb{P}^r)$. Similarly there will be d_e negative weights on $H^0(C_e, f_e^\star T\mathbb{P}^r)$. We find that the number of negative weights coming from the second term of the complex (13) equals

$$\sum_w (d_w + 1) + \sum_e d_e - F = d + \mathfrak{s} - F.$$

Summarizing, the combined contributions of the terms in (13) is $d + \mathfrak{s} - \mathfrak{u}$.

We constructed open immersions $\mathcal{Y}_\Gamma \hookrightarrow \widetilde{\mathcal{Y}_\Gamma} \hookrightarrow \overline{\mathcal{Y}}_\Gamma$. The following lemma clarifies the relationship between these spaces.

Lemma 8. *The image of $\mathcal{Y}_\Gamma$ is dense in $\overline{\mathcal{Y}}_\Gamma$. The stack $\overline{\mathcal{Y}}_\Gamma$ is reduced and irreducible.*

Proof. Since $\mathcal{Y}_\Gamma \hookrightarrow \overline{\mathcal{Y}}_\Gamma$ is an open immersion, we only need to show that $\overline{\mathcal{Y}}_\Gamma$ is irreducible. We observe that the smooth stack $\mathcal{Y}_\Gamma$ is irreducible. Indeed, we can prove $\mathcal{Y}_\Gamma$ is connected by analyzing the $\mathbb{C}^\star$ action. Using Lemma 5 all maps flow to one connected fixed locus which is the image of the connected stack:

$$\prod_v \mathcal{M}_{0,n(v)} \times \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w).$$

To prove the irreducibility of $\overline{\mathcal{Y}}_\Gamma$ one uses the same arguments as in Lemma 1.13 in [G1]. By the previous paragraph, it is enough to show that any map f in $\overline{\mathcal{Y}}_\Gamma$ can be deformed to a map with fewer nodes. Picking a map in $\overline{\mathcal{Y}}_\Gamma$ is tantamount to picking maps f_v and f_w in $\overline{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v)$ and in $\overline{\mathcal{M}}_{0,n(w)}(H, d_w)$ for each vertex v and w of γ, with compatible gluing. For each of the vertices v of γ, Gathmann constructed a deformation of f_v over a smooth curve such that the generic fiber has fewer nodes. We attach the rest of the domain of f to the aforementioned deformation. To be able to glue the remaining components at the markings, we match the images of the markings by acting with automorphisms of $\mathbb{P}^r$ which preserve H. The details are identical to those in [G1].

Proposition 2. *The closed cell $\overline{\mathcal{F}^+_\Gamma}$ is the image of the fibered product $\overline{\mathcal{Y}}_\Gamma$ of closed Gathmann and Konstevich spaces to H under the tautological morphisms. Alternatively, it is the generically finite image of the stack $\overline{\mathcal{X}}_\Gamma$.*

Proof. It is enough to show, by taking closures and using Lemma 8, that the stack theoretic image of $\widetilde{\mathcal{Y}}_\Gamma \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is dense in $\mathcal{F}^+_\Gamma$. We observe that the geometric

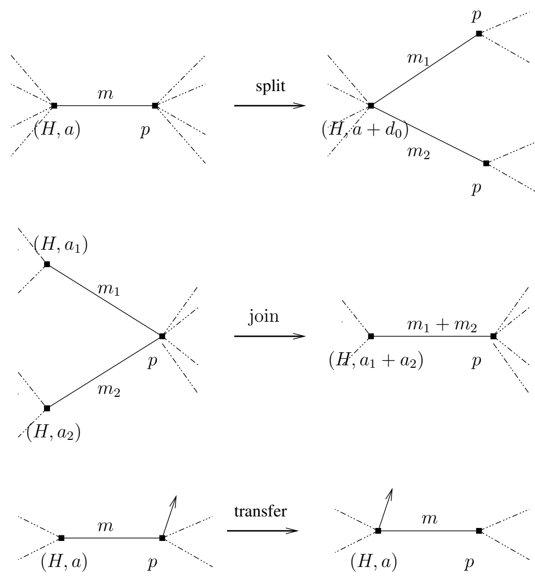

FIGURE 6. Split, joins and transfers

points in the image of $\widetilde{\mathcal{Y}}_\Gamma$ are contained in $\mathcal{F}_\Gamma^+$ because of Lemma 5. Moreover their dimensions match by Lemmas 6 and 7. $\overline{\mathcal{F}_\Gamma^+}$ is reduced and irreducible because $\mathcal{F}_\Gamma$ clearly is, thanks to equation (3). The same is true about $\widetilde{\mathcal{Y}}_\Gamma$. These observations give our claim. The proof of Proposition 3 shows that maps in $\overline{\mathcal{Y}}_\Gamma \setminus \widetilde{\mathcal{Y}}_\Gamma$ cannot flow to a map whose dual graph is Γ. As an afterthought, we obtain that the stack theoretic image of $\widetilde{\mathcal{Y}}_\Gamma$ equals $\mathcal{F}_\Gamma^+$.

3.3. **Filterability.** We will now establish the filterability condition (c) of Lemma 3 which will allow us to prove the tautology of all Chow classes. In this subsection we define an *explicit* partial ordering on the set of graphs indexing the fixed loci.

For any two decorated graphs Γ and Γ' indexing the fixed loci, we decree that $\Gamma' \geq \Gamma$ provided that there is a sequence of combinatorial transformations called *splits*, *joins* and *transfers* changing the graph Γ into Γ'. Each one of these moves is shown in Figure 6. Figure 7 explains the intuition behind this ordering; we exhibit families of maps in a given Białynicki-Birula cell degenerating to a boundary map which belongs to a different cell. The new cell should be higher in our ordering than the original one. In Figure 7, we used the letter a for the degrees, and the letter m for the multiplicity orders with H. Components mapping to H are represented by thick lines.

Explicitly,

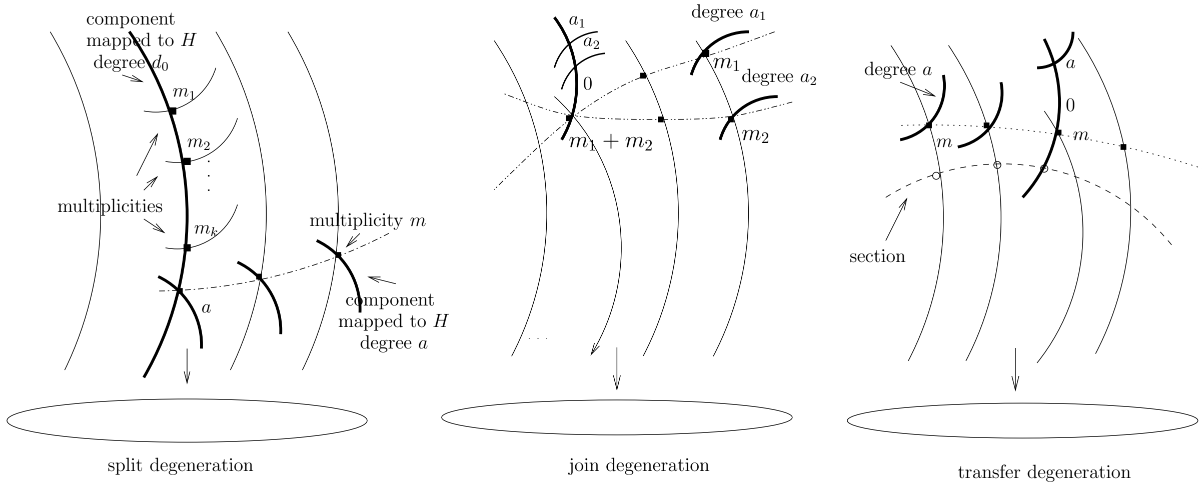

FIGURE 7. Models for the combinatorial moves

- The *split* move takes an edge of degree m and splits it into several edges with positive degrees $m_1, \ldots, m_k$ for $k \geq 1$. The vertex labeled (H, a) is relabeled $(H, a + d_0)$, while the vertex labeled p is replaced by k vertices labeled p. The incoming edges and legs to the vertex p are distributed between the newly created vertices. We require that $m = d_0 + m_1 + \ldots + m_k$. The split move is obtained by degenerating a sequence of maps containing a point mapping to H with multiplicity m. Such a degeneration is constructed in [G1]. The central fiber is a stable map in the boundary of the Gathmann space. There is an "internal" component mapped to H of degree d_0, to which other components are attached, with multiplicities $m_1, \ldots, m_k$ at H. The figure also shows an additional component mapped to H with degree a which is attached to the family.
- The *join* move takes two (or, by applying it successively, several) edges of degrees m_1 and m_2 meeting in vertex labeled p and replaces them by a single edge whose degree is $m_1 + m_2$, also collecting the two vertices labeled H, their degrees and all their incoming edges into a single vertex. Locally, the join move corresponds to a family of maps having two points mapping to H with multiplicities m_1 and m_2 (there are additional components mapping to H with degrees a_1 and a_2 attached at these points). Letting the two points collapse, we obtain a boundary map with a point mapping to H with multiplicity $m_1 + m_2$.
- The *transfer* move can be applied to edges such that the vertex labeled p has an attached leg. We move the leg to the other end of the edge, labeled H. This move can be realized by a family of maps with one marking, containing points

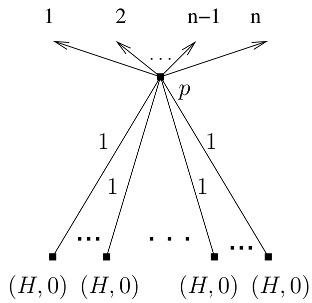

FIGURE 8. The minimal graph

which map to H with multiplicity m. In the limit, the point mapping to H and the marking collapse.

To check that we have indeed defined a partial ordering we introduce the following length function:

$$\begin{aligned} l(\Gamma) &= \sum_e (e-1)\cdot \#\{\text{vertices labelled } (H,e)\} + \#\{\text{vertices labeled } p\} + \\ &+ \#\{\text{legs incident to } H \text{ labeled vertices}\}. \end{aligned}$$

The binary relation " $\geq$ " is indeed anti-symmetric since if

$$\Gamma' > \Gamma \text{ then } l(\Gamma') > l(\Gamma).$$

Moreover, by the definition of the three moves, it is clear that condition (b) is satisfied, the unique minimal element being represented by the graph in Figure 8.

3.4. **The spanning cycles.** We will construct a family of cycles Ξ satisfying the filterability condition (c) of Lemma 3.

To begin with, we compare the cohomology and the Chow groups of the fixed loci, assuming that the filterability condition is satisfied.

Lemma 9. *The rational cohomology and rational Chow groups of $\mathcal{F}_\Gamma$ are isomorphic. The rational cohomology and rational Chow groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ are isomorphic.*

Proof. We will use induction on r. There are two statements to be proved in the above lemma, we call them A_r and B_r respectively. It is proved in [K] that B_0 is true. Lemma 3 shows that $A_r \implies B_r$. We conclude the proof by showing $B_{r-1} \implies A_r$. Indeed, the cohomology of $\mathcal{F}_\Gamma$ can be computed using equation (3):

$$H^\star\Big(\prod_v \overline{\mathcal{M}}_{0,n(v)} \times \prod_w \overline{\mathcal{M}}_{0,n(w)}(H,d_w)\Big)^{A_\Gamma} = \left(\otimes_v H^\star(\overline{\mathcal{M}}_{0,n(v)}) \otimes_w H^\star(\overline{\mathcal{M}}_{0,n(w)}(H,d_w))\right)^{A_\Gamma}.$$

It is remarkable that the same formula holds for the Chow groups. This follows from Theorem 2 in [K]. Our claim is established.

Corollary 2. *Let X be any homogeneous space of the type G/P where G is semisimple algebraic group and P is a parabolic subgroup and $\beta \in A_1(X)$. Then parts (i) and (ii) of Lemma* 3 *are true for $\overline{\mathcal{M}}_{0,n}(X, \beta)$.*

Proof. We use a $\mathbb{T}$-action on X with isolated fixed points. The fixed loci of the induced action on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ are, up to a finite group action, products of the Deligne Mumford spaces $\overline{\mathcal{M}}_{0,n}$. The proof of the above lemma shows that the rational cohomology and Chow groups of the fixed loci are isomorphic. Using Corollary 1 and Proposition 1 we obtain a Białynicki-Birula decomposition on $\overline{\mathcal{M}}_{0,n}(X, \beta)$. We only need to verify the conditions (a) and (b) of Lemma 3 on closed points. Hence we can pass to the coarse moduli schemes (moduli schemes are considered in the sense of Vistoli [V]). The two conditions are satisfied on the projective irreducible [KP] coarse moduli scheme $\overline{M}_{0,n}(X, \beta)$ of $\overline{\mathcal{M}}_{0,n}(X, \beta)$ as shown in [B2]. We conclude observing that the image of $\mathcal{F}_i^+$ in $\overline{M}_{0,n}(X, \beta)$ is the corresponding Bialynicki-Birula cell F_i^+. In fact, one shows that F_i^+ is a coarse moduli scheme for $\mathcal{F}_i^+$. This is because $\mathcal{F}_i^+$ is reduced as $\mathcal{F}_i$ is reduced.

The proof of Lemma 9 also gives the family Ξ. For each graph Γ, we will carry out the following construction:

- for each vertex v, pick a cycle class σ_v on $\overline{\mathcal{M}}_{0,n(v)}$,
- for each vertex w, pick a cycle class σ_w on $\overline{\mathcal{M}}_{0,n(w)}(H, d_w)$,
- average out these choices under the action of A_Γ.

In fact, we can pick explicit representatives for the above classes. Since $A^\star(\overline{\mathcal{M}}_{0,n(v)})$ is generated by boundary classes, we may assume

- σ_v is the locus $\overline{\mathcal{M}}^{\Delta_v}_{0,n(v)}$ of curves with dual graph Δ_v. Here Δ_v is a stable graph with $n(v)$ labeled legs.

In a similar fashion, we could assume, arguing by induction on the dimension of the projective space, that the σ_w's are represented by *tautological* substacks of the form (14) below, with $\mathbb{P}^r$ replaced by H.

To describe the aforementioned tautological substacks, consider the loci of maps in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ with fixed dual graph Δ, such that:

(i) the markings and nodes map to general linear subspaces of $\mathbb{P}^r$ of fixed codimension. Specifically, for each leg l (corresponding to a marking), and each edge e

(corresponding to a node) of Δ, we pick general linear subspaces $H_l \subset \mathbb{P}^r$ and $H_e \subset \mathbb{P}^r$;

(ii) the domain of the map intersects general linear subspaces of $\mathbb{P}^r$. Specifically, for each vertex v corresponding to a component of the domain, we pick a set $\mathcal{H}_v$ of general linear subspaces incident to the map.

We can denote the corresponding substack as

$$\overline{\mathcal{M}}_\Delta \left[\mathcal{H}_v, H_e, H_l\right]. \tag{14}$$

Mark the incidence points in (ii) and write Ω for the resulting dual graph. We can represent (14) as a pushforward under a composition of the marking *forgetful morphism* $\Omega \to \Delta$ and the *gluing morphism* assembling maps with graph Ω of the stacks

$$\bigcap_{f \text{ leg or edge of } \Omega} \mathrm{ev}^{-1} H_f. \tag{15}$$

By Bertini, the latter intersection can be assumed smooth. The cycles (14) give additive generators for $R^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$, see the Appendix in [O].

In the following, ξ will be the A_Γ-average of the cycles

$$\left[\prod_v \overline{\mathcal{M}}^{\Delta_v}_{0,n(v)} \times \prod_w \sigma_w\right]. \tag{16}$$

Proposition 3. *The filterability condition* (c) *is satisfied for the cycles* ξ *defined above.*

Proof. By construction, it is clear that the cycles ξ span the Chow groups of the fixed loci.

We describe the closed points of ξ^+ informally. The dual graph Δ_v determines a domain curve with several irreducible components joined together at nodes; the contact orders with H (which are determined by the incoming edges d_e to v in Γ) are distributed among these components. Then ξ^+ will be a fibered product of tautological stacks σ_w and smaller Gathmann spaces, one for each vertex of the dual graph Δ_v, with multiplicities determined by the degrees d_e lying on that component.

Formally, we begin by adding one leg at each very unstable vertex of Γ, thus obtaining a graph γ without very unstable vertices. Geometrically, this corresponds to marking *all* the smooth points on the domain which map to H. We obtain a forgetful morphism

$$\overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \tag{17}$$

corresponding to the collapsing map $\gamma \to \Gamma$. We may also assume that the cycles σ_w are smooth and of the form (15).

A priori, the only decorations Δ_v carries are the labelled legs. The legs of Δ_v are in one-to-one correspondence to the incoming flags to v in the graph Γ. However, by Section

1.1, all flags of Γ carry degrees, which are the degrees of the incident edges or 0 for the incident legs. In this manner, we enrich the decorations of Δ_v by using these degrees as multiplicities associated to the legs. We denote by $\alpha(v)$ the collection of multiplicities distributed along the legs of Δ_v. We can then form the fibered product $\widetilde{\mathcal{M}}^{\Delta_v,H}_{\alpha(v)}(\mathbb{P}^r, d_v)$ as in Section 3.2 (ii-2).

We let $\xi^+ = \xi \times_{\mathcal{F}_\Gamma} \mathcal{F}^+_\Gamma$. We showed in Proposition 2 that there is a surjective morphism $\widetilde{\mathcal{Y}}_\Gamma \to \mathcal{F}^+_\Gamma$, inducing a surjective morphism $\xi \times_{\mathcal{F}_\Gamma} \widetilde{\mathcal{Y}}_\Gamma \to \xi^+$. We may need to endow these stacks with their reduced structure, but this suffices for the arguments in Chow. By equations (16), (3), (10) and (8), we derive that ξ^+ is the image of the A_Γ-average of

$$\chi^+ = \left(\prod_v \widetilde{\mathcal{M}}^{\Delta_v,H}_{\alpha(v)}(\mathbb{P}^r, d_v) \times_H \prod_w \sigma_w\right)^{E(\Gamma)}. \tag{18}$$

We similarly define $\overline{\chi^+}$ using

$$\left(\prod_v \overline{\mathcal{M}}^{\Delta_v,H}_{\alpha(v)}(\mathbb{P}^r, d_v) \times_H \prod_w \sigma_w\right)^{E(\Gamma)}. \tag{19}$$

We let $\overline{\xi^+}$ be the its image under the forgetful map (17). It is then clear that

$$\chi^+ \hookrightarrow \overline{\chi^+} \hookrightarrow \overline{\mathcal{Y}}_\Gamma$$

where the first map is an open immersion. By the flatness of (17) same is true about

$$\xi^+ \hookrightarrow \overline{\xi^+} \hookrightarrow \overline{\mathcal{F}^+_\Gamma}.$$

We do not know that $\overline{\xi^+}$ is the closure of ξ^+ (we do not know $\overline{\xi^+}$ is irreducible). However, when formulating Lemma 3 we were careful not to include this as a requirement in condition (c).

Finally, we show that a map f contained in the boundary of $\overline{\xi^+} \setminus \xi^+$ flows to a fixed locus indexed by a graph Γ' with $\Gamma' > \Gamma$. We first make a few reductions. Replacing Γ by γ and ξ^+ by χ^+, we may assume Γ has no very unstable vertices. We want to show that the graph of $F = \lim_{t\to 0} f^t$ is obtained from Γ by a sequence of the combinatorial moves which we called joins, splits and transfers.

The datum of a map f is tantamount to giving maps f_v and f_w in the Gathmann spaces $\overline{\mathcal{M}}^{\Delta_v,H}_{\alpha(v)}(\mathbb{P}^r, d_v)$ and the substacks σ_w with compatible gluing. The unstable vertices w require special care as they only give points on the domain of C_v not actual maps. We have seen that the limit F of f^t is obtained from gluing the individual limits F_v and $F_w = f_w$ (for stable w's) of f^t_v and $f^t_w = f_w$. The dual graphs are also obtained by gluing. Since to compute the limit we consider each vertex at a time, we may further assume that Γ consists in one vertex labeled v to which we attach legs and unstable vertices w. Thus, we may take Γ to be the graph in figure 9.

Now recall that Δ_v encodes the domain type of the nodal map f_v with $n(v)$ markings. The markings of f_v are distributed on the components of the domain and come with multiplicities encoded in the graph Δ_v as above. As we can treat the components individually, we may assume Δ_v has only one vertex. Moreover, the map f_v has to be in the boundary of the Gathmann space

$$\overline{\mathcal{M}}^H_{\alpha(v)}(\mathbb{P}^r, d_v) \setminus \widetilde{\mathcal{M}}^H_{\alpha(v)}(\mathbb{P}^r, d_v).$$

Changing notation slightly, we prove the following. We consider a multindex $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots, 0)$ summing up to d. We consider maps $(f, C, x_1, \ldots, x_n, y_1, \ldots, y_m)$ in the Gathmann space $\overline{\mathcal{M}}^H_{\alpha}(\mathbb{P}^r, d)$. Hence $f^! H = \sum_i \alpha_i x_i$. If f were an element in $\widetilde{\mathcal{M}}^H_{\alpha}(\mathbb{P}^r, d)$ then its limit F would have the dual graph Γ shown in Figure 9. This graph has one vertex v labelled p, n edges labeled $\alpha_1, \ldots, \alpha_n$ joining v to unstable vertices w labeled $(H, 0)$.

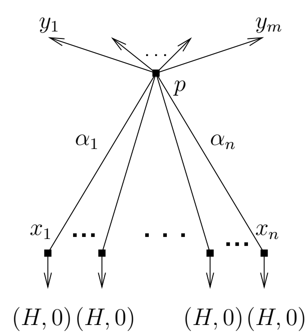

FIGURE 9. The graph Γ of a limit for a generic map in the Gathmann space.

Lemma 10. *Let f be a map in the boundary of the Gathmann stack $\overline{\mathcal{M}}^H_{\alpha}(\mathbb{P}^r, d) \setminus \widetilde{\mathcal{M}}^H_{\alpha}(\mathbb{P}^r, d)$. Let F be the limit of torus flow of f as constructed in Lemma* 5. *Then the dual graph Γ_F of F can be obtained from the graph Γ above by splits, joins and transfers.*

Proof. Equivalently, we will show how undoing the combinatorial moves we can go from the graph Γ_F to Γ. The map f will have components which are contained in H and which are responsible for the different dual graph. Let C_0 be a positive dimensional connected component of $f^{-1}(H)$ on which the map has total degree d_0, and let $C_1, \ldots, C_k$ be the irreducible components joined to C_0, having multiplicities $m_1, \ldots, m_k$ with H at the nodes. Figure 1 shows such a map. In any case, C_0 will contain some of the markings mapping to H, say x_i for $i \in I$, and some of the remaining markings y_j for $j \in J$. The contribution of the components $C_0 \cup C_1 \ldots \cup C_k$ to the dual graph Γ_F, as computed by corollary 5, is shown in the first graph of figure 10. The figure also shows the moves (in reverse order) we apply to this portion of Γ_F to obtain its corresponding contribution to Γ. The rest of the graph Γ_F is attached to the portion shown there and is carried along when performing the combinatorial moves. Observe that existence of the join move is guaranteed by the equation $d_0 + \sum m_i = \sum_{i \in I} \alpha_i$ which follows by considering intersection multiplicities with H. Applying this procedure successively to each vertex (H, d_0) which is not unstable, we arrive in the end at a graph with only unstable H-vertices and all whose edges marked by α's, i.e. at the graph Γ. This completes the proof.

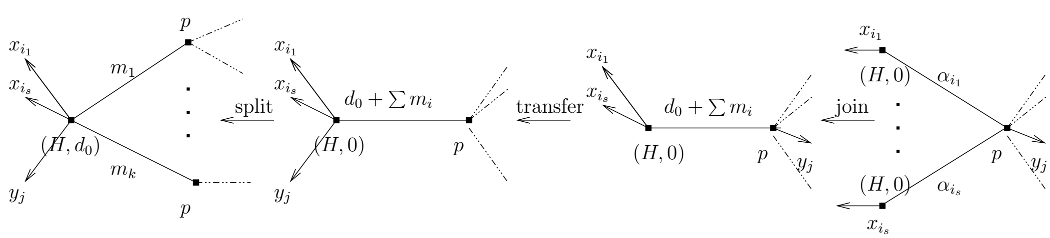

Figure 10. The combinatorial moves comparing Γ_F to Γ.

4. The Chow classes are tautological

In this section we tie the loose ends and prove the main result, Theorem 1. Items (1)-(5) are contained in Proposition 1 and the proof of Corollary 2, Proposition 3, Proposition 2, Lemma 7 and Lemma 9 respectively. Item (6) is a consequence of the proof of Proposition 3 and equations (9) and (19).

The last item (7) uses Lemma 3 (iii). To apply the lemma, we recall the cycles (19) which we prove to be tautological. It suffices to establish the following result.

Lemma 11. (i) *Let $i : H \to \mathbb{P}^r$ be a hyperplane and let $i : \overline{\mathcal{M}}_{0,n}(H, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ denote the induced map. The pushforward map*

$$i_\star : A_\star(\overline{\mathcal{M}}_{0,n}(H, d)) \to A_\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$$

preserves the tautological classes.

(ii) *For each n-multindex α, the class of the Gathmann space $\left[\overline{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)\right]$ is tautological.*

Proof. Consider the bundle $\mathcal{B} = R\pi_\star ev^\star \mathcal{O}_{\mathbb{P}^r}(1)$ where ev and π are the universal evaluation and projection morphisms. This is a rank $d+1$ vector bundle on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. As usual, the equation of H gives a section of $\mathcal{B}$ which vanishes precisely on $\overline{\mathcal{M}}_{0,n}(H, d)$.

We claim that

$$R^\star(\overline{\mathcal{M}}_{0,n}(H, d)) \subset i^\star R^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)).$$

This follows by inspecting the additive generators (14) of the tautological rings. Thus any tautological class α on $\overline{\mathcal{M}}_{0,n}(H, d)$ is the restriction of a tautological class β on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Therefore,

$$i_\star\alpha = i_\star i^\star\beta = \beta \cdot c_{d+1}(\mathcal{B}).$$

It suffices to prove that $c_{d+1}(\mathcal{B})$ is tautological. A computation identical to Mumford's [M] using Grothendieck-Riemann-Roch shows:

$$ch(\mathcal{B}) = \pi_\star\left(e^{ev^\star H}\cdot\left(\frac{c_1(\omega_\pi)}{e^{c_1(\omega_\pi)}-1} + i_\star P(\psi_\star, \psi_\bullet)\right)\right). \tag{20}$$

Here P is a universal polynomial whose coefficients can be explicitly be written down in terms of the Bernoulli numbers. The morphism i is the codimension 2 inclusion of the nodes of the fibers of the universal curve $\pi : \overline{\mathcal{M}}_{0,n\cup\{\diamond\}}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Under the standard identifications, this can be expressed as images of fibered products :

$$i : \overline{\mathcal{M}}_{0,S_1\cup\{\star\}}(\mathbb{P}^r, d_1) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,\{\star,\diamond,\bullet\}}(\mathbb{P}^r, 0) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,\{\bullet\}\cup S_2}(\mathbb{P}^r, d_2) \to \overline{\mathcal{M}}_{0,S_1\cup S_2\cup\{\diamond\}}(\mathbb{P}^r, d)$$

for all partitions $S_1 \cup S_2 = \{1, \ldots, n\}$ and $d_1 + d_2 = d$. The classes $\psi_\star$ and $\psi_\bullet$ in the equation (20) are the cotangent lines at the markings $\star$ and $\bullet$ which are joined at a node.

To prove out claim, we need to argue that $c_1(\omega_\pi)$ and ψ are tautological. This follows from the results of [Pa], where it is shown that all codimension 1 classes are tautological.

Part (ii) is a consequence of equation (4), using induction on the multindex α. The correction terms are pushforwards of classes on the boundary strata. These classes are either lower dimensional Gathmann spaces or Kontsevich spaces to H which are tautological by induction and by part (i) of the lemma respectively.

Lemma 11 gives the last claim (7) of Theorem 1 when the graph Γ has only one vertex. The general case poses the following difficulty: we do not know that the stacks $\overline{\chi^+}$ obtained in Proposition 3 are of the expected dimension. The argument of Lemma 6 only gives this claim for the open substack χ^+ defined in (19). Thus, we cannot immediately conclude that the Chow classes of $\overline{\chi^+}$ are the *refined Gysin gluing* of the the fiber product factors. However, the difference between the two classes is sum of boundary terms, which flow to $\mathcal{F}_{\Gamma'}$ with $\Gamma' > \Gamma$ as shown in Proposition 3. An induction on the graphs Γ finishes the argument.

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