Math 200 - Homework 6. Due Friday, Nov. 5.

From the textbook solve the following:

2. Page 256-259, problems 8, 15, 25, 27, 32.

Assume $A$ is a commutative ring with unity.

4. Show that an ideal $a \neq (1)$ is an intersection of prime ideals iff $a = \sqrt{a}$.

5. The prime spectrum of a ring $X = \text{Spec}(A)$ is the set of all prime ideals $p$ of $A$. For each $E \subset \text{Spec}(A)$, define $V(E)$ to be the set of all prime ideals that contain $E$. Show that

   (i) If an ideal $a$ is generated by $E$, then
   
   $V(E) = V(a) = V(\sqrt{a})$.
   
   (ii) $V(\bigcup_{i \in I} E_i) = \bigcap_{i \in I} V(E_i)$
   
   (iii) $V(a \cap b) = V(a) \cup V(b)$
   
   (iv) Show that the sets $V(E)$ for $E \subset A$ satisfy the axioms for closed sets of a topological space. This gives $\text{Spec}(A)$ a topology, called the Zariski topology.

   (v) Draw pictures of the topological spaces $\text{Spec} \mathbb{C}[x]$ and $\text{Spec} \mathbb{Z}$.

6. For each $f \in A$, write $X_f$ for the complement of $V(f)$ in $X = \text{Spec} A$. The sets $X_f$ are called the basic open sets. Show that

   (i) $X_{fg} = X_f \cap X_g$
   
   (ii) $X_f = X$ iff $f$ is a unit
   
   (iii) $X_f = \emptyset$ iff $f$ is nilpotent
   
   (iv) more generally, $X_f = X_g$ iff $\sqrt{(f)} = \sqrt{(g)}$
   
   (v) $X$ is a quasicompact topological space i.e. every open covering of $X$ has a finite subcover
   
   (vi) more generally, each $X_f$ is quasicompact.

7. Let $\phi : A \to B$ be a ring homomorphism, and let $X = \text{Spec}A$ and $Y = \text{Spec}B$ be the prime spectra of $A$ and $B$. Define

   $\phi^* : Y \to X, \ \phi^*(q) = \phi^{-1}(q)$
i.e. $f^*$ is the map taking prime ideals of $B$ to their preimages in $A$, which are points of $X$.

(i) Show that $f^*$ is continuous. In particular, check $(\phi^*)^{-1}(X_f) = Y_{\phi(f)}$.

(ii) If $\psi : B \to C$ is another ring homomorphism, check that

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$