Math 200 - Homework 6. Due Friday, Nov. 5.

From the textbook solve the following:

1. Page 250, problems 30, 31.

2. Page 256-259, problems 8, 15, 25, 27, 32.

3. Page 267-268, problems 5, 8.

Assume A is a commutative ring with unity.

4. Show that an ideal $\mathfrak{a} \neq (1)$ is an intersection of prime ideals iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

5. The prime spectrum of a ring $X = \text{Spec}(A)$ is the set of all prime ideals $\mathfrak{p}$ of A. For each $E \subset \text{Spec}\ (A)$, define $V(E)$ to be the set of all prime ideals that contain E. Show that

(i) If an ideal $\mathfrak{a}$ is generated by E, then

$$V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}}).$$

(ii) $V\left(\cup_{i\in I} E_i\right) = \cap_{i\in I} V(E_i)$
(iii) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$
(iv) Show that the sets $V(E)$ for $E \subset A$ satisfy the axioms for closed sets of a topological space. This gives Spec (A) a topology, called the Zariski topology.
(v) Draw pictures of the topological spaces Spec $\mathbb{C}[x]$ and Spec $\mathbb{Z}$.

6. For each $f \in A$, write X_f for the complement of $V(f)$ in $X = \text{Spec}\ A$. The sets X_f are called the basic open sets. Show that

(i) $X_{fg} = X_f \cap X_g$
(ii) $X_f = X$ iff f is a unit
(iii) $X_f = \emptyset$ iff f is nilpotent
(iv) more generally, $X_f = X_g$ iff $\sqrt{(f)} = \sqrt{(g)}$
(v) X is a quasicompact topological space i.e. every open covering of X has a finite subcover
(vi) more generally, each X_f is quasicompact.
(vii) show that X_f is the prime spectrum of the ring A_f.

7. Let $\phi : A \to B$ be a ring homomorphism, and let $X = \text{Spec}A$ and $Y = \text{Spec}B$ be the prime spectra of A and B. Define

$$\phi^\star : Y \to X, \ \ \phi^\star(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$$

i.e. $f^\star$ is the map taking prime ideals of B to their preimages in A, which are points of X.

(i) Show that $f^\star$ is continuous. In particular, check $(\phi^\star)^{-1}(X_f) = Y_{\phi(f)}$.

(ii) If $\psi : B \to C$ is another ring homomorphism, check that

$$(\psi \circ \phi)^\star = \phi^\star \circ \psi^\star.$$