Section 0.4.

0.4.9. a) \((fog o h)(3) = f(g(-1)) = f(-3) = 8\)

\[\begin{align*}
\text{b) } fog o h(1) &= f(g(-2)) \\
&= f(-5) \\
&= 25
\end{align*}\]

0.4.12. a) \((1, 20)\).

\[\begin{align*}
\text{b) One needs to find } x \text{ such that } \\
\ln \circ \ln(x) &> 0 \\
\text{i.e. } \ln(x) &> 1 \\
\text{and thus the domain is } x > e. \text{ (e, } \infty)\n\end{align*}\]

Section 0.6.

0.6.3. The graph of the function \(y = \tan\left(\frac{\pi x}{2}\right)\) is given by:

\[\begin{align*}
\text{Diagram showing the graph of } y = \tan\left(\frac{\pi x}{2}\right)\text{ with key points marked.}
\end{align*}\]
If we restrict the domain to \((-1, 1)\), we get a bijection between \((-1, 1)\) and \(\mathbb{R}\). Any explanation is fine. Hence both sets are uncountable.

0.6.5. a) face era to the right: Clear: if \(a \in A\), \(g(f(a)) \neq g(f(f(a)))\) and so on. As to the left, it all depends on the image of \(f\) and \(g\). If \(a \in A\), then either \(\exists b \in B\) s.t. \(g(b) = a\) or not.

If not we are in case 2. If yes, keep going like this. This process will either stop or go on for ever.

b) Clear by part a.

Let \(x\) be a \((f, g)\)-chain and look at its end points:

- right: Go on far enough according to the process explained in (a)
- left: Go as far as you can to the left as explained above.

The uniqueness follows from the fact that both \(f\) and \(g\) are one-to-one and thus there is a unique way to extend a given \((f, g)\)-chain on both sides.
c) $h$ is well-defined:

It is well-defined since $1, 2, 3$ are mutually exclusive cases.

$h$ is one-to-one: Check all possibilities:

1) $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ since $f$ is one-to-one.

2) $g^{-1}(a_1) = g^{-1}(a_2) \Rightarrow a_1 = a_2$ since $g$ is a function.

3) $f(a_1) = g^{-1}(a_2) \Rightarrow g \circ f(a_1) = a_2$.

We are in the case where $a_2$ belongs to a maximal chain of type 3, i.e.

$b, \ldots, b', a_2, \ldots$

where $b$ is not in the image of $f$.

On the other hand, $a_1$ belongs to max. chain of type 1 or 2, i.e.

$b'', a_1, \ldots$

or

$a, \ldots, b', a_1, \ldots$

where $a$ is not in the image of $g$.

Consider then $a_2$

$\ldots, b'', a_1, f(a_1), g \circ f(a_1), \ldots$

by "gluing" the 2 sequences, we get a contradiction, and thus this case is impossible.

4) $f(C_2) = g^{-1}(a_1)$: similar as above.
h is onto: Let \( b \in B \) and consider 
\[ a = g(b) \in A. \]

There are 3 cases:

1) \( a \) belongs to a max. chain of type 3 in which case 
\[ h(a) = g^{-1}(a) = g^{-1}(g(b)) = b. \]

2) \( a \) belongs to a max. chain of type 2 \( \tau \), in which case we have a \( \tau_{(f, g)} \)-chain 
\[ a'', \ldots, a', b, a, \ldots \]
where \( a'' \) is not in the image of \( g \).

Remark that \( a'' \neq a \) since \( a = g(b) \) is in the image of \( g \).

Thus, \( h(a') = f(a') = b \)
(\( a' \) also belongs to the same max. \( \tau_{(f, g)} \)-chain \( \tau \) as \( a \) and therefore we are in case 2 as well).

3) \( a \) belongs to a max. chain of type 1.

Similar as (2).