Math 31BH - Homework 4. Due Tuesday, February 8.

1. (Wednesday, January 26.) Chain rule. From the textbook, solve 1.8.2, 1.8.10(a), 1.8.11.
2. (Wednesday, February 3.) Euler’s identity. A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree $d$ if
   \[ f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n) \]
   for all real numbers $t, x_1, \ldots, x_n$.
   (i) Show that $f(x_1, \ldots, x_n) = x_1^d + \ldots + x_n^d$ is homogeneous of degree $d$.
   (ii) Show that if $f$ is homogeneous of degree $d$, then
   \[ x_1 \cdot \frac{\partial f}{\partial x_1} + \ldots + x_n \cdot \frac{\partial f}{\partial x_n} = df. \]
   
   Hint: Differentiate the defining equality with respect to $t$ and then set $t = 1$.
3. (Wednesday, February 3.) Laplacian in polar coordinates. Using the chain rule, express the Laplacian of a function $f$ in polar coordinates
   \[ \Delta f = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta \theta}. \]
4. (Wednesday, February 3.) The Laplacian, harmonic functions and orthogonal matrices. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function and let $A$ be an orthogonal $2 \times 2$ matrix. Show that
   \[ g(x) = f(Ax) \]
   is a harmonic function.
5. (Friday, February 5.) Pathological functions. From the textbook, solve 1.9.1 and 1.9.2.
6. (Friday, February 5.) Pathological functions and second order derivatives. Put
   \[ f(x, y) = \begin{cases} 
   xy \cdot \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
   0 & \text{if } (x, y) = (0, 0). 
   \end{cases} \]
   (i) Calculate the first derivative $f_x$ at $(0, 0)$ directly from the definition.
   (ii) Calculate the first derivative $f_x$ at any other point $(x, y) \neq (0, 0)$.
   (iii) Differentiate once more i.e. calculate $f_{xy}(0, 0)$ using the definition. Confirm that $f_{xy}(0, 0) = 1$.
   (iv) Repeat for the derivative $f_y(0, 0)$ then $f_{yx}(0, 0)$. Confirm that $f_{yx}(0, 0) = -1$. Observe that
   \[ f_{xy}(0, 0) \neq f_{yx}(0, 0). \]
7. (Friday, February 5.) Spaces of matrices. From the textbook, solve 1.8.13 and 1.10.30.