PRACTICE PROBLEMS FOR MIDTERM I

Problem 1.
Find the limits
(i) \( \lim_{x,y \to 0} \frac{x^2+y^6}{x^2+y^6} \);
(ii) \( \lim_{x,y,z \to 0} \frac{x^6+y^6+z^6}{x^2+y^2+z^2} \);
(iii) \( \lim_{x,y \to 0} \frac{\sin(x^2+y^2)}{\sqrt{x^2+2y^2}} \).

Problem 2.
Consider the function \( f(x,y) = (x - 1)^2 + (y - 1)^2 \).
(i) Draw the level diagram for \( f \).
(ii) Draw the graph of \( f \).
(iii) Using the \( \epsilon - \delta \) definition, prove that \( f \) is continuous at the origin.

Problem 3.
Consider the function \( f(x,y) = \frac{x^2}{y^2} \).
(i) Carefully draw the level curves of \( f \).
(ii) Single out the level curve passing through \((1,-1)\) and draw the gradient of the function at \((1,-1)\).
(iii) Find the equation of the tangent line to the level curve through \((1,-1)\) at the point \((1,-1)\).
(iv) Compute the directional derivative of \( f \) at \((1,-1)\) in the direction \( u = \left( \frac{4}{5}, \frac{3}{5} \right) \). Use this calculation to estimate \( f((1,-1) + .01u) \).
(v) Find the unit direction \( v \) of steepest descent for the function \( f \) at \((1,-1)\). What is the rate of decrease in that direction?
(vi) Find the two unit directions \( w \) for which the derivative \( f_w(1,-1) = 0 \).

Problem 4.
Consider the function \( f(x,y) = \sqrt{\ln(e^{2x}y^4)} \).
(i) Write down the tangent plane to the graph of \( f \) at \((2,1)\).
(ii) Find the approximate value of the number \( \sqrt{\ln(e^{4.1}(1.02)^3)} \).

Problem 5.
(i) Assume \( F \) is a closed subset of \( \mathbb{R}^n \). Show that if \( x_n \) is a sequence of elements of \( F \) converging to \( x \) in \( \mathbb{R}^n \), then \( x \) is an element of \( F \) as well.
(ii) Is this conclusion true if \( F \) is not necessarily closed?

Problem 6. Give an example of a continuous function \( f : \mathbb{R} \to \mathbb{R} \) and a compact set \( K \subset \mathbb{R} \) such that the preimage of \( K \) is not compact.
Solutions

1.

(i) The limit does not exist. Indeed, we first calculate the limit when \( x = y \to 0 \). The limit equals

\[
\lim_{x \to 0} \frac{x^2 + x^6}{x^6 + x^2} = 1.
\]

Now make \( x = 0 \) and \( y \to 0 \). We have

\[
\lim_{y \to 0} \frac{y^6}{y^2} = \lim_{y \to 0} y^4 = 0.
\]

The answers are different, so the limit does not exist.

(ii) We have

\[
0 \leq \frac{x^6 + y^6 + z^6}{x^2 + y^2} = x^4 \cdot \frac{x^2}{x^2 + y^2 + z^2} + y^4 \cdot \frac{y^2}{x^2 + y^2 + z^2} + z^4 \cdot \frac{z^2}{x^2 + y^2 + z^2}
\]

\[
\leq x^4 \cdot 1 + y^4 \cdot 1 + z^4 \cdot 1 \to 0.
\]

The limit equals 0.

(iii) We have

\[
\left| \frac{\sin(x^2 + y^2)}{\sqrt{x^2 + 2y^2}} \right| \leq \frac{\sin(x^2 + y^2)}{\sqrt{x^2 + 2y^2}} = \frac{\sin z}{\sqrt{z}} \to 0
\]

for \( z = x^2 + y^2 \to 0 \). The last limit is computed by L’Hospital

\[
\lim_{z \to 0} \frac{\sin z}{\sqrt{z}} = \lim_{z \to 0} \frac{\cos z}{\frac{1}{2} \sqrt{z}} = \lim_{z \to 0} 2\sqrt{z} \cos z = 2 \cdot 0 \cdot 1 = 0.
\]

The original limit also equals 0.

2.

(i) The level curves are \((x - 1)^2 + (y - 1)^2 = c\) which are circles of radius \(\sqrt{c}\) centered at \((1, 1)\).

(ii) The graph is a paraboloid with 'vertex' at \((1, 1, 0)\).

(iii) Note first that \(f(0,0) = 2\). We pick \(\epsilon > 0\). We need \(\delta\) such that

\[
||(x,y) - (0,0)|| < \delta \implies ||f(x,y) - f(0,0)|| < \epsilon.
\]

This in turn becomes

\[
||(x - 1)^2 + (y - 1)^2 - 2|| < \epsilon \iff ||(x^2 - 2x + 1) + (y^2 - 2y + 1) - 2|| < \epsilon \iff |x(x - 2) + y(y - 2)| < \epsilon.
\]

We will pick \(\delta \leq 1\). In this fashion

\[
||(x,y)|| < \delta \implies x^2 + y^2 < \delta^2 \implies |x| < \delta, |y| < \delta \implies |x| < 1, |y| < 1 \implies |x - 2| < 3, |y - 2| < 3.
\]

Thus

\[
|x(x - 2) + y(y - 2)| \leq |x||x - 2| + |y||y - 2| < \delta \cdot 3 + \delta \cdot 3 = 6\delta.
\]

If we pick \(\delta \leq \epsilon/6\), we are done. Thus a good choice for \(\delta\) is \(\delta = \min(1, \frac{\epsilon}{6})\).

3.

(i) The level curves with level \(c\) are

\[
\frac{x^2}{y^2} = c \implies x^2 = cy^4 \implies x = \pm \sqrt{c}y^2.
\]

Only levels \(c \geq 0\) are in fact possible. The cases \(c \neq 0\) and \(c = 0\) give different shapes for the level curves. The level curves are two parabolas for \(c \neq 0\) and the line \(x = 0\) for \(c = 0\). In fact, one should also remove the origin \((0,0)\) from all the level curves because the function is not defined there.
(ii) The level is \( f(1, -1) = 1 \). The level curve is

\[
f(x, y) = f(1, -1) = 1 \implies x^2 = y^4 \implies x = \pm y^2.
\]

The level curve is a union of two parabolas through the origin. The gradient

\[
\nabla f = \left( \frac{2x}{y^4}, \frac{-4x^2}{y^5} \right) \implies \nabla f(1, -1) = (2, 4)
\]

is normal to the parabolas.

(iii) The slope of the gradient is \( \frac{4}{2} = 2 \). The tangent line is perpendicular to the gradient hence it has slope \( -\frac{1}{2} \). The equation of the tangent line is

\[
y + 1 = \frac{1}{2} (x - 1) \implies 2(y + 1) = -(x - 1) \implies x + 2y = -1.
\]

(iv) We compute

\[
f_u = \nabla f \cdot u = (2, 4) \cdot \left( \frac{4}{5}, \frac{3}{5} \right) = 4.
\]

For the approximation, we have \( f(1, -1) = 1 \) and

\[
f((1, -1) + .01u) \approx f(1, -1) + .01f_u = 1 + .01 \cdot 4 = 1.04.
\]

(v) The direction of steepest decrease is opposite to the gradient. We need to divide by the length to get a unit vector:

\[
v = -\frac{\nabla f}{||\nabla f||} = -\frac{(2, 4)}{\sqrt{2^2 + 4^2}} = \left( -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right).
\]

The rate of decrease is the directional derivative

\[
f_v = \nabla f \cdot v = (2, 4) \cdot \frac{1}{\sqrt{5}}(-1, -2) = -\frac{10}{\sqrt{5}} = -2\sqrt{5}.
\]

(vi) Write

\[
w = (w_1, w_2).
\]

We have

\[
f_w = \nabla f \cdot w = (2, 4) \cdot w = 2w_1 + 4w_2 = 0 \implies w_1 = -2w_2.
\]

Since \( w \) has unit length

\[
w_1^2 + w_2^2 = 1 \implies (-2w_2)^2 + w_2^2 = 1 \implies w_2 = \pm \frac{1}{\sqrt{5}}.
\]

Therefore

\[
w = \pm \left( -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right).
\]

4.

(i) Using the chain rule, we compute

\[
f_x = \frac{1}{2} \frac{2e^{2x}}{\sqrt{\ln(e^{2x}y^3)}} = \frac{1}{2} \frac{1}{\sqrt{\ln(e^{2x}y^3)}} \implies f_x(2, 1) = \frac{1}{\sqrt{\ln e^4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.
\]

Similarly,

\[
f_y = \frac{1}{2} \frac{3y^2e^{2x}}{\sqrt{\ln(e^{2x}y^3)}} = \frac{3}{2y} \frac{1}{\sqrt{\ln(e^{2x}y^3)}} \implies f_y(2, 1) = \frac{3}{2} \frac{1}{\sqrt{\ln e^4}} = \frac{3}{2} \cdot \frac{1}{\sqrt{4}} = \frac{3}{4}.
\]

We compute

\[
f(2, 1) = \sqrt{\ln e^4} = \sqrt{4} = 2.
\]
The tangent plane is
\[ z - 2 = \frac{1}{2}(x - 2) + \frac{3}{4}(y - 1) \implies z = \frac{1}{2}x + \frac{3}{4}y + \frac{1}{4}. \]

(ii) The number we are approximating is
\[ f(2.05, 1.02) \approx \frac{1}{2} \cdot 2.05 + \frac{3}{4} \cdot 1.02 + \frac{1}{4} = 2.04. \]

5.

(i) We argue by contradiction. Assume \( x \notin F \). Then \( x \in \mathbb{R}^n \setminus F \). Since \( F \) is closed, \( \mathbb{R}^n \setminus F \) is open. This means that there is a small ball \( B_\epsilon(x) \) which is contained in \( \mathbb{R}^n \setminus F \). Of course, this ball \( B_\epsilon(x) \) does not intersect \( F \). Hence it cannot contain any of the members of the sequence \( x_n \) which are all in \( F \). However, by the definition of convergence, we know that for \( n \) large enough,
\[ ||x_n - x|| < \epsilon \implies x_n \in B_\epsilon(x). \]
This is a contradiction. Therefore \( x \in F \).

(ii) The conclusion is false if \( F \) is not closed. For instance, let \( x_n = \frac{1}{n} \), and let \( F = (0, 1) \). Clearly, \( x_n \) converges to \( x = 0 \) but \( x \notin F \).

6. Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x^2 + 1} \). The preimage of the compact set \( K = [0, 1] \) is the entire real line \( \mathbb{R} \) since \( 0 < f(x) < 1 \) is true for all \( x \). The preimage is not compact (since \( \mathbb{R} \) is not bounded).