Problem 1.

Find the limits below or explain why they do not exist:

(i) \( \lim_{x,y \to 0} \frac{(x^2+y^2)^2}{2x^2+3y^2} \)

We note that

\[
0 \leq \frac{(x^2+y^2)^2}{2x^2+3y^2} \leq \frac{(2x^2+3y^2)^2}{2x^2+3y^2} = 2x^2 + 3y^2 \to 0.
\]

Therefore, the original limit equals 0 as well.

(ii) \( \lim_{x,y \to 0} \frac{x^3y}{x^4+y^4} \)

The limit does not exist. Indeed, approaching 0 by keeping \( x = y \to 0 \), the fraction equals

\[
\frac{x^3 \cdot x}{x^4 + x^4} = \frac{1}{2}.
\]

On the other hand, approaching 0 by keeping \( x = 0, y \to 0 \), the fraction equals 0. Since the two answers are different, the limit does not exist.
Problem 2

Consider the function

\[ f(x, y) = x^2 \sin(2y - 2x) \]

and the point \( P(1, \frac{\pi}{2} + 1) \).

(i) Find the gradient of \( f \) at the point \( P \).

We compute the derivatives

\[ f_x = 2x \sin(2y - 2x) - 2x^2 \cos(2y - 2x) \]
\[ f_y = 2x^2 \cos(2y - 2x) \]

\[ \Rightarrow f_x(1, \frac{\pi}{2} + 1) = 2 \sin \pi - 2 \cos \pi = 2 \]
\[ f_y(1, \frac{\pi}{2} + 1) = 2 \cos \pi = -2. \]

Then \( \nabla f(P) = (2, -2) \).

(ii) Calculate the directional derivative of \( f \) at \( P \) in the direction

\[ \vec{u} = \frac{3i + 4j}{5}. \]

We have

\[ f_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = (2, -2) \cdot \left( \frac{3}{5}, \frac{4}{5} \right) = 2 \cdot \frac{3}{5} - 2 \cdot \frac{4}{5} = -\frac{2}{5}. \]

(iii) Find the (unit) direction of steepest decrease for the function \( f(x, y) \) at \( P \). What is the rate of decrease?

We have

\[ \vec{u} = -\frac{\nabla f}{\|\nabla f\|} = \frac{(2, -2)}{\sqrt{2^2 + (-2)^2}} = \left( \frac{2}{\sqrt{8}}, -\frac{2}{\sqrt{8}} \right) = \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right). \]

The rate of change equals

\[ f_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = -\nabla f(P) \cdot \frac{\nabla f(P)}{\|\nabla f(P)\|} = -\|\nabla f(P)\| = -\sqrt{8}. \]

(iv) Find a formula, valid near \( P \), which expresses the change \( \Delta f \) in \( f \) induced by a small change \( \Delta x \) in \( x \) and a small change \( \Delta y \) in \( y \).

We have

\[ \Delta f = 2\Delta x - 2\Delta y. \]
(v) Find the equation to the tangent plane to the graph of $f$ at the point $(P, f(P))$

Note that $f(P) = \sin \pi = 0$. Since $f_x(P) = 2$, $f_y(P) = -2$, we find

$$z = 2(x - 1) + 2(y - 1 - \frac{\pi}{2}) \implies z = 2x + 2y - 4 - \pi.$$
Problem 3.

Consider the function

\[ f(x, y) = \begin{cases} 
|y|, & \text{if } |x| \leq |y| \\
|x|, & \text{if } |y| < |x| 
\end{cases} \]

(i) Draw the cross section to the graph of \( f \) corresponding to \( x = 1 \).

Assume \( x = 1 \). We have \( f(1, y) = |y| \) when \( |y| \geq 1 \) and \( f(1, y) = 1 \) if \( |y| < 1 \). The graph \( z = f(1, y) \) consists of three straight line segments:

- the first lies on the diagonal in the second quadrant,
- the horizontal line \( z = 1 \) when \(-1 < y < 1\)
- the third piece is contained in the diagonal of the first quadrant.

(ii) Draw the contour diagram of \( f \) showing at least the levels 1, 2 and 3.

To draw the curve of level \( z = 1 \), we solve for \( f(x, y) = 1 \). When \( |x| \leq |y| \), we have

\[ f(x, y) = 1 \implies |y| = 1 \implies y = \pm 1 \text{ and } -1 \leq x \leq 1. \]

When \( |x| > |y| \), we have

\[ f(x, y) = 1 \implies |x| = 1 \implies x = \pm 1 \text{ and } -1 < y < 1. \]

The level curve is a square of side 2 centered at the origin with vertices at \((\pm 1, \pm 1)\).

For the level 2, we get a square as well, this time of side 4 but still centered at the origin.

For level 3, we still get a square of side 6 centered at the origin.

(iii) Draw and describe the graph of \( f \).

The graph is an upside down infinite pyramid whose horizontal cross sections are squares, and whose vertex is at \((0, 0, 0)\).
Problem 4.

(i) Consider the function

\[ F : \mathbb{R}^3 \to \mathbb{R}^2, \quad F(x, y, z) = (x + y^2, y + z^2). \]

Write down the total derivative \( Df \) at \((1, 1, 1)\).

Consider the first component \( f = x + y^2 \). Then the partial derivatives of \( f \) equal \( f_x = 1, f_y = 2y = 2, f_z = 0 \).

We similarly compute the other derivatives. The Jacobian matrix is

\[
Df(1, 1, 1) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}
\]

(ii) Find the tangent plane to the surface \( xy^3 + yz^2 + x^2y = 3 \) at the point \((1, 1, 1)\). Estimate the value of \( z \) when \( x = 1.01 \) and \( y = 1.01 \).

Write \( f(x, y, z) = xy^3 + yz^2 + x^2y \). We find

\[
\begin{align*}
f_x &= y^3 + 2xy \quad \implies \quad f_x(1, 1, 1) = 3 \\
f_y &= 3xy^2 + z^2 + x^2 \quad \implies \quad f_y(1, 1, 1) = 5 \\
f_z &= 2yz \quad \implies \quad f_z(1, 1, 1) = 2.
\end{align*}
\]

The tangent plane equals

\[ 3(x - 1) + 5(y - 1) + 2(z - 1) = 0. \]

When \( x = 1.01 \) and \( y = 1.02 \) we find

\[ 3 \cdot .01 + 5 \cdot .01 + 2(z - 1) = 0 \quad \implies \quad z = .96. \]
Problem 5.

(i) Using greek letters, prove that the sequence
\[ x_n = \frac{2n^2 - 1}{n^2 + 1} \]
converges and find its limit.

We claim that the limit equals 2. Fix \( \epsilon > 0 \). We need \( N \) such that if \( n \geq N \) we have
\[ |x_n - 2| < \epsilon. \]

We calculate
\[ |x_n - 2| = \left| \frac{2n^2 - 1}{{n^2 + 1} - 2} \right| = \frac{3}{n^2 + 1} < \epsilon \iff \frac{3}{\epsilon} < n^2 + 1 \iff \sqrt{\frac{3}{\epsilon} - 1} < n. \]

We can use any \( N > \sqrt{\frac{3}{\epsilon} - 1} \). The quantity under the square root may be negative if \( \epsilon > 3 \). If this is the case, any \( N \) works.

(ii) Complete the following sentence:
Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a function. We say that \( \lim_{x \to a} f(x) = L \) if . . .

for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that if \( ||x - a|| < \delta \), \( x \neq a \), then \( ||f(x) - L|| < \epsilon \).
Problem 6.

Show that if $K \subset \mathbb{R}^n$ is compact and $F \subset K$ is closed, then $F$ is compact as well.

Pick a sequence $\{x_n\}$ of elements of $F$. We show we can find a subsequence which converges to an element $x \in F$.

Since $F \subset K$, we have $x_n \in K$, for all $n$. Since $K$ is compact, we can find a subsequence of $\{x_n\}$ which converges to some $x \in K$. However, since $F$ is closed, the limit $x \in F$. Therefore, the subsequence is convergent in $F$, which is what we wanted.