Math 52, Problem Set 3. Due Wednesday, Oct 17.


2. (Thursday—Friday: Line integrals) Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where
\[
\mathbf{F} = 2x \mathbf{i} + (x - y) \mathbf{j}
\]
and \( C \) is the clockwise triangle with vertices \((0, 0), (1, 0)\) and \((0, 1)\).

3. (Thursday—Friday: Line integrals) Let \( \mathbf{F} = 3 \mathbf{i} + 2 \mathbf{j} \). How would you place a unit segment \( C \) such that the value of
\[
\int_C \mathbf{F} \cdot d\mathbf{r}
\]
is maximum? How about minimum? What are the min/max values?

4. (Thursday—Friday: Line integrals) Let \( C \) be the counterclockwise circle of radius \( a \) centered at the origin. Consider the fields
\[
\mathbf{F} = x \mathbf{i} + y \mathbf{j}, \quad \mathbf{G} = y \mathbf{i} - x \mathbf{j}.
\]
Without parametrizing \( C \), but using only the geometric properties of the fields, compute the line integrals
\[
\int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{and} \quad \int_C \mathbf{G} \cdot d\mathbf{r}.
\]

5. (Monday—Tuesday: Gradient fields) Page 399, problems 4 and 6. You do not need to find a potential function in case we do not cover that topic in class.

6. (Monday—Tuesday: Path independence, gradient fields) Consider the following field in the \((x, y)\) plane:
\[
\mathbf{F} = -\nabla (\ln r).
\]
(a) Let \( C \) be any path joining two points \( P_1 \) and \( P_2 \) at distances \( r_1 \) and \( r_2 \) from the origin. Show that \( \int_C \mathbf{F} \cdot d\mathbf{r} \) depends only on \( r_1 \) and \( r_2 \) but not on the path \( C \).
(b) For what position of a line segment \( L \) of length 1 do we expect
\[
\int_L \mathbf{F} \cdot d\mathbf{r} = 0.
\]
(c) Compute the field \( \mathbf{F} \) in rectangular coordinates.
(d) Compute the line integral
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
where \( C \) is a circle of radius \( a \) centered at the origin. Use two different methods. First, use the results of part (a). Second, use an explicit computation in rectangular coordinates, using a convenient parametrization of the circle.

7. (Monday-Tuesday: Path independence, gradient fields)
Let \( f(x, y) = x^4y + y^2 \), and \( C \) be the parabola \( y^2 = x \), between \((1, -1)\) and \((1, 1)\), directed upwards. Calculate \( \mathbf{F} = \nabla f \). Then determine the integral
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
in three different ways:
(i) directly;
(ii) by using path-independence to replace \( C \) by a simpler path;
(iii) by using the Fundamental Theorem for line integrals.

8. (Monday-Tuesday: Gradient fields)
Find the values of \( a, b \) for which the field \( \mathbf{F} = (ax^2y - 3y - xy^2 + 1)i + (x^3 - y - bx - x^2y)j \) is conservative.