Math 145 - Midterm Solutions

Problem 1. (*10 points.*) *Let $n \geq 2$, and let $S = \{a_1, \ldots, a_n\}$ be a finite set with n elements in $\mathbb{A}^1$.*

(i) *Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take X to be the affine algebraic set given by the equations*

$$X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.$$

Show that the projection onto the first coordinate

$$\pi : X \to \mathbb{A}^1 \setminus S,\ (X_0, \ldots, X_n) \mapsto X_0$$

is an isomorphism.

(ii) *Show that the ring of regular functions on $\mathbb{A}^1 \setminus \{0\}$ is isomorphic to $k\left[t, \frac{1}{t}\right]$, the ring of polynomials in t and $\frac{1}{t}$.*

(iii) *Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$, for instance by proving that their rings of regular functions are not isomorphic.*

Answer:

(i) If $(X_0, \ldots, X_n) \in X$ we have $X_i(X_0 - a_i) = 1$ hence $X_0 \neq a_i$ for all $1 \leq i \leq n$. Therefore

$$\pi : X \to \mathbb{A}^1 \setminus S$$

is well defined. Let

$$\phi : \mathbb{A}^1 \setminus S \to X, t \mapsto \left(t, \frac{1}{t - a_1}, \ldots, \frac{1}{t - a_n}\right).$$

Both π and ϕ are rational maps, regular everywhere. It is clear that

$$\pi \circ \phi = \text{identity and } \phi \circ \pi = \text{ identity }.$$

Therefore, π and ϕ are inverse isomorphisms.

(ii) It is clear that t and $\frac{1}{t}$ are both regular functions on $\mathbb{A}^1 \setminus \{0\}$, and so is any polynomial in t and $\frac{1}{t}$. Conversely, let $f \in K(\mathbb{A}^1 \setminus \{0\})$. View f as a rational function on $\mathbb{A}^1$. Consider the ideal of denominators I_f of the function f on $\mathbb{A}^1$. Clearly

$$\mathcal{Z}(I_f) \subset \{0\} \implies t \in \sqrt{I_f} \implies t^n \in I_f \implies t^n f \in k[t].$$

Therefore

$$f = \frac{g(t)}{t^n} = \sum_{i=0}^{m} a_i t^{i-n}.$$

The last expression is a polynomial in t and $\frac{1}{t}$.

(iii) Assume there is an isomorphism

$$\Phi : A(X) \to k[t, t^{-1}].$$

Since

$$X_i(X_0 - a_i) = 1 \text{ in } A(X),$$

it follows that

$$\Phi(X_i)\Phi(X_0 - a_i) = 1.$$

Writing

$$\Phi(X_i) = \frac{g_i(t)}{t^{\alpha_i}},\ \Phi(X_0 - a_i) = \frac{h_i(t)}{t^{\beta_i}},$$

for some polynomials g_i, h_i, we obtain

$$g_i(t)h_i(t) = t^{\alpha_i + \beta_i}.$$

This implies that h_i is of the form ct^m, or equivalently

$$\Phi(X_0 - a_i) = c_i t^{m_i}$$

for some $m_i \in \mathbb{Z}$ and $c_i \in k$. Subtracting the relations for i and j, it follows that

$$a_j - a_i = \Phi(a_j - a_i) = c_j t^{m_j} - c_i t^{m_i}.$$

Comparing degrees, we see that this implies $m_i = m_j = 0$, as $a_i \neq a_j$ for $i \neq j$. In turn, we obtain

$$\Phi(X_0 - a_i) = c_i.$$

Since $\Phi(c_i) = c_i$, we contradicted the injectivity of Φ. This shows that Φ cannot be an isomorphism completing the proof.

Problem 2. (*10 points.*) *Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:*

$$Z_n = \mathcal{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathcal{Z}(y^n - x^n(x+1)).$$

Show that Z_n and W_n are birational but not isomorphic.

(i) *Show that*

$$f : \mathbb{A}^1 \to Z_n, \;\; f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}.$$

Similarly, show that

$$g : \mathbb{A}^1 \to W_n, \;\; g(t) = (t^n - 1, t^{n+1} - t).$$

is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and W_n where g becomes an isomorphism.

(ii) *Using (i), explain why Z_n and W_n are birational. Write down a birational isomorphism between $Z_n \to W_n$.*

(iii) *Assume that there exists an isomorphism*

$$h : Z_n \to W_n$$

such that $h((0,0)) = (0,0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}.$$

Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$.

(iv) *(Entirely optional. Extra credit: 5 points) Prove that h cannot exist even in the case when h may not send the origin to itself. (You may need to prove a stronger version of Problem 1.)*

(v) *Show that Z_1 and W_1 are isomorphic. Write down an isomorphism between them.*

Answer:

(i) It is clear that

$$f : \mathbb{A}^1 \to Z_n, \;\; t \to (t^n, t^{n+1})$$

is a well defined morphism. Consider the morphism

$$f^{-1} : Z_n \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$$

given by

$$(x, y) \to \frac{y}{x}.$$

A direct computation shows that f^{-1} is the inverse morphism of f. Similarly,

$$g : \mathbb{A}^1 \to W_n,\ t \to (t^n - 1, t^{n+1} - t)$$

is a well defined morphism. Its inverse morphism is

$$g^{-1} : W_n \setminus \{0\} \to \mathbb{A}^1 \setminus S,\ (x, y) \to \frac{y}{x}.$$

Here, S is the set of all n roots of unity. The two morphisms g and g^{-1} establish an isomorphism between

$$\mathbb{A}^1 \setminus S \to W_n \setminus \{0\}.$$

(ii) Part (i) shows that both Z_n and W_n are birational to $\mathbb{A}^1$ so they are birational to each other. An explicit birational isomorphism is

$$g \circ f^{-1} : Z_n \dashrightarrow W_n.$$

A direct computation shows

$$g \circ f^{-1}(x, y) = \left(\frac{y^n}{x^n}, \frac{y^{n+1}}{x^{n+1}} - \frac{y}{x} \right).$$

(iii) If $h : Z_n \to W_n$ is an isomorphism sending the origin to itself, then

$$h : Z_n \setminus \{0\} \to W_n \setminus \{0\}$$

is also an isomorphism. By part (i), $g^{-1} \circ h \circ f$ induces an isomorphism between the quasi-affine sets

$$\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus S.$$

Such an isomorphism cannot exist by Problem 1.

(v) Let

$$F : Z_1 \to W_1, (x, y) \to (x, y + x).$$

This is an isomorphism between Z_1 and W_1 with inverse $F^{-1}(x, y) = (x, y - x)$.

Problem 3. (*5 points.*) *Let $X \subset \mathbb{P}^2$ be a projective variety. A morphism $f : \mathbb{P}^1 \to X$ is a polynomial map*

$$f([x : y]) = (f_0([x : y]), f_1([x : y]), f_2([x : y])),$$

where f_0, f_1, f_2 are homogeneous polynomials of the same degree, such that $f(\mathbb{P}^1) \subset X$.

Prove the following facts about lines and conics in projective plane:

(i) *For any line $L \subset \mathbb{P}^2$, there is a bijective morphism*

$$f : \mathbb{P}^1 \to L.$$

(ii) *For any irreducible conic $C \subset \mathbb{P}^2$, there is a morphism*

$$f : \mathbb{P}^1 \to C.$$

You may wish to change coordinates so that your conic has a convenient expression. Can you assume that f is bijective?

Answer:

(i) Let L be the line $\alpha x + \beta y + \gamma z = 0$. We may assume that $\gamma \neq 0$, eventually relabeling the coordinates if necessary. Set

$$f : \mathbb{P}^1 \to L,\ [x : y] \to \left[x : y : -\frac{\alpha}{\gamma} x - \frac{\beta}{\gamma} y \right],$$

and

$$g : L \to \mathbb{P}^1\ [x : y : z] \to [x : y].$$

It is easy to see that both f and g are well defined inverse isomorphisms.

(ii) Changing coordinates, we may assume the conic C is given by the equation

$$xz = y^2.$$

Set

$$f : \mathbb{P}^1 \to C,\ [s : t] \to [s^2 : st : t^2].$$

We have seen in class that

$$g : C \to \mathbb{P}^1, g([x : y : z]) = \begin{cases} [x : y] \text{ if } (x, y) \neq (0, 0) \\ [y : z] \text{ if } (y, z) \neq (0, 0) \end{cases}$$

is an inverse morphism of f.

Problem 4. *Part A. (2 points.) Show that any two distinct points in $\mathbb{P}^2$ lie on a unique line. Therefore, two distinct lines in $\mathbb{P}^2$ intersect in exactly one point.*

Part B. (3 points.) Show that a line and an irreducible conic in $\mathbb{P}^2$ cannot intersect in 3 points. (We will see later that they intersect in exactly 2 points, if counted with multiplicity.)

Part C. (10 points.)

(i) *Four points in $\mathbb{P}^2$ are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if $p_1, \ldots, p_4$ are points in general position, there exists a linear change of coordinates*

$$T : \mathbb{P}^2 \to \mathbb{P}^2$$

with

$$T([1 : 0 : 0]) = p_1,\ T([0 : 1 : 0]) = p_2,\ T([0 : 0 : 1]) = p_3,\ T([1 : 1 : 1]) = p_4.$$

(ii) *Given five distinct points in $\mathbb{P}^2$, no three of which are collinear, show that there is an unique irreducible projective conic passing though all five points.*

(iii) *Deduce that two distinct irreducible conics in $\mathbb{P}^2$ cannot intersect in 5 points.*

Answer: Part A. Let $P_1 = [a_1 : b_1 : c_1]$ and $P_2 = [a_2 : b_2 : c_2]$. The matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

has rank 2. Indeed, the columns are not proportional since P_1 and P_2 are distinct points in $\mathbb{P}^2$. Therefore, the rank cannot be smaller than 1. This implies that the null space of A is one dimensional. Pick $[\alpha, \beta, \gamma]$ any nonzero vector in the null space. Then

$$\alpha a_i + \beta b_i + \gamma c_i = 0,$$

so the line

$$\alpha x + \beta y + \gamma z = 0$$

passes through P_1 and P_2. The same proof shows the uniqueness of the line.

Part B. Changing coordinates, we may assume that the conic C is

$$xz = y^2.$$

Let L be the line

$$\alpha x + \beta y + \gamma z = 0.$$

If $\beta = 0$, it is clear that C and L intersect at the points

$$[-\gamma : \pm\sqrt{\alpha\gamma} : \alpha].$$

If $\beta \neq 0$, then

$$y = -\frac{\alpha}{\beta}x - \frac{\gamma}{\beta}z$$

which gives

$$xz = \frac{1}{\beta^2}(\alpha x + \gamma z)^2.$$

If $x = 0$, then $y = z = 0$ which is not allowed. Therefore, we may assume $x \neq 0$. Dividing by x^2 we obtain the quadratic equation in $\frac{z}{x}$:

$$\frac{\gamma^2}{\beta^2}\left(\frac{z}{x}\right)^2 + \left(\frac{2\alpha\gamma}{\beta^2} - 1\right)\frac{z}{x} + \frac{\alpha^2}{\beta^2} = 0.$$

Letting λ_1, λ_2 be the solutions of this equation, we see that the intersection points are

$$\left[1 : -\frac{\alpha}{\beta} - \frac{\gamma}{\beta}\lambda_i : \lambda_i\right].$$

Part C.

(i) Let $p_i = [a_i : b_i : c_i]$ for $1 \leq i \leq 4$. Define

$$A = \begin{pmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ \beta a_2 & \beta b_2 & \beta c_2 \\ \gamma a_3 & \gamma b_3 & \gamma c_3 \end{pmatrix},$$

where α, β, γ will be specified later. In fact, we will require that α, β, γ solve the system

$$\alpha a_1 + \beta b_1 + \gamma c_1 = a_4,$$
$$\alpha a_2 + \beta b_2 + \gamma c_2 = b_4,$$
$$\alpha a_3 + \beta b_3 + \gamma c_3 = c_4.$$

A solution exists since the matrix of coefficients

$$B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is invertible. Indeed, the rows of B are independent. Otherwise, a nontrivial linear relation between the rows would give a line on which the points p_1, p_2, p_3 lie. Thus B is invertible. Now, the system above has the solution

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = B^{-1} \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}.$$

Note that the same argument shows that A is invertible. Let

$$S : \mathbb{P}^2 \to \mathbb{P}^2$$

be the linear transformation defined by A. Then S is invertible. A direct computation shows

$$S([1:0:0]) = p_1, S([0:1:0]) = p_2, S([0:0:1]) = p_3, S([1:1:1]) = p_4.$$

The proof is completed letting T be the inverse of S.

(ii) After a linear change of coordinates, we may assume that the five points are $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$ and $[u : v : w]$. The equation $f(x, y, z)$ of any conic passing through the first three points can't contain x^2, y^2, z^2, so

$$f(x, y, z) = ayz + bxz + cxy.$$

The remaining two points impose the conditions

$$a + b + c = avw + buw + cuv = 0.$$

Letting

$$A = \begin{pmatrix} 1 & 1 & 1 \\ vw & uw & uv \end{pmatrix},$$

we see that (a, b, c) must be in the null space of A. The rank of A is 2 (the rank cannot be 1 since then the rows would be proportional, hence $u = v = w$ which is not allowed). Therefore, the null space of this matrix is one dimensional, hence the conic passing through the 5 points is unique. The conic cannot be reducible since then it would be union of two lines. One of the lines would have to contain 3 points but that contradicts the general position assumption.

(iii) By Part B, a conic and a line cannot intersect in 3 points. Therefore, any 5 points on an irreducible conic are in general position. Now, the claim is evident by (ii).

Problem 5. (*10 points.*) *Let $\lambda \in k \setminus \{0, 1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation*

$$y^2 - x(x-1)(x-\lambda) = 0.$$

Show that E_λ is not birational to $\mathbb{A}^1$. In fact, show that there are no non-constant rational maps

$$F : \mathbb{A}^1 \dashrightarrow E_\lambda.$$

(i) *Write*

$$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)} \right)$$

where the pairs of polynomials (f, g) and (p, q) have no common factors. Conclude that

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}$$

is an equality of fractions that cannot be further simplified. By analyzing the factorization into irreducibles (in our case linear factors) of both numerators and denominators, conclude that $f, g, f - g, g - \lambda g$ must be perfect squares.

(ii) *Prove the following:*

Lemma: If f, g are polynomials in $k[t]$ such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f - g, f - \lambda g$ are perfect squares, then f and g must be constant.

(iii) *Consider the elliptic curve $\overline{E}_\lambda \subset \mathbb{P}^2$:*

$$y^2 z = x(x-z)(x-\lambda z).$$

Show that there are no nonconstant morphisms

$$\mathbb{P}^1 \to \overline{E}_\lambda \subset \mathbb{P}^2.$$

Therefore, elliptic curves are not rational curves, and are not isomorphic to lines or conics.

Answer:

(i) We have

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}.$$

Since p, q are relatively prime, the right hand side cannot be further simplified. Similarly, $f, g, f - g, f - \lambda g$ cannot have any common factors. Indeed, a common factor for instance of f and $f - g$, will necessarily have to divide $f - (f - g) = g$ as well. But this is impossible since f and g are coprime. Therefore the right hand side cannot be further simplied as well. Thus, for some constant $a \in k$, we must have

$$ap^2 = f(f-g)(f-\lambda g), \; aq^2 = g^3.$$

The exponents of the irreducible factors of the left hand sides of the above equations must be even. Therefore, the same must be true about the right hand side. This immediately implies that g is a square. But since f, $f-g$, $f-\lambda g$ have no common factors, it follows that the exponents of the irreducible factors of each f, $f-g$ and $f-\lambda g$ must be even as well. Thus f, $f-g$, $f-\lambda g$ must be squares.

(ii) Pick f and g such that $\max(\deg f, \deg g)$ is minimal among all pairs (f, g) which satisfy the requirement that $f, g, f-g, f-\lambda g$ are squares for some $\lambda \neq 0, 1$. We may assume that f, g are coprime since otherwise we can reduce their degree by dividing by their gcd which is also a square. Write

$$f = u^2, g = v^2,$$

where u, v are coprime. Then

$$f - g = (u-v)(u+v)$$

is a square. Note that $u-v$ and $u+v$ cannot have common factors since such factors will have to divide both

$$\frac{1}{2}((u-v)+(u+v)) = u \text{ and } \frac{1}{2}((u-v)-(u+v)) = v$$

which is assumed to be false. Thus $u-v$ and $u+v$ are coprime, and since their product $f-g$ is a square, it follows that $u-v, u+v$ must be square. The same argument applied to

$$f - \lambda g = (u - \sqrt{\lambda} v)(u + \sqrt{\lambda} v)$$

shows that $u - \sqrt{\lambda} v, u + \sqrt{\lambda} v$ are squares. Let

$$\tilde{u} = \frac{1+\sqrt{\lambda}}{2}(u+v), \tilde{v} = \frac{\sqrt{\lambda}-1}{2}(u-v).$$

Clearly, $\tilde{u}, \tilde{v}$ are squares. A direct computation shows that

$$\tilde{u} - \tilde{v} = u + \sqrt{\lambda} v,$$

which is also a square. Finally,

$$\tilde{u} - \left(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}\right)^2 \tilde{v} = \frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}(u - \sqrt{\lambda} v)$$

is a square. Setting

$$\mu = \left(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}\right)^2,$$

we see that $\tilde{u}, \tilde{v}, \tilde{u}-\tilde{v}, \tilde{u}-\mu\tilde{v}$ are squares. Note that $\mu \neq 0, 1$ for $\lambda \neq 0, 1$. Furthermore,

$$\max(\deg \tilde{u}, \deg \tilde{v}) = \frac{1}{2} \max(\deg f, \deg g).$$

Unless f and g are irreducible, this contradicts the assumption that f, g are of minimal degree.

(iii) Assume that

$$F : \mathbb{P}^1 \to \overline{E}_\lambda \subset \mathbb{P}^2$$

is a morphism with

$$F([x:y]) = [f_0([x:y]) : f_1([x:y]) : f_2([x:y])],$$

where

$$f_1^2 f_2 = f_0(f_0 - f_1)(f_0 - \lambda f_1).$$

If $f_2 \equiv 0$, it follows from the equation above that $f_0 = 0$, hence F is the constant $[0 : 1 : 0]$. Let us assume $f_2 \neq 0$. Define

$$f(t) = \left(\frac{f_0(t,1)}{f_2(t,1)}, \frac{f_1(t,1)}{f_2(t,1)} \right).$$

It is clear that f defines a rational map

$$f : \mathbb{A}^1 \dashrightarrow E_\lambda$$

which must be constant by (ii). This implies that $f_0(t,1)/f_2(t,1)$ and $f_1(t,1)/f_2(t,1)$ are constants. It follows that all points $[t : 1]$ take the same value under F. Similarly, F sends all $[1 : t]$ to the same value, and therefore F must be constant.