Math 145, Midterm Exam, Due May 7 in class.

You may assume that the ground field is $k = \mathbb{C}$.

Problem 1. (*10 points.*) Let $n \geq 2$, and let $S = \{a_1, \ldots, a_n\}$ be a finite set with n elements in $\mathbb{A}^1$.

(i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take X to be the affine algebraic set given by the equations

$$X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.$$

Show that the projection onto the first coordinate

$$\pi : X \to \mathbb{A}^1 \setminus S, \ (X_0, \ldots, X_n) \mapsto X_0$$

is an isomorphism.

(ii) Show that the ring of regular functions on $\mathbb{A}^1 \setminus \{0\}$ is isomorphic to $k\left[t, \frac{1}{t}\right]$, the ring of polynomials in t and $\frac{1}{t}$.

(iii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$, for instance by proving that their rings of regular functions are not isomorphic. (A different method will work just as well.)

Hint: Assume that

$$\Phi : A(X) \to k\left[t, t^{-1}\right]$$

is an isomorphism. Observe that $X_0 - a_i$ are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_0 - a_i) = t^{m_i}$ for some integers m_i. Furthermore, show that $m_i = \pm 1$, for instance by observing that $X_0 - a_i$ are not higher powers of any elements in $A(X)$. (This may be slightly trickier to prove.) Possibly using the fact that Φ is bijective, derive a contradiction.

Problem 2. (*10 points.*) Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:

$$Z_n = \mathcal{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathcal{Z}(y^n - x^n(x+1)).$$

Show that Z_n and W_n are birational but not isomorphic.

(i) Show that

$$f : \mathbb{A}^1 \to Z_n, \ \ f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}.$$

Similarly, show that

$$g : \mathbb{A}^1 \to W_n, \ \ g(t) = (t^n - 1, t^{n+1} - t).$$

is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and W_n where g becomes an isomorphism.

(ii) Using (i), explain why Z_n and W_n are birational. Write down a birational isomorphism between $Z_n \to W_n$.

(iii) Assume that there exists an isomorphism

$$h : Z_n \to W_n$$

such that $h((0,0)) = (0,0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}.$$

Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$.

(iv) (*Entirely optional. Extra credit: 5 points*) Prove that h cannot exist even in the case when h may not send the origin to itself. (You may need to prove a stronger version of Problem 1.)

(v) Show that Z_1 and W_1 are isomorphic. Write down an isomorphism between them.

Problem 3. (*5 points.*) Let $X \subset \mathbb{P}^2$ be a projective variety. A morphism $f : \mathbb{P}^1 \to X$ is a polynomial map

$$f([x:y]) = (f_0([x:y]), f_1([x:y]), f_2([x:y])),$$

where f_0, f_1, f_2 are *homogeneous* polynomials of the *same degree*, such that $f(\mathbb{P}^1) \subset X$.

Prove the following facts about lines and conics in projective plane:

(i) For any line $L \subset \mathbb{P}^2$, there is a bijective morphism

$$f : \mathbb{P}^1 \to L.$$

(ii) For any irreducible conic $C \subset \mathbb{P}^2$, there is a morphism

$$f : \mathbb{P}^1 \to C.$$

You may wish to change coordinates so that your conic has a convenient expression. Can you assume that f is bijective?

Remark: A curve X which is the image of a morphism $f : \mathbb{P}^1 \to X \subset \mathbb{P}^2$ is called a rational curve. This is the same thing as saying that X admits a polynomial parametrization. In particular, we showed that lines and conics in $\mathbb{P}^2$ are rational curves in $\mathbb{P}^2$.

Problem 4. *Part A.* (*2 points.*) Show that any two distinct points in $\mathbb{P}^2$ lie on a unique line. Therefore, two distinct lines in $\mathbb{P}^2$ intersect in exactly one point.

Part B. (*3 points.*) Show that a line and an irreducible conic in $\mathbb{P}^2$ cannot intersect in 3 points. (We will see later that they intersect in exactly 2 points, if counted with multiplicity.)

Part C. (*10 points.*)

(i) Four points in $\mathbb{P}^2$ are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if $p_1, \ldots, p_4$ are points in general position, there exists a linear change of coordinates

$$T : \mathbb{P}^2 \to \mathbb{P}^2$$

with

$$T([1:0:0]) = p_1,\ T([0:1:0]) = p_2,\ T([0:0:1]) = p_3,\ T([1:1:1]) = p_4.$$

(ii) Given five distinct points in $\mathbb{P}^2$, no three of which are collinear, show that there is an unique irreducible projective conic passing though all five points. You may want to use part (i) to assume that four of the points are $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$.

(iii) Deduce that two distinct irreducible conics in $\mathbb{P}^2$ cannot intersect in 5 points. (We will see later that they intersect in exactly 4 points counted with multiplicity.)

Remark: For any degree d, fix $3d-1$ points in $\mathbb{P}^2$ in "general position". You may ask how many rational curves of degree d in $\mathbb{P}^2$ pass through these $3d-1$ points. We have shown above that there is $N_1 = 1$ line through 2 points, and $N_2 = 1$ conic through 5 points. The next few numbers are

$$N_3 = 12, N_4 = 620, N_5 = 87,304, N_6 = 26,312,976, N_7 = 14,616,808,192.$$

Thus, there are are 12 rational cubics through 8 points, 620 rational quartics through 11 points and so on. A general answer for arbitrary d was found in 1994 using ideas from physics/string theory. The area of algebraic geometry that computes these numbers is called enumerative geometry/Gromov-Witten theory.

Problem 5. (*10 points.*) Let $\lambda \in k \setminus \{0, 1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation

$$y^2 - x(x-1)(x-\lambda) = 0.$$

Show that E_λ is not birational to $\mathbb{A}^1$. In fact, show that there are no non-constant rational maps

$$F : \mathbb{A}^1 \dashrightarrow E_\lambda.$$

(i) Write

$$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)}\right)$$

where the pairs of polynomials (f, g) and (p, q) have no common factors. Conclude that

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}$$

is an equality of fractions that cannot be further simplified. By analyzing the factorization into irreducibles (in our case linear factors) of both numerators and denominators, conclude that $f, g, f-g, g-\lambda g$ must be perfect squares.

(ii) Prove the following:

Lemma: If f, g are polynomials in $k[t]$ such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f - g, f - \lambda g$ are perfect squares, then f and g must be constant.

Hint: Pick f and g such that $\max(\deg f, \deg g)$ is minimal among all pairs (f, g) which satisfy the requirement that $f, g, f-g, f-\lambda g$ are squares for some $\lambda \neq 0, 1$. Write $f = u^2, g = v^2$. Considering the factorizations of $f - g$ and $f - \lambda g$, prove that $u - v, u + v, u - \sqrt{\lambda} v, u + \sqrt{\lambda} v$ are also squares. Construct two constant coefficient linear combinations $\tilde{u}, \tilde{v}$ of the polynomials u and v such that $\tilde{u}, \tilde{v}, \tilde{u} - \tilde{v}, \tilde{u} - \mu\tilde{v}$ are squares for some constant $\mu \in k \setminus \{0, 1\}$. Note that $\max(\deg \tilde{u}, \deg \tilde{v}) < \max(\deg f, \deg g)$ unless f, g are constant. Why is this a contradiction?

Remark: We will see later that any cubic curve can be written in the form

$$y^2 - x(x-1)(x-\lambda) = 0, \text{ or } y^2 - x^3 = 0 \text{ or } y^2 - x^2(x-1) = 0,$$

after a change of coordinates. The latter curves are Z_2 and W_2 in Problem 2, so they are birational to $\mathbb{A}^1$.

(iii) Consider the elliptic curve $\overline{E}_\lambda \subset \mathbb{P}^2$:

$$y^2 z = x(x-z)(x-\lambda z).$$

Show that there are no nonconstant morphisms

$$\mathbb{P}^1 \to \overline{E}_\lambda \subset \mathbb{P}^2.$$

You may wish to prove first that any such morphism gives rise to a rational map

$$\mathbb{A}^1 \dashrightarrow E_\lambda.$$

Therefore, elliptic curves are not rational curves, and are not isomorphic to lines or conics.