

TAUTOLOGICAL RELATIONS ON THE STABLE MAP SPACES

DRAGOS OPREA

ABSTRACT. The cohomology of the spaces of rational stable maps to flag varieties is generated by tautological classes. We study relations between the tautological generators. We conjecture that all relations between these generators are tautological, *i.e.* they are essentially obtained from Keel's relations on $\overline{\mathcal{M}}_{0,n}$ with the aid of the pushforwards by the natural morphisms. We check this claim on the open part of the moduli space of stable maps to \mathbb{P}^r , and also in low codimensions on the compactified moduli spaces.

A well-known result of Keel states that the cohomology of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ is additively generated by the boundary classes of curves with fixed dual graph [8]. In addition, all relations between the tautological generators are interpreted in terms of the cross-ratio of 4 points on \mathbb{P}^1 *i.e.*, these relations come from $\overline{\mathcal{M}}_{0,4}$ via

$$\overline{\mathcal{M}}_{0,n} \rightarrow \overline{\mathcal{M}}_{0,4}.$$

This result has implications, for instance, in the study of the tree-level cohomological field theories [12].

In our previous paper [15], we partially extended Keel's theorem to the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ of marked genus zero stable morphisms to flag varieties X . In particular, we determined additive generators for the rational cohomology of the spaces $\overline{\mathcal{M}}_{0,n}(X, \beta)$. Unfortunately, a complete set of relations between the tautological generators analogous to Keel's result is not yet available. Such relations may be of interest for intersection theoretic computations in the tautological rings, *e.g.* for the calculation of genus 0 Gromov-Witten invariants.

In this paper, we conjecture that “non-trivial” relations between the genus 0 tautological classes are tautological, *i.e.* they are essentially consequences of Keel's relations, in a way which will be made precise shortly.

0.1. The tautological generators. Our setup will be as follows. We consider X a convex complex projective manifold, typically a flag variety over the complex numbers. We denote by $\overline{\mathcal{M}}_{0,S}(X, \beta)$ the moduli stack of S -pointed genus zero stable maps to X in the homology class $\beta \in H_2(X, \mathbb{Z})$. We use the notation $\overline{\mathcal{M}}_{0,n}(X, \beta)$ when the marking set is $S = \{1, 2, \dots, n\}$.

The tautological systems encode the classes on the moduli spaces $\overline{\mathcal{M}}_{0,n}(X, \beta)$ which are defined using minimal information about the geometry of the parametrized morphisms, making use only of the inductive structure of the Kontsevich-Manin spaces. To make the definition precise, observe the following natural morphisms:

- (i) forgetful morphisms: $\pi : \overline{\mathcal{M}}_{0,S}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,T}(X, \beta)$ defined whenever $T \subset S$;
- (ii) evaluation morphisms to the target space, $ev_i : \overline{\mathcal{M}}_{0,S}(X, \beta) \rightarrow X$ for all $i \in S$;
- (iii) gluing morphisms which produce maps with nodal domains,

$$gl : \overline{\mathcal{M}}_{0,S_1 \cup \{\bullet\}}(X, \beta_1) \times_X \overline{\mathcal{M}}_{0,\{\bullet\} \cup S_2}(X, \beta_2) \rightarrow \overline{\mathcal{M}}_{0,S_1 \cup S_2}(X, \beta_1 + \beta_2).$$

Definition 1. The genus 0 tautological rings $R^*(\overline{\mathcal{M}}_{0,n}(X, \beta))$ are the smallest system of *subrings* of the rational Chow rings $A^*(\overline{\mathcal{M}}_{0,n}(X, \beta))$ such that:

- (i) The evaluation classes $ev_i^* \alpha$ where $\alpha \in A^*(X)$ are in the system.
- (ii) The system is closed under pushforwards by the gluing and forgetful morphisms.

We will consider a certain collection of tautological classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ indexed by *weighted graphs*.

Definition 2. A weighted graph $(\Gamma, \mathfrak{w}, \mathfrak{f})$ consists of:

- (i) A *stable* graph Γ of genus 0, decorated by degrees and n labeled legs. The following notations will be used:
 - We write $V(\Gamma)$, $E(\Gamma)$, $L(\Gamma)$, $H(\Gamma)$ for the vertices, edges, legs, and half-edges of Γ .
 - β_\bullet denotes the degree map

$$\beta_\bullet : V(\Gamma) \rightarrow H_2(X, \mathbb{Z}), \quad v \mapsto \beta_v.$$

The total degree of Γ is denoted by

$$\beta = \sum_v \beta_v.$$

- For each vertex v of Γ we write n_v for its total valency. Stability of Γ means that vertices of degree 0 have valency at least 3.
- (ii) The weights \mathfrak{w} are an assignment of cohomology classes on X to the half-edges and legs of Γ :

$$\mathfrak{w} : H(\Gamma) \cup L(\Gamma) \rightarrow H^*(X).$$

The role of the weights is to keep track of the incidence conditions with cycles in X .

- We define weight of an edge e with two half edges h_1 and h_2 as the product

$$\mathfrak{w}(e) = \mathfrak{w}(h_1) \cdot \mathfrak{w}(h_2).$$

- (iii) the "forgetting" data \mathfrak{f} is a subset of the legs $\mathfrak{f} \subset L(\Gamma)$. Its role is to remember whether the incidence points with the fixed cycles, determined by the weights, come from markings of the domain curve or not.

The graph Γ determines the boundary stratum $\overline{\mathcal{M}}(\Gamma)$ which fits in the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n}(X, \beta) & \xleftarrow{gl_\Gamma} & \overline{\mathcal{M}}(\Gamma) & \longrightarrow & \prod_v \overline{\mathcal{M}}_{0,n_v}(X, \beta_v) \\ & & \downarrow & & \downarrow \\ & & X^{E(\Gamma)} & \xrightarrow{\Delta_\Gamma} & X^{H(\Gamma)} \end{array}$$

Here Δ_Γ is the diagonal inclusion of the cartesian product of copies of X , while gl_Γ is the inclusion of the boundary stratum into the moduli space. This induces a gluing map ζ_Γ by composition of the gluing pushforward, the Gysin morphism and the exterior product (which we will omit from the notation):

$$(1) \quad \zeta_\Gamma : \bigotimes_v A_\star(\overline{\mathcal{M}}_{0,n_v}(X, \beta_v)) \rightarrow A_\star(\overline{\mathcal{M}}_{0,n}(X, \beta)), \quad \zeta_\Gamma = (gl_\Gamma)_\star \Delta_\Gamma^!$$

Note moreover that the weighted graph also determines a forgetful map

$$\pi_\Gamma : \overline{\mathcal{M}}_{0,L(\Gamma)}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,L(\Gamma)\setminus\mathfrak{f}}(X, \beta).$$

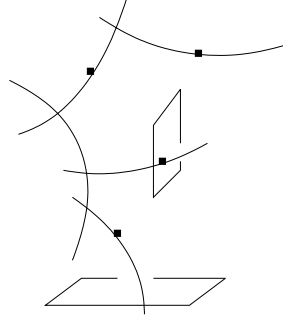
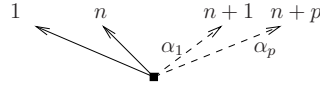
The tautological cohomology class $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ is a class on the moduli space $\overline{\mathcal{M}}_{0,L(\Gamma)\setminus\mathfrak{f}}(X, \beta)$ defined as:

$$(2) \quad [\Gamma, \mathfrak{w}, \mathfrak{f}] = (\pi_\Gamma)_\star \zeta_\Gamma \left(\prod_{v \in V(\Gamma)} \left(\prod_{\text{flags } f \text{ adjacent to } v} ev_f^\star \mathfrak{w}(f) \cap [\overline{\mathcal{M}}_{0,n_v}(X, \beta_v)] \right) \right).$$

With this notation understood, the main result of [15] is the following

Theorem 1. *The classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ are additive generators for the cohomology of the spaces $\overline{\mathcal{M}}_{0,n}(X, \beta)$.*

Remark 1. The classes corresponding to the graphs in the figure below are (via gluing) the building blocks of our system of tautological generators. Here Γ is a one vertex graph with $n + p$ legs such that p of them, carrying weights $\alpha_1, \alpha_2, \dots, \alpha_p \in H^\star(X)$, form the forgetting data \mathfrak{f} ; the remaining weights are trivial. We will write

FIGURE 1. The tautological generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$.FIGURE 2. The graph corresponding to the κ classes.

$\kappa_n(\alpha_1, \dots, \alpha_p)$ for the corresponding class:

$$(3) \quad \kappa_n(\alpha_1, \dots, \alpha_p) = \pi_*(ev_{n+1}^* \alpha_1 \cdot \dots \cdot ev_{n+p}^* \alpha_p \cap [\overline{\mathcal{M}}_{0,n}(X, \beta)]).$$

0.2. Tautological relations. We now indicate how relations between the tautological classes are obtained via a very simple algorithmic procedure.

We start by succinctly reviewing the degree 0 case considered by Keel [8]. The additive generators $[\overline{\mathcal{M}}(\Gamma)]$ of $H^*(\overline{\mathcal{M}}_{0,n})$ are the classes of rational curves with dual graph Γ . Relations are obtained by cross ratio. For example, in codimension 1, we first fix markings i, j, k, l , and distribute them on the branches of a nodal stable curve in two ways: $(ij)(kl)$ and $(ik)(jl)$. The remaining markings are distributed arbitrarily on the branches. This yields two sums of boundary divisors which are linearly equivalent. We pictorially show this relation below. Moreover, it is easy to see how this generalizes to arbitrary codimension strata, by adding more branches. It is shown in [12] that these are all possible relations between the additive generators $[\overline{\mathcal{M}}(\Gamma)]$.

We propose a description of the additive structure of the tautological systems of the stable map spaces. The only difference from the case studied by Keel is the presence of the incidence conditions. This introduces additional bookkeeping complications, mostly coming from degree 0, evaluation at divisor classes, and from relations in the cohomology of the target.

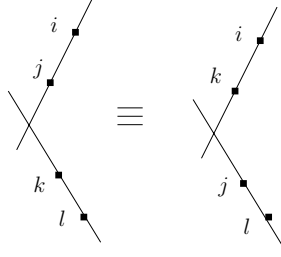


FIGURE 3. Keel's relations on $\overline{\mathcal{M}}_{0,n}$.

We regard the tautological systems as a sequence of abelian groups connected by forgetful morphisms:

$$\dots \xrightarrow{\pi_*} R^*(\overline{\mathcal{M}}_{0,n}(X, \beta)) \xrightarrow{\pi_*} R^*(\overline{\mathcal{M}}_{0,n-1}(X, \beta)) \xrightarrow{\pi_*} \dots$$

The additive generators of these groups are symbols $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ indexed by weighted graphs such that $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ lives on the moduli space of maps to X of degree equal to the total degree of Γ , and whose markings are $L(\Gamma) \setminus \mathfrak{f}$. We require:

- (automorphisms) The class $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ only depends on the isomorphism class of the triple $[\Gamma, \mathfrak{w}, \mathfrak{f}]$.

The forgetting maps π_* are defined as follows:

- Let Γ be a stable graph with cohomology weights \mathfrak{w} and forgetting data \mathfrak{f} , and let $\mathfrak{l} \notin \mathfrak{f}$ be a leg of Γ inducing a forgetful morphism π . Let $\tilde{\mathfrak{f}} = \mathfrak{f} \cup \{\mathfrak{l}\}$ be new forgetting data for Γ . Then:

$$\pi_* [\Gamma, \mathfrak{w}, \mathfrak{f}] = [\Gamma, \mathfrak{w}, \tilde{\mathfrak{f}}].$$

Note that for now we do not worry about forgetting a destabilizing marking, we will take care of this issue shortly. (In the figures below, the legs which are forgotten are indicated by dashed lines.)

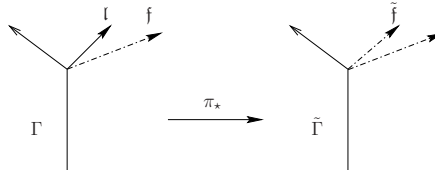


FIGURE 4. The forgetting maps

We have the following relations between our generators:

- (no incidences) Assume $\mathfrak{w}(\mathfrak{l}) = 1 \in H^0(X)$ for a leg $\mathfrak{l} \in \mathfrak{f}$, and which does not destabilize Γ by forgetting. Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = 0.$$

- (divisor) Let \mathfrak{l} be a leg attached to a vertex of degree β , which is part of the forgetting data \mathfrak{f} . Assume $\mathfrak{w}(\mathfrak{l}) = \mathfrak{d}$ is a divisor with $\beta \cdot \mathfrak{d} \neq 0$. We let $\tilde{\Gamma}$ be the graph obtained by forgetting the leg \mathfrak{l} , with weights $\tilde{\mathfrak{w}} = \mathfrak{w} \setminus \{\mathfrak{d}\}$, and forgetting data $\tilde{\mathfrak{f}} = \mathfrak{f} \setminus \{\mathfrak{l}\}$. Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = (\beta \cdot \mathfrak{d}) [\tilde{\Gamma}, \tilde{\mathfrak{w}}, \tilde{\mathfrak{f}}]$$

- (forgetting destabilizing legs) Assume $\mathfrak{w}(\mathfrak{l}) = 1$ for a leg \mathfrak{l} which is part of the forgetting data \mathfrak{f} and which does destabilize Γ after forgetting. Let $\tilde{\Gamma}$ be the graph obtained by forgetting \mathfrak{l} and stabilizing: the graph $\tilde{\Gamma}$ has one less vertex, and one less leg. Let $\tilde{\mathfrak{w}}$ be the induced weights (the weights of the new flags are defined by multiplying the weights of the collapsed flags) and let $\tilde{\mathfrak{f}}$ be forgetting data. Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = [\tilde{\Gamma}, \tilde{\mathfrak{w}}, \tilde{\mathfrak{f}}].$$

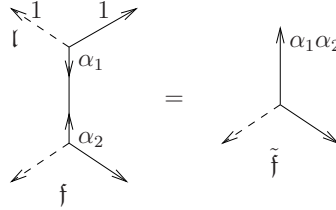


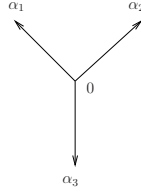
FIGURE 5. Forgetting destabilizing legs

- (mapping to a "point") Let $\pi = (\pi_1, \pi_2) : X \rightarrow Y_1 \times Y_2$ be the embedding of X into a product of two flag varieties which is obtained by remembering complementary steps in the flags of X (for example Y_2 can be a point). Assume Γ contains two legs \mathfrak{l}_1 and \mathfrak{l}_2 adjacent to a vertex \mathfrak{v} whose degree $\beta_{\mathfrak{v}} \in \pi_2^* H^*(Y_2)$. Let \mathfrak{w}_1 and \mathfrak{w}_2 be weights of Γ which differ only in the assignment of the cohomology class $\pi_1^* \alpha$, $\alpha \in H^*(Y_1)$ to the legs \mathfrak{l}_1 and \mathfrak{l}_2 respectively, otherwise being identical. Then:

$$[\Gamma, \mathfrak{w}_1, \mathfrak{f}] = [\Gamma, \mathfrak{w}_2, \mathfrak{f}].$$

It is a consequence of this relation and gluing that for each degree 0 vertex \mathfrak{v} in Γ , the corresponding generator only depends on the product of

weights incident to \mathfrak{v} . For instance, in the figure below, the corresponding generator depends on the product $\alpha_1 \cdot \alpha_2 \cdot \alpha_3$.



- (pullbacks from X) Let $\mathfrak{w}, \mathfrak{w}_1, \mathfrak{w}_2$ be weights of the graph Γ which agree everywhere except for a leg \mathfrak{l} for which we have:

$$\mathfrak{w}(\mathfrak{l}) = \mathfrak{w}_1(\mathfrak{l}) + \mathfrak{w}_2(\mathfrak{l}).$$

Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = [\Gamma, \mathfrak{w}_1, \mathfrak{f}] + [\Gamma, \mathfrak{w}_2, \mathfrak{f}].$$

More importantly, Keel's theorem gives a way of obtaining non-trivial relations. We first fix a Keel relation on $\overline{\mathcal{M}}_{0,n}$, pull it back to the space of stable maps $\overline{\mathcal{M}}_{0,n}(X, \beta)$, intersect it with a monomial in evaluation classes and use the forgetful pushforward. We chose to split this procedure in several steps: at first we ignore the evaluation monomials, which we can add in later by gluing degree 0 tripods and forgetting. For the same reason, we only need codimension 1 Keel relations:

- (Keel relations/WDVV) The pullback of a Keel relation under the stabilization map

$$st : \overline{\mathcal{M}}_{0,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,n}$$

is a relation between the classes $[\Gamma, \emptyset, \emptyset]$. Here, we agree that for each graph stable graph Γ of degree 0, we have

$$st^* [\overline{\mathcal{M}}(\Gamma)] = \sum_{\tilde{\Gamma}} [\tilde{\Gamma}, \emptyset, \emptyset]$$

the sum being taken over all possible ways of decorating Γ by degrees summing up to β .

We can get even more relations by gluing. To this end, we define:

- (gluing generators) Let $(\Gamma_1, \mathfrak{w}_1, \mathfrak{f}_1)$ and $(\Gamma_2, \mathfrak{w}_2, \mathfrak{f}_2)$ be weighted graphs and let $\mathfrak{l}_1 \in L(\Gamma_1) \setminus \mathfrak{f}_1$ and $\mathfrak{l}_2 \in L(\Gamma_2) \setminus \mathfrak{f}_2$ be two unforgotten legs. The glued graph:

$$(\Gamma_1, \mathfrak{w}_1, \mathfrak{f}_1) \star (\Gamma_2, \mathfrak{w}_2, \mathfrak{f}_2)$$

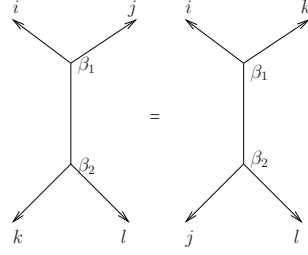


FIGURE 6. The pullbacks of Keel relations.

is obtained by joining Γ_1 and Γ_2 along an edge whose half edges are \mathfrak{l}_1 and \mathfrak{l}_2 . The weights and forgetting data are obtained by collecting the weights and forgetting data of Γ_1 and Γ_2 .

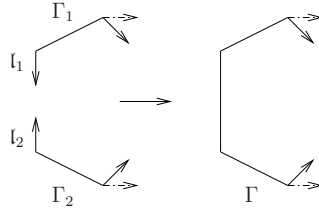


FIGURE 7. Gluing graphs.

We can now define gluing of relations as follows:

- (gluing relations) Let $(\Gamma_i, \mathfrak{w}_i, \mathfrak{f}_i)$ be data indexing generators on the same moduli space; that is, the total degree of Γ_i is independent of i , and the marking sets $L(\Gamma_i) \setminus \mathfrak{f}_i$ are independent of i . Let $\mathfrak{l} \in L(\Gamma_i) \setminus \mathfrak{f}_i$. Let $(\Gamma, \mathfrak{w}, \mathfrak{f})$ be another weighted graph and let $\widehat{\mathfrak{l}} \in L(\Gamma) \setminus \mathfrak{f}$. Assume we have a relation:

$$\sum_i [\Gamma_i, \mathfrak{w}_i, \mathfrak{f}_i] = 0.$$

Then, a new relation is obtained by gluing along \mathfrak{l} and $\widehat{\mathfrak{l}}$:

$$\sum_i [\Gamma, \mathfrak{w}, \mathfrak{f}] \star [\Gamma_i, \mathfrak{w}_i, \mathfrak{f}_i] = 0.$$

We can now define the tautological relations.

Definition 3. The tautological relations are the smallest system of relations between the generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ with the following properties:

- All relations listed above are already in the system.
- The system of relations is closed under the morphisms π_\star .

- The system of relations is closed under gluing.

We make two immediate observations. First, the Keel relations did not involve assigning weights to the legs. However, this can be achieved via the gluing and forgetting equations. Indeed, assigning a class α to the leg l is equivalent to gluing

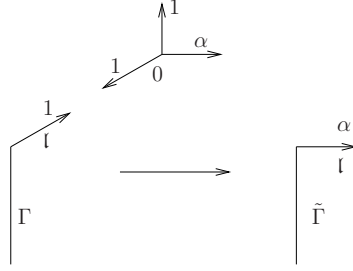


FIGURE 8. Adding in weights via gluing and forgetting.

in a tripod of degree 0 along the leg l , with weights $(1, 1, \alpha)$, and then forgetting one of the markings of the newly added vertex (and stabilizing). These Keel relations with weights assigned to the legs are therefore tautological. The same argument shows that:

- multiplication of tautological relations by $ev_l^* \alpha$ still gives a tautological relation.

Secondly,

- the system of tautological relations between $[\Gamma, \mathbf{w}, \mathbf{f}]$ is closed under pullback by the forgetful morphisms π .

We agree that:

$$\pi^* [\Gamma, \mathbf{w}, \mathbf{f}] = \sum_{\tilde{\Gamma}} [\tilde{\Gamma}, \mathbf{w}, \mathbf{f}]$$

the sum being taken over all possible graphs $\tilde{\Gamma}$ obtained from Γ by attaching a leg at any of its vertices. This is again a consequence of gluing a tripod of degree 0 and forgetting a destabilizing leg.

As a consequence, we can generalize Keel's relations accounting for $n+4$ attached legs with arbitrary weights, the last n of which are distributed arbitrarily among two vertices, and the first 4 being distributed among the vertices as $(ij)(kl)$ and $(ik)(jl)$ (see the figure).

We believe that the tautological systems are *generally* insensitive to the geometry of the target space. More formally, we pose the following:

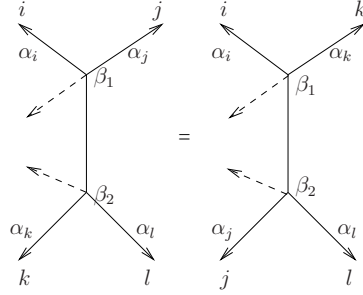


FIGURE 9. Generalized Keel relations.

Conjecture 1. *All relations between the additive generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ are obtained from the tautological relations above.*

It is remarkable that all relations we could find in the literature are of this nature [16], [10], [3], [5], [4]. However, these relations are highly non-trivial since they are incarnations, via the pushforward morphisms, of relations on different moduli spaces between classes of higher codimension. We have a similar universal way of producing relations involving the cotangent lines. In their study [14], Mustata and Mustata also propose a set of generators and a set of relations for the tautological rings. Their master relations in Lemma 2.1 and Lemma 2.2 are tautological, if one enlarges the definition to include ψ classes; the relations giving the κ -class decomposition in Corollary 2.4 are tautological as well (these remarks follow by the methods of Section 2 below).

We already made several checks of the Conjecture above in low codimensions and low degrees. We verified the cases when the target is $X = \mathbb{P}^r$, $d = 1$ and n arbitrary, $d \leq 3$ and $n \leq 1$ in complex codimension up to 4, also $X = \mathbb{P}^1$, and all $d \leq 5$ and $n \leq 1$, codimension 2 for any flag X , $n \leq 3$. In this note, we explain the following two results:

Theorem 2. *All relations between the tautological generators of the open moduli spaces $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ are restrictions of the tautological relations.*

Theorem 3. *Conjecture 1 is true for all SL flags manifolds in codimension 1. It also holds in codimension 2 for projective spaces.*

1. STABLE MAPS TO PROJECTIVE SPACES.

In this section we determine the Chow groups of the *open* stratum of irreducible maps of degree $d \geq 1$ to \mathbb{P}^r . We prove a "Gorenstein" property of the tautological rings reminiscent of Faber's conjecture.

- Proposition 1.** (1) *The Chow rings of $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ can be described explicitly in terms of the tautological classes in definition 1 - the precise description is given in lemmas 1 - 3. They are isomorphic to the Chow rings of certain projective manifolds of lower dimension which do not depend on the degree $d \geq 1$.*
- (2) *All relations between the tautological generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ are tautological in the sense of Definition 3.*

Remark 2. When $n = 0$, Pandharipande proved the fact below [17]. Our goal here is to obtain similar results for more markings.

Fact 1 (Pandharipande). The Chow ring of $M_{0,0}(\mathbb{P}^r, d)$ is isomorphic to the Chow ring of the Grassmannian $\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$. The (restrictions of the) classes $\kappa(H^{i+1}, H^{j+1})$, where $0 \leq i \leq j \leq r - 1$, form a set of generators.

To prove the first item in proposition 1, we follow the same line of reasoning as the original paper, *claiming no new ideas*. We include these computations because they fit quite naturally with our arguments, and because they are necessary in proving the second part of the proposition. Moreover, we would like to point out consistency with the results of [7] where the Betti numbers of the relevant spaces are determined.

1.1. Three marked points. Let us start with the case when $n \geq 3$. As in section 2, $A^*(M_{0,n}(\mathbb{P}^r, d))$ is isomorphic to $A^*(\text{Map}_d(\mathbb{P}^1, \mathbb{P}^r))$ via the natural projection $p : M_{0,n}(\mathbb{P}^r, d) \rightarrow \text{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$.

The last group can be computed from the image of any compactification of $\text{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$. There is an obvious candidate for a compactification, namely the projective space $\mathbb{P}(V)$ where $V = \bigoplus_0^r H^0(\mathbb{P}^1, \mathcal{O}(d))$. We need to identify the image of the restriction map:

$$i^* : A^k(\mathbb{P}(V)) \rightarrow A^k(\text{Map}_d(\mathbb{P}^1, \mathbb{P}^r)).$$

We claim that the image is 1 dimensional for $k \leq r - 1$ and zero otherwise. If h is the hyperplane class on $\mathbb{P}(V)$, it is enough to show that $i^*h^{r-1} \neq 0$ and $i^*h^r = 0$.

We use the different compactification of $\text{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$ by the Kontsevich-Manin space of maps $\overline{M}_{0,3}(\mathbb{P}^r, d)$, letting \tilde{D} be the disjoint union of the boundary divisors. We apply the exact sequence:

$$A^{k-1}(\tilde{D}) \rightarrow A^k(\overline{M}_{0,3}(\mathbb{P}^r, d)) \rightarrow \text{Image } i^* \rightarrow 0.$$

We define the following class on the Kontsevich-Manin space: $L_i = ev_i^*H$ where H is the hyperplane class in \mathbb{P}^r . In lemma 2 we show that the class:

$$L_1^r + L_1^{r-1}L_2 + \dots + L_1L_2^{r-1} + L_2^r$$

is supported on the boundary. We let j denote the inclusion of $\text{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$ into the Kontsevich-Manin space. Since $j^*L_1 = j^*L_2 = i^*h$ we see that $i^*h^r = 0$.

It remains to show that $i^*h^{r-1} \neq 0$, or that i^* is nonzero in degree $r-1$. By the exact sequence above, it suffices to show that the class L_1^{r-1} is not supported on the boundary.

To see this, we follow an idea of Pandharipande [17] to reduce to the case of maps of degree 1. Let ν be a self-morphism of \mathbb{P}^r of degree d . Let C be the universal curve over $\overline{M}_{0,3}(\mathbb{P}^r, 1)$. We have a diagram:

$$\begin{array}{ccc} C & \xrightarrow{ev} & \mathbb{P}^r \xrightarrow{\nu} \mathbb{P}^r \\ \downarrow \pi & & \\ \overline{M}_{0,3}(\mathbb{P}^r, 1) & & \end{array}$$

which induces a map $\tau : \overline{M}_{0,3}(\mathbb{P}^r, 1) \rightarrow \overline{M}_{0,3}(\mathbb{P}^r, d)$. Letting \tilde{L}_1 be the evaluation class on the space of degree 1 maps, we have:

$$\tau^*L_1 = \tau^*ev_1^*H = ev_1^*\nu^*H = d \cdot ev_1^*H = d\tilde{L}_1.$$

Assuming that L_1^{r-1} is supported on the boundary of $\overline{M}_{0,3}(\mathbb{P}^r, d)$ then we conclude that the class \tilde{L}_1^{r-1} is supported on the boundary of $\overline{M}_{0,3}(\mathbb{P}^r, 1)$. Therefore, $j^*\tilde{L}_1^{r-1} = 0$ on $M_{0,3}(\mathbb{P}^r, 1)$. Here j denotes, just as for degree d , the inclusion $M_{0,3}(\mathbb{P}^r, 1) \rightarrow \overline{M}_{0,3}(\mathbb{P}^r, 1)$.

We will now derive the contradiction by looking at the space of degree 1 maps. Denote again by i the inclusion of $M_{0,3}(\mathbb{P}^r, 1)$ into its "obvious" compactification \mathbb{P}^{2r+1} . In fact, $M_{0,3}(\mathbb{P}^r, 1)$ can be described as $\mathbb{P}^{2r+1} \setminus S$, where S is the subvariety corresponding to $r+1$ -tuples of polynomials of degree 1 with a common root. Then S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^r$. The restriction map $i^* : A^{r-1}(\mathbb{P}^{2r+2}) \rightarrow A^{r-1}(M_{0,3}(\mathbb{P}^r, 1))$ is an isomorphism. It is clear that $j^*\tilde{L}_1 = i^*h$, where h is the hyperplane class on \mathbb{P}^{2r+1} . We obtain the contradiction $0 = j^*\tilde{L}_1^{r-1} = i^*h^{r-1} \neq 0$. Our claim is now proved.

Combining the claim with the observation opening this subsection, we obtain:

Lemma 1. *When $n \geq 3$, $A^*(M_{0,n}(\mathbb{P}^r, d))$ is isomorphic to $A^*(\mathbb{P}^{r-1})$, and $L_1 = ev_1^*H$ is a multiplicative generator.*

1.2. Two marked points. In this subsection, we will compute $A^*(M_{0,2}(\mathbb{P}^r, d))$ using the ideas of Pandharipande [17].

We let ν be a degree d self-morphism of \mathbb{P}^r . As before, composition with ν induces a morphism $\tau : M_{0,2}(\mathbb{P}^r, 1) \rightarrow M_{0,2}(\mathbb{P}^r, d)$.

Observe that $M_{0,2}(\mathbb{P}^r, 1) = \mathbb{P}^r \times \mathbb{P}^r \setminus \Delta$ where Δ is the diagonal. Moreover, $\overline{M}_{0,2}(\mathbb{P}^r, 1) = \text{Bl}_\Delta(\mathbb{P}^r \times \mathbb{P}^r)$ is the blowup along the diagonal. The two hyperplane classes on the two factors of $\mathbb{P}^r \times \mathbb{P}^r$, as well as on the blow up, will be denoted by h_1 and h_2 . The evaluation classes \tilde{L}_1 and \tilde{L}_2 coincide with h_1 and h_2 on $\overline{M}_{0,2}(\mathbb{P}^r, 1)$. Notice that $\tau^*L_i = d \cdot h_i$.

Letting $s_r = \sum_{i+j=r} h_1^i h_2^j$, we see that $A^*(\mathbb{P}^r \times \mathbb{P}^r \setminus \Delta) = \mathbb{C}[h_1, h_2]/(h_1^{r+1}, h_2^{r+1}, s_r)$. It is clear that τ induces a homomorphism:

$$(4) \quad \tau^* : A^*(M_{0,2}(\mathbb{P}^r, d)) \rightarrow \mathbb{C}[h_1, h_2]/(h_1^{r+1}, h_2^{r+1}, s_r) = A^*(M_{0,2}(\mathbb{P}^r, 1)).$$

The map τ^* is surjective since we saw h_1 and h_2 are contained in its image.

We seek to show that τ^* is an isomorphism. To this end, we will analyze $M_{0,2}(\mathbb{P}^r, d)$ differently, by exhibiting this space as a quotient. Let $\mathbb{P}^1 = \mathbb{P}(V)$ where $V \cong \mathbb{C}^2$ is a two dimensional vector space with the natural action of the torus $T = \mathbb{C}^* \times \mathbb{C}^*$. Let

$$U \hookrightarrow \bigoplus_0^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \bigoplus_0^r \text{Sym}^d(V^*)$$

be the open subvariety corresponding to $(r+1)$ -tuples of degree d polynomials with no common vanishing.

The torus T acts with finite stabilizers on U and the geometric quotient is $M_{0,2}(\mathbb{P}^r, d)$ [17]. There is an isomorphism:

$$A^*(M_{0,2}(\mathbb{P}^r, d)) = A_T^*(U) = A^*(U_T).$$

Here $U_T = U \times_T ET$ is the Borel construction. In the topological category, we take $BT = \mathbb{P} \times \mathbb{P}$, where \mathbb{P} is the infinite projective space, while $ET \rightarrow BT$ is the T -bundle whose associated vector bundle is $\mathbb{S} = p_1^* \mathcal{O}_{\mathbb{P}}(-1) \oplus p_2^* \mathcal{O}_{\mathbb{P}}(-1)$. Here, we write p_1 and p_2 for the two projections. In the algebraic category, for the purposes of finding classes of *fixed* codimension, we will pass to projective spaces \mathbb{P} of large, but finite, dimension.

Now, we observe that U_T sits naturally as a subspace of the following bundle $p : B \rightarrow \mathbb{P} \times \mathbb{P}$:

$$B = \bigoplus_0^r \text{Sym}^d V^* \times_T ET = \bigoplus_0^r \text{Sym}^d(p_1^* \mathcal{O}_{\mathbb{P}}(1) \oplus p_2^* \mathcal{O}_{\mathbb{P}}(1)) = \bigoplus_0^r \bigoplus_{i+j=d} p_1^* \mathcal{O}_{\mathbb{P}}(i) \oplus p_2^* \mathcal{O}_{\mathbb{P}}(j).$$

We let D be the complement of U_T in this bundle. We let \mathbf{h}_1 and \mathbf{h}_2 denote the two generators of $A^*(\mathbb{P} \times \mathbb{P})$. The pullbacks $p^*\mathbf{h}_1$ and $p^*\mathbf{h}_2$ are the generators for B .

There is an exact sequence:

$$(5) \quad A_{\dim-k}(D) \xrightarrow{i} A^k(B) \xrightarrow{j^*} A^k(U_T) \rightarrow 0$$

Therefore, $A^*(U_T)$ is spanned by the restrictions $j^*p^*\mathbf{h}_1$ and $j^*p^*\mathbf{h}_2$.

We consider the following element in the Chow group of $\mathbb{P} \times \mathbb{P}$: $\mathbf{s}_j = \sum_{a+b=j} \mathbf{h}_1^a \mathbf{h}_2^b$. We claim that the elements $p^*\mathbf{s}_{k+r}$ are in the image of the inclusion map i so they are also in the kernel of j^* for all $k \geq 0$.

The proof of the claim is almost identical to Pandharipande's argument. We introduce the following notation. We let π be the projection $\mathbb{P}(\mathcal{S}) \rightarrow \mathbb{P} \times \mathbb{P}$. We look at the total space Q of the bundle π^*B which sits over $\mathbb{P}(\mathcal{S})$, and comes equipped with the pullback of the tautological bundle $p^*\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1)$:

$$\begin{array}{ccc} Q = \pi^*B & \xrightarrow{p} & \mathbb{P}(\mathcal{S}) \\ \downarrow \pi & & \downarrow \pi \\ B & \xrightarrow{p} & \mathbb{P} \times \mathbb{P}. \end{array}$$

Just as in [17] one arrives at the following diagram:

$$\begin{array}{ccc} Q \setminus \overline{D} & \xrightarrow{ev} & \mathbb{P}^r \\ \downarrow \pi & & \\ B \setminus D = U_T & \xrightarrow{\epsilon} & U/T = M_{0,2}(\mathbb{P}^r, d). \end{array}$$

Indeed, unwinding the definitions, we obtain a natural evaluation map $ev : Q \setminus \overline{D} \rightarrow \mathbb{P}^r$ undefined over the common vanishing \overline{D} of $r+1$ canonical sections of $p^*\mathcal{O}_{\mathbb{P}(\mathcal{S})}(d)$. It is clear that \overline{D} maps birationally to D via π . The classifying map ϵ to the coarse moduli scheme is the natural quotient map.

We observed that $[\overline{D}] = c_1(p^*\mathcal{O}_{\mathbb{P}(\mathcal{S})}(d))^{r+1}$. Hence, the class $\pi_* (c_1(p^*\mathcal{O}_{\mathbb{P}(\mathcal{S})}(d))^{r+1} \alpha)$ is supported on D , so it is in the image of i , for all classes α on Q . We apply this observation to the class:

$$\alpha = p^*c_1(\mathcal{O}_{\mathbb{P}(\mathcal{S})}(d))^k.$$

The following class is contained in the image of i :

$$\begin{aligned} \pi_* \left(p^*c_1(\mathcal{O}_{\mathbb{P}(\mathcal{S})}(d))^{r+1+k} \right) &= d^{r+k+1} p^* \pi_* (c_1(\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1))^{r+k+1}) = p^* \pi_* (c_1(\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1))^{r+k+1}) = \\ &= p^* \left(\frac{1}{c(\mathcal{S})} \right)_{r+k} = p^* \left(\frac{1}{(1-\mathbf{h}_1)(1-\mathbf{h}_2)} \right)_{r+k} = p^*\mathbf{s}_{r+k}. \end{aligned}$$

Using what we just proved together with the exact sequence (5), we obtain a surjection:

$$(6) \quad j^* : \mathbb{C}[p^*\mathbf{h}_1, p^*\mathbf{h}_2] / (p^*\mathbf{s}_{k+r})_{k \geq 0} \rightarrow A^*(M_{0,2}(\mathbb{P}^r, d)).$$

The reader can check that there is an obvious isomorphism between the right hand side of (4) and the left hand side of (6). Hence both (4) and (6) are isomorphisms.

We know that $h_1^i h_2^j$ ($0 \leq i, j \leq r$) span the right hand side of (4) with the relation $s_r = 0$ and $h_1^i h_2^j = d^{-i-j} \tau^* j^*(L_1^i L_2^j)$. Therefore,

Lemma 2. *The map τ^* induces a ring isomorphism between $A^*(M_{0,2}(\mathbb{P}^r, d))$ and $A^*(\mathbb{P}^r \times \mathbb{P}^r \setminus \Delta) = \mathbb{C}[h_1, h_2] / (h_1^{r+1}, h_2^{r+1}, \sum_{i+j=r} h_1^i h_2^j)$. The class*

$$(7) \quad \sum_{i+j=r} ev_1^* H^i \cdot ev_2^* H^j$$

is supported on the boundary.

1.3. One marked point. The discussion for one marked point is similar. We first observe that $\overline{M}_{0,0}(\mathbb{P}^r, 1) = \mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$ and $\overline{M}_{0,1}(\mathbb{P}^r, 1) = \mathbb{P}(\mathbf{S})$ where \mathbf{S} is the tautological bundle over the Grassmannian.

We fix ν a degree d self-morphism of \mathbb{P}^r , and as usual, we use composition with ν to get degree d maps from degree 1 maps. We obtain two morphisms:

$$\tau : \mathbf{G}(\mathbb{P}^1, \mathbb{P}^r) \rightarrow M_{0,0}(\mathbb{P}^r, d) \text{ and } \tau : \mathbb{P}(\mathbf{S}) \rightarrow M_{0,1}(\mathbb{P}^r, d).$$

We conclude that there is a diagram:

$$\begin{array}{ccc} A^*(M_{0,0}(\mathbb{P}^r, d)) & \xrightarrow{\tau^*} & A^*(\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)) \\ \pi^* \downarrow & & \pi^* \downarrow \\ A^*(M_{0,1}(\mathbb{P}^r, d)) & \xrightarrow{\tau^*} & A^*(\mathbb{P}(\mathbf{S})). \end{array}$$

The lower horizontal arrow:

$$(8) \quad \tau^* : A^*(M_{0,1}(\mathbb{P}^r, d)) \rightarrow A^*(\mathbb{P}(\mathbf{S}))$$

is surjective. Indeed, the Chow ring of the projective bundle $\mathbb{P}(\mathbf{S})$ is generated by the Chow ring of the base $\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$ together with the additional class $\lambda = c_1(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(1))$. The class λ is in the image of τ^* . Indeed, pulling back under the two evaluation maps: $ev : \mathbb{P}(\mathbf{S}) = M_{0,1}(\mathbb{P}^r, 1) \rightarrow \mathbb{P}^r$ and $ev : M_{0,1}(\mathbb{P}^r, d) \rightarrow \mathbb{P}^r$ it is clear that:

$$\tau^* L_1 = \tau^* ev^* H = ev^* \nu^* H = d \cdot ev^* H = d\lambda.$$

We also know by fact 1 that the upper arrow τ^* is surjective. This proves the claim.

The next step involves the computation in equivariant Chow groups. We keep the notation of the previous subsection. We observe that $M_{0,1}(\mathbb{P}^r, d) = U/N$, where N is the group of 2×2 upper triangular matrices acting on $V \cong \mathbb{C}^2$. We obtain the isomorphism:

$$A^*(M_{0,1}(\mathbb{P}^r, d)) = A_N^*(U) = A^*(U_N) = A^*(U \times_N EN).$$

We denote by \mathbb{G} the infinite Grassmannian of 2 dimensional planes, and by \mathbb{S} the tautological rank 2 bundle. BN can be identified with the projective bundle $\mathbb{P}(\mathbb{S})$. We let $\pi : \mathbb{P}(\mathbb{S}) \rightarrow \mathbb{G}$ denote the projection. Also $V \times_N EN = \pi^*\mathbb{S}$. For our purposes, we will pass to finite dimensional truncations of \mathbb{G} by large dimensional Grassmannians.

We can view $U \times_N EN$ as a subvariety of the bundle:

$$\bigoplus_0^r \text{Sym}^d(V^*) \times_N EN = \bigoplus_0^r \text{Sym}^d \pi^*\mathbb{S}^*.$$

Let B denote this bundle and let D denote the complement of U_N in B . We obtain an exact sequence:

$$(9) \quad A_{\dim-k}(D) \xrightarrow{i} A^k \left(\bigoplus_0^r \text{Sym}^d(\pi^*\mathbb{S}^*) \right) = A^k(\mathbb{P}(\mathbb{S})) \xrightarrow{j^*} A^k(M_{0,1}(\mathbb{P}^r, d)) \rightarrow 0.$$

We denote by $p : B \rightarrow \mathbb{P}(\mathbb{S})$ and $q : \mathbb{P}(\pi^*\mathbb{S}) \rightarrow \mathbb{P}(\mathbb{S})$ the two projections. We let Q be the total space of the bundle q^*B so that we obtain a commutative diagram:

$$\begin{array}{ccccc} Q = q^*B & \xrightarrow{p} & \mathbb{P}(\pi^*\mathbb{S}) & & \\ & & \downarrow q & \uparrow q & \\ M_{0,1}(\mathbb{P}^r, d) & \xleftarrow{\epsilon} & B & \xrightarrow{p} & \mathbb{P}(\mathbb{S}) \xrightarrow{\pi} \mathbb{G}. \end{array}$$

Just as before, the space $\mathbb{P}(\pi^*\mathbb{S})$ comes equipped with a bundle $\mathcal{O}(d)$, which gives by pullback the bundle $p^*\mathcal{O}(d)$ over Q .

Unwinding the definitions, we obtain an evaluation morphism $ev : Q \setminus \overline{D} \rightarrow \mathbb{P}^r$ undefined over the common vanishing \overline{D} of $r+1$ sections of the line bundle $p^*\mathcal{O}(d)$ on Q . Hence, $[\overline{D}] = c_1(p^*\mathcal{O}(d))^{r+1}$. Therefore, the image of i contains the classes:

$$q_*(c_1(p^*\mathcal{O}(d))^{r+k+1}) = p^*q_*(c_1(\mathcal{O}(d))^{r+k+1}) = d^{r+k+1}p^*\pi^*s_{r+k}(\mathbb{S}), \quad k \geq 0,$$

where s_{r+k} are the Segre classes. We use the exact sequence (9) to obtain a surjection:

$$A^*(\mathbb{P}(\mathbb{S})) / (\pi^*s_{r+k}(\mathbb{S}))_{k \geq 0} \rightarrow A^*(M_{0,1}(\mathbb{P}^r, d)).$$

We recall the surjection (8) and the isomorphism:

$$A^*(\mathbb{P}(\mathbf{S})) / (\pi^* s_{r+k}(\mathbf{S}))_{k \geq 0} \rightarrow A^*(\mathbb{P}(\mathbf{S})).$$

This follows from the usual description of the Chow rings of projective bundles and the fact that over the base we have the analogous isomorphism:

$$A^*(\mathbb{G}) / (s_{r+k}(\mathbf{S}))_{k \geq 0} \rightarrow A^*(\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)).$$

This is enough to infer that the map τ^* is an isomorphism. However, we push the analysis further. We see that $A^*(\mathbb{P}(\mathbf{S}))$ is generated by $A^*(\mathbf{G})$ together with the class $\lambda = c_1(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(1))$ which satisfies the relation $\lambda^2 + c_1(\mathbf{S})\lambda + c_2(\mathbf{S}) = \mathbf{0}$. We have:

$$\lambda = \frac{1}{d}\tau^*L_1, \quad \frac{1}{d^2}\tau^*\kappa(H^2) = -c_1(\mathbf{S}), \quad \frac{1}{d^4}\tau^*\kappa(H^2)^2 - \frac{1}{d^3}\tau^*\kappa(H^3) = c_2(\mathbf{S}).$$

The last two follow from a fact explained in [17], namely that the Chern classes of the quotient bundles $c_1(Q)$ and $c_2(Q)$ are the pullbacks of $\frac{1}{d^2}\kappa(H^2)$ and $\frac{1}{d^3}\kappa(H^3)$ under τ . We derive:

Lemma 3. *$A^*(M_{0,1}(\mathbb{P}^r, d))$ is generated multiplicatively by $L_1 = ev_1^*H$ and the pullback of $A^*(M_{0,0}(\mathbb{P}^r, d))$ as determined in 1. τ^* establishes an isomorphism with $A^*(\mathbb{P}(\mathbf{S}))$, where \mathbf{S} is the tautological bundle over the Grassmannian $\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$. The codimension 2 class:*

$$(10) \quad ev_1^*H^2 - \frac{1}{d}ev_1^*H \cdot \kappa(H^2) + \frac{1}{d^2}\kappa(H^2)^2 - \frac{1}{d}\kappa(H^3)$$

is supported on the boundary.

1.4. The tautological relations on the open stratum. We will show that all relations between the tautological classes $[\Gamma, \mathbf{w}, \mathbf{f}]$ on $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ are tautological in the sense of definition 3.

We start our analysis starts with the case $n = 0$. All tautological classes $[\Gamma, \mathbf{w}, \mathbf{f}]$ on $\mathcal{M}_{0,0}(\mathbb{P}^r, d)$ are restrictions of the classes $\kappa(H^{i_1}, \dots, H^{i_m})$. According to Fact 1, the two point classes $\kappa(H^{l_1}, H^{l_2})$ are additive generators without relations. It suffices to express $\kappa(H^{i_1+1}, \dots, H^{i_i+1})$ in terms of the two point classes by means of the tautological relations. This can be thought of as an instance of Kontsevich-Manin reconstruction, and it can be proved in the same manner.

We use the Keel relations on $\overline{\mathcal{M}}_{0,n+4}(\mathbb{P}^r, d)$ with cohomology weights $H^{i_1}, \dots, H^{i_4}, H^{j_1}, \dots, H^{j_n}$ assigned to the legs of the graph. Using invariance under the forgetful morphisms, we conclude that the following equation on $\mathcal{M}_{0,0}(\mathbb{P}^r, d)$:

$$\kappa(H^{i_1+i_4}, H^{i_2}, H^{i_3}, H^{j_1}, \dots, H^{j_n}) + \kappa(H^{i_1}, H^{i_4}, H^{i_2+i_3}, H^{j_1}, \dots, H^{j_n}) =$$

$$(11) \quad = \kappa(H^{i_1+i_3}, H^{i_2}, H^{i_4}, H^{j_1}, \dots, H^{j_n}) + \kappa(H^{i_1}, H^{i_3}, H^{i_2+i_4}, H^{j_1}, \dots, H^{j_n})$$

is tautological. We have made use of the mapping to a point and forgetting destabilizing legs relations. Setting $i_4 = 1$, and using the divisor equation, we express

$$\kappa(H^{i_1+1}, H^{i_2}, H^{i_3}, H^{j_1}, \dots, H^{j_n}) - \kappa(H^{i_1}, H^{i_2+1}, H^{i_3}, H^{j_1}, \dots, H^{j_n})$$

in terms of classes with fewer insertions. This works as long as we have at least at least 3 insertions. This system of equations together determine uniquely the κ classes with several insertions in terms of the two point classes $\kappa(H^{l_1}, H^{l_2})$.

When $n = 1$, the same reasoning applies. We only need to write down tautological equations expressing the classes $L_1^e \cdot \kappa(H^{i_1}, \dots, H^{i_l})$ in terms of the generators with $e \leq 1$ and $l \leq 2$. We can multiply (11) by $ev_1^* H^e$ to get tautological relations which reduce us to the case $l \leq 2$.

The last step of the reduction consists in proving that the equation (10) is tautological. In fact, we claim the following tautological equation on $\mathcal{M}_{0,1}(\mathbb{P}^r, d)$:

$$(12) \quad ev_1^* \alpha - \frac{1}{d} ev_1^* H \cdot \kappa(\alpha) + \frac{1}{d^2} \kappa(\alpha, H^2) - \frac{1}{d} \kappa(\alpha H) = 0.$$

Indeed, consider the Keel relation on $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d)$ with distribution of the markings (12)(34) and (13)(24) among two vertices, such that the weights of the legs are $1, \alpha, H, H$ respectively. We then forget the the last three markings via the morphism $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d) \rightarrow \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, d)$, and restrict to the open part $\mathcal{M}_{0,1}(\mathbb{P}^r, d)$. The statement then follows using the divisor relation, and the contracting unstable tripods relations.

When $n = 2$, we need to prove that the generators

$$ev_1^* H^{i_1} \cdot ev_2^* H^{i_2} \cdot \kappa(H^{j_1}, \dots, H^{j_l})$$

can be expressed in terms of the classes $ev_1^* H^{i_1} \cdot ev_2^* H^{i_2}$ via tautological equations. This is a consequence of the above discussion and of equation (13). To include more insertions in the κ classes we multiply (13) below by a monomial in the evaluation classes and apply the forgetful morphisms. The identity (13) below also shows that equation (7) is tautological; to this end, we specialize to $k = 0$ and $l = r$, then use the pullback from the target relations.

We claim that the following equation on $\mathcal{M}_{0,2}(\mathbb{P}^r, d)$:

$$(13) \quad \sum_{i=0}^l ev_1^* H^{k+l-i} \cdot ev_2^* H^{k+i} = \frac{1}{d} ev_1^* H^k \cdot ev_2^* H^k \cdot \kappa(H^{l+1})$$

is tautological. It is clear that the case $l = 0$ is just the divisor equation. The case $l = 1$ is a tautological equation since it is obtained by multiplication by evaluation

classes of the tautological equation:

$$(14) \quad ev_1^*H + ev_2^*H = \frac{1}{d}\kappa(H^2).$$

In turn, this is obtained from the Keel equation on $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d)$, splitting the legs in the two configurations (12)(34) and (13)(24) among two vertices. The weights on the legs are $(1, 1, H, H)$. We then pushforward the relation by the forgetful morphism $\pi : \overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d) \rightarrow \overline{\mathcal{M}}_{0,2}(\mathbb{P}^r, d)$, and use the divisor equation to obtain (14). Similarly, one proves that the equation:

$$(15) \quad ev_1^*H \cdot ev_2^*H = \frac{1}{d^2}\kappa(H^2, H^2) - \frac{1}{d}\kappa(H^3)$$

is a tautological identity on $\mathcal{M}_{0,2}(\mathbb{P}^r, d)$.

As a corollary of (12) and (14) the following more general tautological equation on $\mathcal{M}_{0,2}(\mathbb{P}^r, d)$ holds true:

$$(16) \quad ev_1^*H^l + ev_2^*H^l = \frac{2}{d}\kappa(H^{l+1}) - \frac{1}{d^2}\kappa(H^2, H^l)$$

Finally, equation (13) for $(k+1, l-2)$ and (15) imply the statement for (k, l) if one observes that:

$$\kappa(H^{l+1}) - \frac{1}{d}\kappa(H^2, H^l) + \frac{1}{d^2}\kappa(H^2, H^2, H^{l-1}) - \frac{1}{d}\kappa(H^3, H^{l-1}) = 0.$$

This again is a tautological equation obtained from Keel's relation (11) with $i_1 = 1, i_2 = l-1, l_3 = 1, i_4 = 2$ and the divisor equation.

Finally, let $n \geq 3$. The system of equations (16) and (13) for all pairs of indices (i, j) now becomes solvable. We obtain the following tautological equation on $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$:

$$(17) \quad ev_1^*H^l = \dots = ev_n^*H^l = \frac{1}{d(l+1)}\kappa(H^{l+1}).$$

For $k=0, l=r$, we obtain that

$$ev_1^*H^r = 0$$

is a tautological equation on the open part.

We have seen in the case $n=2$ that all classes $[\Gamma, \mathbf{w}, \mathbf{f}]$ can be expressed via tautological equations in terms of the evaluation classes at the first 2 markings. In turn, these can be expressed via the tautological relation (17) in terms of evaluation classes with one marking. In the light of lemma 1, the proof of proposition 1 is complete.

2. SMALL CODIMENSION EXAMPLES

We will now indicate a proof of theorem 3 concerning the relations in small codimension. We first consider the case of codimension 1 classes for any target which is an SL flag variety relying on the results of [15]. Next, the analysis in codimension 2 when the target is a projective space makes use of a Betti number computation involving localization. The technique should be more useful for general computations.

2.1. Codimension 1. Throughout this subsection X is an SL flag variety whose Betti numbers in codimension 2 and 4 are h^2 and h^4 . Let

$$V \otimes \mathcal{O}_X = \mathcal{Q}_0 \rightarrow \mathcal{Q}_1 \rightarrow \dots \rightarrow \mathcal{Q}_l \rightarrow 0$$

be the tautological quotients and let β be a class with $d_i = \beta \cdot c_1(\mathcal{Q}_i) > 0$. The tautological generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ are

- the boundary divisors,
- the classes $\kappa(\alpha)$, where $\alpha \in A^2(X)$ can be either $c_1(\mathcal{Q}_i) \cdot c_1(\mathcal{Q}_j)$ or $c_2(\mathcal{Q}_i)$,
- the evaluation classes $ev_i^* c_1(\mathcal{Q}_j)$.

We seek to prove:

Proposition 2. *When X is any SL flag variety, all relations between the codimension 1 tautological classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ are tautological.*

The discussion in [15] essentially established this claim. It was established in [15] that the dimension of the complex codimension 1 cohomology of $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is

$$(18) \quad \left[2^{n-1}(d_1 + 1) \dots (d_l + 1) + \frac{1}{2} \right] - 1 - \binom{n}{2} + h^4 - \binom{h^2}{2}.$$

It remains to exhibit enough independent tautological relations between the tautological generators to obtain the above dimension. This is essentially done in [15]. For instance, we proved that relations between the boundary divisors are Keel relations. It suffices to show all relations between the κ 's and evaluation classes used in [15] are tautological.

- First, we claimed in [15] that when $n \geq 3$, the evaluation classes can be expressed in terms of the κ 's via the following tautological equation (14):

$$(19) \quad ev_i^* c_1(\mathcal{Q}_k) + ev_j^* c_1(\mathcal{Q}_k) = \frac{1}{c_1(\mathcal{Q}_k) \cdot \beta} \kappa(c_1(\mathcal{Q}_k)^2) + \text{boundaries} .$$

When $n \geq 3$, this system can be solved to express the evaluation classes in terms of the κ 's via the tautological relations.

- When $n = 1$ and $n = 2$, we claimed in [15] that the span of $ev_i^* \mathcal{Q}_j$ is one dimensional modulo boundaries and κ 's. We need to explain this using the tautological equations. We need to prove that the tautological equations express any evaluation class $ev_i^* H'$ in terms of a fixed one, say $ev_1^* H$. Here H, H' are two divisors, such as the Chern classes of the quotient bundles \mathcal{Q}_i and \mathcal{Q}_j on X . The tautological equation (19) reduces our analysis to one marking $i = 1$.

We produce a tautological equation which expressed $ev_1^* H'$ in terms of κ classes and the evaluation $ev_1^* H$. Such an equation is not so easy to come across. The reason is the fact that our relation is an incarnation of a Keel relation in codimension 5 on the space $\overline{\mathcal{M}}_{0,5}(X, \beta)$. The legs are split as (15)(23) and (12)(53) among two vertices, and the weights assigned to the legs are $(1, H', H, H', H)$ respectively. After forgetting the first 4 markings, using the divisor equation, the no incidence equation, the unstable tripod equation, we obtain the following tautological relation:

$$\frac{1}{\beta \cdot H} ev_1^* H - \frac{1}{\beta \cdot H'} ev_1^* H' = \frac{1}{(\beta \cdot H)^2} \kappa(H^2) - \frac{1}{(\beta \cdot H)(\beta \cdot H')} \kappa(HH') + \text{boundaries.}$$

- The final ingredient needed in [15] was the fact that $\kappa(H \cdot H')$ can be expressed in terms of $\kappa(H^2)$ and $\kappa(H'^2)$. This can be seen by adding the above equations when H, H' are interchanged. As a consequence, we obtain:

$$\frac{1}{(\beta \cdot H)^2} \kappa(H^2) + \frac{1}{(\beta \cdot H')^2} \kappa(H'^2) = 2 \frac{1}{(\beta \cdot H)(\beta \cdot H')} \kappa(HH') + \text{boundaries.}$$

This completes the proof of the proposition.

2.2. The codimension 2 classes. In this subsection we show that

Proposition 3. *All relations between the codimension 2 tautological generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are tautological.*

To begin with we observe that the results of [15] essentially establish this claim for $n \leq 3$. For instance, when $n = 0$, we showed that

Lemma 4. *The collections of codimension 2 classes on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ below form a basis for the codimension 2 Chow group.*

- the boundary classes of maps whose domain has at least three components. These correspond to graphs with three vertices and no legs, no weights, and no forgetting data.

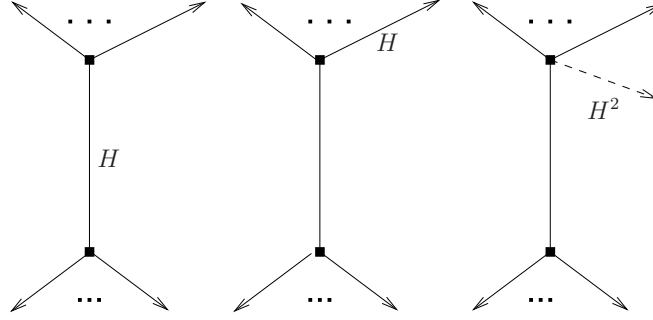


FIGURE 10. The codimension 2 tautological generators for graphs with two vertices.

- the nodal classes of maps whose node is mapped to a codimension 1 subspace; these classes correspond to graphs with two vertices, and the weight of the edge is H .
- (when $r \geq 2$) classes of nodal maps, one component passing through a fixed codimension 2 subspace; these correspond to graphs with two vertices, and a forgotten leg with weight H^2 .
- (when $r \geq 2$) classes of maps whose images pass through two general codimension two subspaces; these correspond to graphs with one vertex with two forgotten legs with weight H^2 .
- (when $r \geq 3$) the class of maps intersecting a codimension 3 subspace; these correspond to graphs with one vertex and one forgotten leg with weight H^3 .

We will omit the case when $n = 1$ and $n = 2$. The essential part of the proof of proposition 3 consists in checking the case $n \geq 3$. We will do so only in the case $r \geq 3$.

- (i) First, the proposition 1 expresses all generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ where Γ has only one vertex of degree d as a sum of $ev_1^* H^2$ with boundary classes via the tautological equations. Therefore, we retain only one generator of this form.
- (ii) The codimension 2 classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ for which Γ has two vertices are obtained when:
 - (a) an edge of Γ is decorated with the weight H , there are no other weights or forgotten legs;
 - (b) a leg of Γ is decorated with the weight H , there are no other weights or forgotten legs;
 - (c) a forgotten leg of Γ is decorated with the weight H^2 , there are no additional weights or forgotten legs.

Moreover, the tautological equations (14) and (17) show (via gluing) that we only need to consider the classes in (a) and the classes in (c) when the forgotten leg is incident to a vertex with at most 2 unforgotten legs. A simple count gives

$$(20) \quad 2^n + (d-1)(2^{n-1} + n + 1)$$

generators as above. In addition, there are

$$(21) \quad \frac{n(n-3)}{2}$$

independent Keel relations between them.

(iii) Finally, we consider the classes for which Γ has two vertices. We will first discuss the case $d = 1$. An easy count gives

$$(22) \quad \frac{3^{n+1} + 3}{2} + \frac{n(n+3)}{2} - 2^n(n+3)$$

boundary terms. We will exhibit

$$(23) \quad 2^{n-2}(n^2 - n - 8) + 2 - \frac{n(3n^3 - 10n^2 + 21n - 86)}{24}$$

independent relations between them.

Indeed, we think of the graph Γ as a vertex v with $k + 1$ legs (one of them is distinguished) glued to a graph with 2 vertices and $n - k + 1$ legs (one of them is distinguished). Once v and its incident flags are fixed, we obtain

$$\frac{(n - k + 1)^2 - 3(n - k + 1)}{2}$$

Keel relations coming from the $n - k + 1$ legs distributed between the remaining two vertices. This is however not entirely correct since the resulting relations may not be independent. We count the relations differently: first, there are relations obtained when v has degree 1. Secondly, there are relations obtained when v has degree 0, but these may not be independent from those exhibited before. Instead, a second set of relations are pulled back from $\overline{\mathcal{M}}_{0,n}$. These can be expressed in terms of Keel's relations. Moreover, they are independent from the relations of the first kind. Indeed, linear combinations of the first set of relations do not involve graphs whose middle vertex has degree 1, while any non-trivial combination of an independent set of relations of the second kind would. We count relations of the first type. We must have $k \leq n - 3$, because the remaining two degree 0 vertices

are stable. We obtain

$$(24) \quad \sum_{k \leq n-3} \binom{n}{k} \cdot \frac{(n-k+1)^2 - 3(n-k+1)}{2}$$

independent relations. The relations coming from $\overline{\mathcal{M}}_{0,n}$ are counted next. There are

$$\frac{3^n + 1}{2} - 2^{n-1}(n+3) + \frac{(n+1)(n+2)}{2}$$

codimension 2 boundaries in $\overline{\mathcal{M}}_{0,n}$. The recursions in [12] give

$$h^4(\overline{\mathcal{M}}_{0,n}) = \frac{3^n + 1}{2} - 2^{n-3}(n^2 + 3n + 4) + \frac{n(n-1)(3n^2 - 7n + 26)}{24}.$$

Therefore, we obtain

$$(25) \quad 2^{n-3}(n^2 - n - 8) + \frac{(n+1)(n+2)}{2} - \frac{n(n-1)(3n^2 - 7n + 26)}{24}$$

independent relations between these boundaries. Equation (23) follows from (24) and (25).

Finally, we match the number of generators modulo the number of relations obtained in items (i) – (iii) above to the actual dimension of $H^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1))$. This proves that there are no other relations we need to account for.

The Betti number in question is computed via localization. We use a torus action on \mathbb{P}^r fixing a point p and a hyperplane H :

$$t \cdot [z_0 : \dots : z_r] = [z_0 : tz_1 : \dots : tz_r].$$

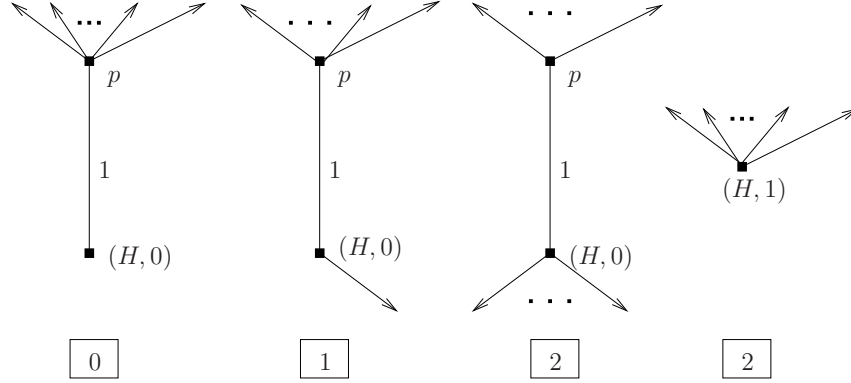
This induces an action on the moduli space of stable maps by translating the image.

There is a way of bookkeeping the fixed loci in terms of decorated graphs Γ whose edges correspond to the non-contracted components of the fixed map f and whose vertices correspond to the connected components of $f^{-1}(p) \cup f^{-1}(H)$ and are decorated by legs and degree labels. In degree 1 these graphs are particularly simple.

To compute the Betti numbers from those of the fixed loci we need to compute the number of negative weights on the normal bundle of the Γ -indexed locus. There is a general formula derived in [15] for this number:

$$1 + \mathfrak{s}(\Gamma) - \mathfrak{u}(\Gamma).$$

Here $\mathfrak{s}(\Gamma)$ and $\mathfrak{u}(\Gamma)$ are the number of H labeled vertices in the graph Γ which have valency at least 3 or positive degree, respectively have degree 0 and no legs attached.

FIGURE 11. The fixed loci on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)$.

We sum the following three contributions of the following fixed loci shown in figure 2.2:

- The fixed locus $\overline{\mathcal{M}}_{0,n+1} \times H$ with no negative weights on the normal bundle. It corresponds to the graph with one edge labeled 1, all the n legs being attached to the vertex labeled p . The contribution of this locus to the Betti number is

$$(26) \quad \begin{aligned} h^4(\overline{\mathcal{M}}_{0,n+1} \times H) &= h^4(\overline{\mathcal{M}}_{0,n+1}) + h^2(\overline{\mathcal{M}}_{0,n+1}) + 1 = \\ &= \frac{3^{n+1} + 1}{2} - 2^{n-2}(n^2 + 5n + 4) + \frac{n(n+1)(3n^2 - n + 10)}{24}. \end{aligned}$$

- There are n loci corresponding to graphs with one edge, the vertex labeled $(H, 0)$ supporting an attached leg. These fixed loci have 1 negative weight on their normal bundle and are isomorphic to $\overline{\mathcal{M}}_{0,n} \times H$. Their contribution to the Betti number is

$$(27) \quad n \cdot h^2(\overline{\mathcal{M}}_{0,n} \times H) = n \left(2^{n-1} - \frac{n(n-1)}{2} \right).$$

- There are $2^n - 1 - n$ fixed loci corresponding to graphs with an edge of degree 1 and the vertex labeled $(H, 0)$ supporting at least two legs. There is an additional fixed locus corresponding to the graph with one vertex labeled $(H, 1)$ with n legs attached. All these fixed loci have 2 negative weights on their normal bundle. Their total contribution to the Betti number is

$$(28) \quad 2^n - n.$$

Now, the computation for d arbitrary is similar, and we will not reproduce it here entirely. The count obtained in items (i) – (iii) needs to be modified in two

places. First, the total number of boundary graphs with two edges is:

$$(29) \quad \frac{(d+1)(d+2)}{2} \cdot \frac{3^n + 1}{2} - 2^{n-1}(n+3)(d+1) + \frac{(n+1)(n+2)}{2} - \left\lfloor \frac{d+1}{2} \right\rfloor \cdot \left\lfloor \frac{d+2}{2} \right\rfloor$$

Secondly, there are

$$(30) \quad (d-1) \cdot \left(\sum_{k \leq n-3} \binom{n}{k} \cdot \frac{(n-k+1)^2 - 3(n-k+1)}{2} \right)$$

independent relations between these classes *in addition* to (23).

The count of generators modulo relations is then matched to the Betti number. The dimension of the cohomology $H^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$ is computed via localization, or better via the Deligne spectral sequence. We obtain complexes:

$$(31) \quad 0 \rightarrow \bigoplus_{\alpha} H^0(D_{\alpha}^{(2)}) \rightarrow \bigoplus_{\alpha} H^2(D_{\alpha}^{(1)}) \rightarrow H^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \rightarrow 0.$$

Here, $D_{\alpha}^{(2)}$ are the codimension 2 boundary strata corresponding to dual graphs without automorphisms. Their number is obtained from formula (29) subtracting $\left\lfloor \frac{d}{2} \right\rfloor$. The middle terms are isomorphic to fibered products

$$\overline{\mathcal{M}}_{0,A \cup \{\star\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,B \cup \{\star\}}(\mathbb{P}^r, d_B)$$

for all possible splittings of the markings and degrees such that the corresponding graphs are stable. The dimensions of the middle terms are computed from (18). The alternating sums of the dimensions of the terms in (31) can be read off from the virtual Poincare polynomial $P(\mathcal{M}_{0,n}(\mathbb{P}^r, d))$. This can be computed from the associated graded of the Hodge weight filtration:

$$(32) \quad P(M) = \sum_{i,j} (-1)^{i+j} \dim Gr_j^W(H_c^i(M)) q^j$$

The expression in [7] proves that the alternating sums of dimensions is 1. We conclude that

$$\begin{aligned} h^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) &= \frac{(d-1)(d+4)}{2} \cdot \frac{3^n - 1}{2} - 2^{n-3}(d-1)(n^2 + 6n + 9) + \\ &+ \left\lfloor \frac{d+1}{2} \right\rfloor \cdot \left\lfloor \frac{d+2}{2} \right\rfloor + \left\lfloor \frac{d}{2} \right\rfloor - 1 + (26) + (27) + (28). \end{aligned}$$

Putting everything together, we find the claimed result.

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DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CALIFORNIA, SAN DIEGO

E-mail address: doprea@math.ucsd.edu