Math 10C - Winter 2010 - Midterm I

Name: ________________________________

Student ID: __________________________

Section time: _________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books or telephones. You may use a “cheat sheet” of notes which should be a page, front only.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 45 points. You have 50 minutes to complete the test.

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Problem 1. [8 points.]

Consider the function

\[ f(x, y) = x^2 + y^2 - 1. \]

(i) [4 points] Draw the contour diagram for \( f(x, y) \) and clearly label the level curves. Show the contours for at least three levels.

The level curve at level \( c \) is

\[ f(x, y) = c \implies x^2 + y^2 - 1 = c \implies x^2 + y^2 = c + 1. \]

This is a circle with center at the origin and radius \( \sqrt{c+1} \).

For instance you may pick the following three levels:
- \( c = 0 \) which gives a circle of radius 1 with center at the origin. This circle is labeled by the level 0;
- \( c = 3 \) which gives a circle of radius 2 labeled by the level value 3;
- \( c = 8 \) which gives a circle of radius 3 labeled by the level value 8;

There are other choices for levels, so other answers may be correct as well.

(ii) [4 points] Draw the graph of \( z = f(x, y) \).

The graph is a paraboloid whose lowest point is \((0, 0, -1)\).
Problem 2. [12 points.]

(i) [7 points] Find the equation of the plane that passes through the points \((1, 1, 0)\), \((2, 1, -1)\) and \((2, 0, 4)\).

We consider the points \((1, 1, 0)\) and \((2, 1, -1)\). Their second coordinates agree, so we can use these points to compute the \(m\)-slope:

\[
m = \frac{-1 - 0}{2 - 1} = -1.
\]

Similarly, we can use the points \((2, 1, -1)\) and \((2, 0, 4)\) to compute the \(n\)-slope:

\[
n = \frac{4 - (-1)}{0 - 1} = -5.
\]

Finally, we can write down the equation of the plane when we know the slopes \(m\) and \(n\) and one of the points \((1, 1, 0)\):

\[
z = -(x - 1) - 5(y - 1) \implies z = -x - 5y + 6.
\]

(ii) [5 points] Compute the sum

\[
\frac{4}{5^5} + \frac{4}{5^6} + \frac{4}{5^7} + \ldots + \frac{4}{5^{2010}}.
\]

Please write your answer in a simple form.

This is a geometric series. The first term equals \(a = \frac{4}{5^5}\), the common ratio is \(r = \frac{1}{5}\). The number of terms in the series is \(n = 2010 - 5 + 1 = 2006\). The sum equals

\[
\frac{4}{5^5} \cdot \frac{1 - \left(\frac{1}{5}\right)^{2006}}{1 - \frac{1}{5}} = \frac{4}{5^5} \cdot 5 \cdot \left(1 - \frac{1}{5^{2006}}\right) = \frac{1}{5^4} \left(1 - \frac{1}{5^{2006}}\right).
\]
Problem 3. [10 points.]

(i) [4 points] The degree three Taylor polynomial of a function \( f(x) \) at \( a = 0 \) equals
\[
1 + 2x - 3x^2 - \frac{1}{3}x^3.
\]
Calculate the third derivative \( f'''(0) \).

The Taylor polynomial equals
\[
f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.
\]
We match the coefficients of \( x^3 \), hence
\[
\frac{f'''(0)}{6} = -\frac{1}{3} \implies f'''(0) = -2.
\]

(ii) [6 points] Find the degree 2 Taylor polynomial at \( a = 2 \) for the function
\[
g(x) = \frac{1}{x - 1}.
\]
We compute \( g(2) = 1 \). Writing
\[
g(x) = (x - 1)^{-1}
\]
we see that
\[
g'(x) = -(x - 1)^{-2} \implies g'(2) = -1
\]
\[
g''(x) = 2(x - 1)^{-3} \implies g''(2) = 2.
\]
The Taylor polynomial is
\[
g(2) + g'(2)(x - 2) + \frac{g''(2)}{2}(x - 2)^2 = 1 - (x - 2) + (x - 2)^2.
\]
Problem 4. [15 points.]

The outcome \( x \) of an experiment is always between 1 and 3. The probability density function equals
\[
f(x) = \frac{C}{x^2}, \text{ for } 1 \leq x \leq 3.
\]

(i) [4 points] Find the constant \( C \).

Any pdf integrates to 1. In our case
\[
\int_{1}^{3} \frac{C}{x^2} \, dx = 1 \implies -C \bigg|_{x=3}^{x=1} = 1 \implies -\frac{C}{3} + C = 1 \implies \frac{2C}{3} = 1 \implies C = \frac{3}{2}.
\]

(ii) [4 points] Determine the cumulative distribution function.

We integrate the pdf to get the cdf. We compute
\[
P(t) = \int_{1}^{t} \frac{3}{2x^2} \, dx = -\frac{3}{2x} \bigg|_{x=1}^{x=t} = -\frac{3}{2t} + \frac{3}{2}.
\]
(iii) [3 points] What is the median outcome of the experiment?

To find the median $T$ we need to solve

$$P(T) = \frac{1}{2} \implies -\frac{3}{2T} + \frac{3}{2} = \frac{1}{2} \implies \frac{3}{2T} = 1 \implies T = \frac{3}{2}.$$ 

(iv) [4 points] Find the mean outcome of the experiment.

The mean is calculated by the integral

$$\text{mean} = \int_1^3 x f(x) \, dx = \int_1^3 \frac{3}{2x} \, dx = \frac{3}{2} \ln x \bigg|_{x=1}^{x=3} = \frac{3}{2} \ln 3.$$