Math 10C - Winter 2010 - Midterm I

Name: ______________________________

Student ID: ___________________________

Section time: _________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books or telephones. You may use a ”cheat sheet” of notes which should be a page, front only.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 45 points. You have 50 minutes to complete the test.

Question	Score	Maximum
1		8
2		12
3		10
4		15
Total		45

Problem 1. [*8 points.*]

Consider the function

$$f(x, y) = x^2 + y^2 - 1.$$

(i) [*4 points*] Draw the contour diagram for $f(x, y)$ and clearly label the level curves. Show the contours for at least three levels.

The level curve at level c is

$$f(x, y) = c \implies x^2 + y^2 - 1 = c \implies x^2 + y^2 = c + 1.$$

This is a circle with center at the origin and radius $\sqrt{c+1}$.

For instance you may pick the following three levels:

- $c = 0$ which gives a circle of radius 1 with center at the origin. This circle is labeled by the level 0;
- $c = 3$ which gives a circle of radius 2 labeled by the level value 3;
- $c = 8$ which gives a circle of radius 3 labeled by the level value 8;

There are other choices for levels, so other answers may be correct as well.

(ii) [*4 points*] Draw the graph of $z = f(x, y)$.

The graph is a paraboloid whose lowest point is $(0, 0, -1)$.

Problem 2. [*12 points.*]

(i) [*7 points*] Find the equation of the plane that passes through the points $(1,1,0)$, $(2,1,-1)$ and $(2,0,4)$.

We consider the points $(1,1,0)$ and $(2,1,-1)$. Their second coordinates agree, so we can use these points to compute the m-slope:

$$m = \frac{-1-0}{2-1} = -1.$$

Similarly, we can use the points $(2,1,-1)$ and $(2,0,4)$ to compute the n-slope:

$$n = \frac{4-(-1)}{0-1} = -5.$$

Finally, we can write down the equation of the plane when we know the slopes m and n and one of the points $(1,1,0)$:

$$z = -(x-1) - 5(y-1) \implies \boxed{z = -x - 5y + 6}.$$

(ii) [*5 points*] Compute the sum

$$\frac{4}{5^5} + \frac{4}{5^6} + \frac{4}{5^7} + \ldots + \frac{4}{5^{2010}}.$$

Please write your answer in a simple form.

This is a geometric series. The first term equals $a = \frac{4}{5^5}$, the common ratio is $r = \frac{1}{5}$. The number of terms in the series is $n = 2010 - 5 + 1 = 2006$. The sum equals

$$\frac{4}{5^5} \cdot \frac{1 - \left(\frac{1}{5}\right)^{2006}}{1 - \frac{1}{5}} = \frac{4}{5^5} \cdot \frac{5}{4} \cdot \left(1 - \frac{1}{5^{2006}}\right) = \frac{1}{5^4}\left(1 - \frac{1}{5^{2006}}\right).$$

Problem 3. [*10 points.*]

(i) [*4 points*] The degree three Taylor polynomial of a function $f(x)$ at $a = 0$ equals

$$1 + 2x - 3x^2 - \frac{1}{3}x^3.$$

Calculate the third derivative $f'''(0)$.

The Taylor polynomial equals

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

We match the coefficients of x^3, hence

$$\frac{f'''(0)}{6} = -\frac{1}{3} \implies \boxed{f'''(0) = -2}.$$

(ii) [*6 points*] Find the degree 2 Taylor polynomial at $a = 2$ for the function

$$g(x) = \frac{1}{x-1}.$$

We compute $g(2) = 1$. Writing

$$g(x) = (x-1)^{-1}$$

we see that

$$g'(x) = -(x-1)^{-2} \implies g'(2) = -1$$
$$g''(x) = 2(x-1)^{-3} \implies g'(2) = 2.$$

The Taylor polynomial is

$$g(2) + g'(2)(x-2) + \frac{g''(2)}{2}(x-2)^2 = 1 - (x-2) + (x-2)^2.$$

Problem 4. [*15 points.*]

The outcome x of an experiment is always between 1 and 3. The probability density function equals

$$f(x) = \frac{C}{x^2}, \text{ for } 1 \leq x \leq 3.$$

(i) [*4 points*] Find the constant C.

Any pdf integrates to 1. In our case

$$\int_1^3 \frac{C}{x^2}\, dx = 1 \implies -\frac{C}{x}\Big|_{x=1}^{x=3} = 1 \implies -\frac{C}{3} + C = 1 \implies \frac{2C}{3} = 1 \implies C = \frac{3}{2}.$$

(ii) [*4 points*] Determine the cummulative distribution function.

We integrate the pdf to get the cdf. We compute

$$P(t) = \int_1^t \frac{3}{2x^2}\, dx = -\frac{3}{2x}\Big|_{x=1}^{x=t} = -\frac{3}{2t} + \frac{3}{2}.$$

(iii) [3 *points*] What is the median outcome of the experiment?

To find the median T we need to solve

$$P(T) = \frac{1}{2} \implies -\frac{3}{2T} + \frac{3}{2} = \frac{1}{2} \implies \frac{3}{2T} = 1 \implies \boxed{T = \frac{3}{2}}.$$

(iv) [4 *points*] Find the mean outcome of the experiment.

The mean is calculated by the integral

$$\text{mean} = \int_1^3 x f(x)\, dx = \int_1^3 \frac{3}{2x}\, dx = \frac{3}{2} \ln x|_{x=1}^{x=3} = \boxed{\frac{3}{2} \ln 3.}$$