Problem 1. [*8 points.*]

Consider the vectors

$$\mathbf{v} = \mathbf{i} + 2\mathbf{j}, \;\; \mathbf{w} = 3\mathbf{i} - \mathbf{j}.$$

(i) *[3 points] Draw the two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *in standard position, and draw their sum* $\mathbf{v} + \mathbf{w}$.
(ii) *[2 points] Compute the vector* $2\mathbf{v} + 3\mathbf{w}$.
(iii) *[3 points] Find the unit vector* $\mathbf{u}$ *in the direction of* $\mathbf{w}$.

Answer: (i) The vector $\mathbf{v} + \mathbf{w}$ in standard position joins the origin to the vertex of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.
(ii) We have

$$2\mathbf{v} + 3\mathbf{w} = 2\,(\mathbf{i} + 2\mathbf{j}) + 3\,(3\mathbf{i} - \mathbf{j}) = 11\,\mathbf{i} + \mathbf{j}.$$

(iii) We have

$$\mathbf{u} = \frac{\mathbf{w}}{||\mathbf{w}||} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{3^2 + (-1)^2}} = \frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}.$$

□

Problem 2. *[10 points.]*

Consider the function

$$f(x, y) = (x - 1)^2 + (y - 1)^2.$$

(i) *[5 points] Draw the contour diagram for* $f(x, y)$ *and clearly label the level curves. Show the contours for at least three levels.*
(ii) *[5 points] Draw the graph of* $z = f(x, y)$.

Answer: (i) The contour diagram consists in circles centered at $(1, 1)$ of radius $\sqrt{c}$ where c is the level. For instance, for levels $c = 1$, $c = 4$, $c = 9$, we draw three circles of center $(1, 1)$ and radii $1, 2$ and 3 respectively.
(ii) The graph $z = f(x, y)$ is a paraboloid whose lowest point is $(1, 1, 0)$.

□

Problem 3. *[10 points.]*

(i) *[5 points] Suppose that* $z = f(x, y)$ *is a linear function of* x *and* y*, with slope* 3 *in the* x *direction and slope* -2 *in the* y *direction. A change of* $.1$ *in* x *and* $-.2$ *in* y *produces what change in* z*?*
(ii) *[5 points] The graph of a linear function* $z = g(x, y)$ *passes through the point* $(1, 2, 5)$*. The graph intersects the* xz *plane along the line* $z = 3x + 6$*. Determine the linear function* g.

Answer: (i) We know that $m = 3$ and $n = -2$. Thus

$$\Delta z = 3\Delta x - 2\Delta y.$$

In our case $\Delta x = .1$ and $\Delta y = -.2$. The change in z is therefore

$$\Delta z = 3(.1) - 2(-.2) = .7.$$

(ii) The graph of the linear function is

$$z = c + mx + ny.$$

The intersection with the xz plane is given by setting $y = 0$. The equation we obtain is $z = c + mx$ which we know should be $z = 3x + 6$. We obtain

$$c = 6, m = 3.$$

Now, since $(1, 2, 5)$ lies on the graph we have

$$5 = c + m + 2n.$$

We determine $n = -2$. The linear function is

$$f(x, y) = 6 + 3x - 2y.$$

□

Problem 4. *[10 points.]*

(i) *[3 points] Write the following sum in a simpler form*

$$\frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \ldots + \frac{3}{2^{2009}}.$$

(ii) *[3 points] Find the value of z for which*

$$1 - z + z^2 - z^3 + \ldots = \frac{3}{4}.$$

(iii) *[4 points] Find the quadratic Taylor polynomial (around* 0*) for the function* $f(x) = \ln(x^2 + 1)$.

Answer: (i) We have a geometric series with first term $a = \frac{3}{2}$ and step $r = 12$. The answer is

$$\frac{3}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^{2009}}{1 - \frac{1}{2}} = 3\left(1 - \frac{1}{2^{2009}}\right).$$

(ii) The left hand side is an infinite geometric series with step $-z$. The sum equals

$$\frac{1}{1-(-z)} = \frac{1}{1+z} = \frac{3}{4} \implies 1 + x = \frac{4}{3} \implies z = \frac{1}{3}.$$

(iii) Letting

$$f(x) = \ln(1 + x^2)$$

we compute

$$f(0) = \ln 1 = 0,$$

$$f'(x) = \frac{2x}{1+x^2} \implies f'(0) = 0$$

$$f''(x) = \frac{(2x)'(1+x^2) - (2x)(1+x^2)'}{(1+x^2)^2} = \frac{2 - 2x^2}{(1+x^2)^2} \implies f''(0) = 2.$$

The second Taylor polynomial equals

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = x^2.$$

□

Problem 5. *[12 points.]*

The outcome x of an experiment has the cumulative density function

$$f(x) = \begin{cases} c\left(1 - \frac{4}{x^2}\right) & \textit{for } x \geq 2, \\ 0 & \textit{for } x < 2. \end{cases}$$

(i) *[3 points] Sketch the graph of $f(x)$. Show that we must have $c = 1$.*
(ii) *[3 points] Determine the probability density function.*
(iii) *[3 points] What is the median outcome of the experiment?*
(iv) *[3 points] Find the mean value for the outcome of the experiment.*

Answer: (i) The graph of f is an increasing function contained between the lines $y = 0$ and $y = c$. Since any cdf must approach the value 1 as $x \to \infty$ we must have $c = 1$.

(ii) The pdf is obtained by taking the derivative

$$p(x) = f'(x) = \begin{cases} \frac{8}{x^3} & \text{for } x \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) The median T is obtained by solving

$$f(T) = \frac{1}{2} \implies 1 - \frac{4}{T^2} = \frac{1}{2} \implies \frac{4}{T^2} = \frac{1}{2} \implies T^2 = 8 \implies T = 2\sqrt{2}.$$

(iv) The mean is computed as

$$\mu = \int_0^\infty x p(x)\, dx = \int_2^\infty x \cdot \frac{8}{x^3}\, dx = \int_2^\infty \frac{8}{x^2}\, dx = -\frac{8}{x}\Big|_{x=2}^{x=\infty} = 0 - \frac{8}{-2} = 4.$$

□