Math 10C - Winter 2010 - Midterm II

Name: ______________________________

Student ID: ___________________________

Section time: __________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books or telephones. You may use a ”cheat sheet” of notes which should be a page, front only.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 45 points. You have 50 minutes to complete the test.

Question	Score	Maximum
1		11
2		12
3		13
4		9
Total		45

Problem 1. [*11 points.*]

Consider the function
$$f(x,y) = e^{-y}x^3.$$

(i) [3] Find the equation of the tangent plane to the graph $z = f(x,y)$ at $(1,0,1)$.

We compute
$$f_x = 3x^2e^{-y} \implies f_x(1,0) = 3$$
$$f_y = -x^3e^{-y} \implies f_y(1,0) = -1.$$

The tangent plane is
$$z - 1 = 3(x-1) - y \implies z = 3x - y - 2.$$

(ii) [3] Using linear approximation, estimate $f(1.01, .01)$.

We use part (i) to estimate
$$z = f(1.01, .01) \approx 3 \cdot 1.01 - .01 - 2 = 1.02.$$

(iii) [5] Calculate the quadratic Taylor polynomial of f near $(1, 0)$.

We calculate

$$f_x = 3x^2e^{-y} \implies f_x(1,0) = 3$$
$$f_y = -x^3e^{-y} \implies f_y(1,0) = -1$$
$$f_{xx} = 6xe^{-y} \implies f_xx(1,0) = 6$$
$$f_{xy} = -3x^2e^{-y} \implies f_{xy}(1,0) = -3$$
$$f_{yy} = x^3e^y \implies f_{yy}(1,0) = 1.$$

The Taylor polynomial is

$$1 + 3(x-1) - y + 3(x-1)^2 - 3(x-1)y + \frac{1}{2}y^2.$$

Problem 2. [*12 points.*]

Consider the function

$$f(x, y) = x^2 \sin(2y - 2x)$$

and the point $P(1, \frac{\pi}{2} + 1)$.

(i) [5] Find the gradient of f at the point P.

We compute the derivatives

$$f_x = 2x\sin(2y - 2x) - 2x^2\cos(2y - 2x) \implies f_x(1, \frac{\pi}{2} + 1) = 2\sin\pi - 2\cos\pi = 2$$

$$f_y = 2x^2\cos(2y - 2x) \implies f_y(1, \frac{\pi}{2} + 1) = 2\cos\pi = -2.$$

Then

$$\nabla f(P) = (2, -2).$$

(ii) [4] Calculate the directional derivative of f at P in the direction

$$\vec{u} = \frac{\vec{3}i + 4\vec{j}}{5}.$$

We have

$$f_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = (2, -2) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = 2 \cdot \frac{3}{5} - 2 \cdot \frac{4}{5} = -\frac{2}{5}.$$

(iii) [3] Find the (unit) direction of steepest increase for the function $f(x, y)$ at P.

We have

$$\vec{u} = \frac{\nabla f}{||\nabla f||} = \frac{(2, -2)}{\sqrt{2^2 + (-2)^2}} = \left(\frac{2}{\sqrt{8}}, -\frac{2}{\sqrt{8}}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Problem 3. [*13 points.*]

Consider the planes

$$x + 5y + 3z = 1 \text{ and } 2x - y + z = 2.$$

(i) [3] For each of the planes, write down a normal vector.

$$\vec{n}_1 = (1, 5, 3), \quad \vec{n}_2 = (2, -1, 1).$$

(ii) [3] Are the two planes perpendicular?

We compute

$$\vec{n}_1 \cdot \vec{n}_2 = (1, 5, 3) \cdot (2, -1, 1) = 1 \cdot 2 + 5 \cdot (-1) + 3 \cdot 2 = 0.$$

Therefore the two normal vectors, hence the two planes, are perpendicular.

(iii) [4] Find a vector parallel to the intersection of the two planes.

The vector parallel to the intersection of the two planes is normal to both $\vec{n}_1$ and $\vec{n}_2$. We can use the cross product to find such a vector

$$\vec{n}_1 \times \vec{n}_2 = (1, 5, 3) \times (2, -1, 1) = (8, 5, -11).$$

(iv) [3] Write down a plane parallel to the plane $x + 5y + 3z = 1$ and passing through $(1, -1, 1)$.

The normal vector to the new plane is still $(1, 5, 3)$. The plane is

$$(x - 1) + 5(y + 1) + 3(z - 1) = 0 \implies x + 5y + 3z = -1.$$

Problem 4. [*9 points.*]

Consider the function

$$z = y^2 \ln x$$

and assume

$$x = e^{2u}, \ \ y = v^2\sqrt{u}.$$

Using the chain rule, calculate the derivatives

$$\frac{\partial z}{\partial u} \text{ and } \frac{\partial z}{\partial v}.$$

Please express your answer in the simplest form.

We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

We compute

$$\frac{\partial z}{\partial x} = \frac{y^2}{x} = \frac{v^4 u}{e^{2u}}$$

$$\frac{\partial z}{\partial y} = 2y \ln x = 2v^2\sqrt{u} \ln e^{2u} = 4v^2 u\sqrt{u}$$

$$\frac{\partial x}{\partial u} = 2e^{2u}$$

$$\frac{\partial y}{\partial u} = \frac{v^2}{2\sqrt{u}}.$$

We get

$$\frac{\partial z}{\partial u} = \frac{v^4 u}{e^{2u}} \cdot 2e^{2u} + 4v^2 u\sqrt{u} \cdot \frac{v^2}{2\sqrt{u}} = 2v^4 u + 2v^4 u = 4v^4 u.$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = 4v^2 u\sqrt{u} \cdot 2v\sqrt{u} = 8v^3 u^2.$$