Math 10C - Winter 2010 - Midterm II

Name: ________________________________

Student ID: __________________________

Section time: _________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books or telephones. You may use a "cheat sheet" of notes which should be a page, front only.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 45 points. You have 50 minutes to complete the test.

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Problem 1. [12 points.]

Consider the function

\[ f(x, y) = y^2 \sin(2x - 2y) \]

and the point \( P(1 + \frac{\pi}{2}, 1) \).

(i) [5] Find the gradient of \( f \) at the point \( P \).

We compute the derivatives

\[ f_x = 2 \cos(2x - 2y) \implies f_x(1 + \frac{\pi}{2}, 1) = 2 \cos \pi = -2 \]

\[ f_y = 2y \sin(2x - 2y) - 2y^2 \cos(2x - 2y) \implies f_y(1 + \frac{\pi}{2}, 1) = 2 \cdot 1 \cdot \sin \pi - 2 \cdot 1^2 \cdot \cos \pi = 2. \]

Then

\[ \nabla f(P) = (-2, 2). \]

(ii) [4] Calculate the directional derivative of \( f \) at \( P \) in the direction

\[ \vec{u} = \frac{4i + 3j}{5}. \]

We have

\[ f_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = (-2, 2) \cdot \left( \frac{4}{5}, \frac{3}{5} \right) = -2 \cdot \frac{4}{5} + 2 \cdot \frac{3}{5} = \frac{-8 + 6}{5} = \frac{-2}{5}. \]
(iii) [3] Find the (unit) direction of steepest increase for the function \( f(x, y) \) at \( P \).

We have

\[
\vec{u} = \frac{\nabla f}{||\nabla f||} = \frac{(-2, 2)}{\sqrt{(-2)^2 + 2^2}} = \left(-\frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).
\]

**Problem 2.** [13 points.]

Consider the planes

\[5x + y + z = 1 \text{ and } x - 6y + z = 2.\]

(i) [3] For each of the planes, write down a normal vector.

The normal vectors are

\[\vec{n}_1 = (5, 1, 1), \quad \vec{n}_2 = (1, -6, 1).\]
(ii) [3] Are the two planes perpendicular?

We compute the dot product
\[ \vec{n}_1 \cdot \vec{n}_2 = (5, 1, 1) \cdot (1, -6, 1) = 5 - 6 + 1 = 0. \]
The two vectors and therefore the two planes are perpendicular.

(iii) [4] Find a vector parallel to the intersection of the two planes.

The vector parallel to the intersection of the two planes is normal to both \( \vec{n}_1 \) and \( \vec{n}_2 \). We can use the cross product to find such a vector
\[ \vec{n}_1 \times \vec{n}_2 = (5, 1, 1) \times (1, -6, 1) = (7, -4, -31). \]

(iv) [3] Find the plane parallel to \( 5x + y + z = 1 \) and passing through \( (1, -2, -1) \).

The normal vector to the new plane is still \( (5, 1, 1) \). The plane is
\[ 5(x - 1) + (y + 2) + (z + 1) = 0 \implies 5x + y + z = 2. \]
Problem 3. [9 points.]

Consider the function

\[ z = x^2 \ln y \]

and assume

\[ x = v^2 \sqrt{u}, \quad y = e^{-2u}. \]

Using the chain rule, calculate the derivatives

\[ \frac{\partial z}{\partial u} \text{ and } \frac{\partial z}{\partial v}. \]

Please express your answer in simplest form.

We have

\[ \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}. \]

We compute

\[ \frac{\partial z}{\partial x} = 2x \ln y = 2v^2 \sqrt{u} \ln e^{-2u} = 2v^2 \sqrt{u} \cdot (-2u) = -4v^2 u \sqrt{u} \]

\[ \frac{\partial z}{\partial y} = \frac{x^2}{y} = \frac{v^4 u}{e^{-2u}} \]

\[ \frac{\partial x}{\partial u} = \frac{v^2}{2 \sqrt{u}} \]

\[ \frac{\partial y}{\partial u} = -2e^{-2u}. \]

We get

\[ \frac{\partial z}{\partial u} = \frac{v^4 u}{e^{-2u}} \cdot (-2e^{-2u}) + (-4v^2 u \sqrt{u}) \cdot \frac{v^2}{2 \sqrt{u}} = -2v^4 u - 2v^4 u = -4v^4 u. \]

Similarly,

\[ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} = -4v^2 u \sqrt{u} \cdot 2v \sqrt{u} = -8v^3 u^2. \]
Problem 4. [11 points.]

Consider the function 

\[ f(x, y) = e^{-x}y^3. \]

(i) [3] Find the equation of the tangent plane to the graph \( z = f(x, y) \) at \((0, 1, 1)\).

We compute

\[ f_x = -e^{-x}y^3 \implies f_x(0, 1) = -1 \]
\[ f_y = 3y^2e^{-x} \implies f_y(1, 0) = 3. \]

The tangent plane is

\[ z - 1 = -x + 3(y - 1) \implies z = -x + 3y - 2. \]

(ii) [3] Using linear approximation, estimate \( f(0.01, .99) \).

We use part (i) to estimate

\[ z = f(.01, .99) \approx - .01 + 3 \cdot .99 - 2 = .96. \]
(iii) [5] Calculate the quadratic Taylor polynomial of $f$ near $(0, 1)$.

**We calculate**

\[
\begin{align*}
    f_x &= -e^{-x}y^3 \implies f_x(0, 1) = -1 \\
    f_y &= 3e^{-x}y^2 \implies f_y(0, 1) = 3 \\
    f_{xx} &= e^{-x}y^3 \implies f_{xx}(0, 1) = 1 \\
    f_{xy} &= -3y^2e^{-x} \implies f_{xy}(0, 1) = -3 \\
    f_{yy} &= 6e^{-x}y \implies f_{yy}(0, 1) = 6.
\end{align*}
\]

**The Taylor polynomial is**

\[
1 - x + 3(y - 1) + \frac{1}{2}(x - 1)^2 - 3x(y - 1) + 3(y - 1)^2.
\]