Math 10C - Winter 2010 - Midterm II

Name: ______________________________

Student ID: ______________________________

Section time: ______________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books or telephones. You may use a ”cheat sheet” of notes which should be a page, front only.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 45 points. You have 50 minutes to complete the test.

Question	Score	Maximum
1		12
2		13
3		9
4		11
Total		45

Problem 1. [*12 points.*]

Consider the function
$$f(x,y) = y^2 \sin(2x - 2y)$$
and the point $P(1 + \frac{\pi}{2}, 1)$.

(i) [5] Find the gradient of f at the point P.

We compute the derivatives
$$f_x = 2\cos(2x - 2y) \implies f_x(1 + \frac{\pi}{2}, 1) = 2\cos\pi = -2$$
$$f_y = 2y\sin(2x - 2y) - 2y^2\cos(2x - 2y) \implies f_y(1 + \frac{\pi}{2}, 1) = 2 \cdot 1 \cdot \sin\pi - 2 \cdot 1^2 \cdot \cos\pi = 2.$$

Then
$$\nabla f(P) = (-2, 2).$$

(ii) [4] Calculate the directional derivative of f at P in the direction
$$\vec{u} = \frac{\vec{4}i + 3\vec{j}}{5}.$$

We have
$$f_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = (-2, 2) \cdot \left(\frac{4}{5}, \frac{3}{5}\right) = -2 \cdot \frac{4}{5} + 2 \cdot \frac{3}{5} = -\frac{2}{5}.$$

(iii) [3] Find the (unit) direction of steepest increase for the function $f(x, y)$ at P.

We have

$$\vec{u} = \frac{\nabla f}{||\nabla f||} = \frac{(-2, 2)}{\sqrt{(-2)^2 + 2^2}} = \left(-\frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Problem 2. [*13 points.*]

Consider the planes

$$5x + y + z = 1 \text{ and } x - 6y + z = 2.$$

(i) [3] For each of the planes, write down a normal vector.

The normal vectors are

$$\vec{n}_1 = (5, 1, 1), \quad \vec{n}_2 = (1, -6, 1).$$

(ii) [3] Are the two planes perpendicular?

We compute the dot product

$$\vec{n}_1 \cdot \vec{n}_2 = (5, 1, 1) \cdot (1, -6, 1) = 5 - 6 + 1 = 0.$$

The two vectors and therefore the two planes are perpendicular.

(iii) [4] Find a vector parallel to the intersection of the two planes.

The vector parallel to the intersection of the two planes is normal to both $\vec{n}_1$ and $\vec{n}_2$. We can use the cross product to find such a vector

$$\vec{n}_1 \times \vec{n}_2 = (5, 1, 1) \times (1, -6, 1) = (7, -4, -31).$$

(iv) [3] Find the plane parallel to $5x + y + z = 1$ and passing through $(1, -2, -1)$.

The normal vector to the new plane is still $(5, 1, 1)$. The plane is

$$5(x - 1) + (y + 2) + (z + 1) = 0 \implies 5x + y + z = 2.$$

Problem 3. [*9 points.*]

Consider the function

$$z = x^2 \ln y$$

and assume

$$x = v^2\sqrt{u}, \quad y = e^{-2u}.$$

Using the chain rule, calculate the derivatives

$$\frac{\partial z}{\partial u} \text{ and } \frac{\partial z}{\partial v}.$$

Please express your answer in simplest form.

We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

We compute

$$\frac{\partial z}{\partial x} = 2x \ln y = 2v^2\sqrt{u} \ln e^{-2u} = 2v^2\sqrt{u} \cdot (-2u) = -4v^2u\sqrt{u}$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{y} = \frac{v^4 u}{e^{-2u}}$$

$$\frac{\partial x}{\partial u} = \frac{v^2}{2\sqrt{u}}$$

$$\frac{\partial y}{\partial u} = -2e^{-2u}.$$

We get

$$\frac{\partial z}{\partial u} = \frac{v^4 u}{e^{-2u}} \cdot (-2e^{-2u}) + (-4v^2u\sqrt{u}) \cdot \frac{v^2}{2\sqrt{u}} = -2v^4u - 2v^4u = -4v^4u.$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} = -4v^2u\sqrt{u} \cdot 2v\sqrt{u} = -8v^3u^2.$$

Problem 4. [*11 points.*]

Consider the function

$$f(x, y) = e^{-x}y^3.$$

(i) [3] Find the equation of the tangent plane to the graph $z = f(x, y)$ at $(0, 1, 1)$.

We compute

$$f_x = -e^{-x}y^3 \implies f_x(0, 1) = -1$$
$$f_y = 3y^2e^{-x} \implies f_y(1, 0) = 3.$$

The tangent plane is

$$z - 1 = -x + 3(y - 1) \implies z = -x + 3y - 2.$$

(ii) [3] Using linear approximation, estimate $f(.01, .99)$.

We use part (i) to estimate

$$z = f(.01, .99) \approx - \cdot .01 + 3 \cdot .99 - 2 = .96.$$

(iii) [5] Calculate the quadratic Taylor polynomial of f near $(0, 1)$.

We calculate

$$\begin{aligned} f_x = -e^{-x}y^3 &\implies f_x(0,1) = -1 \\ f_y = 3e^{-x}y^2 &\implies f_y(0,1) = 3 \\ f_{xx} = e^{-x}y^3 &\implies f_{xx}(0,1) = 1 \\ f_{xy} = -3y^2e^{-x} &\implies f_{xy}(0,1) = -3 \\ f_{yy} = 6e^{-x}y &\implies f_{yy}(0,1) = 6. \end{aligned}$$

The Taylor polynomial is

$$1 - x + 3(y-1) + \frac{1}{2}(x-1)^2 - 3x(y-1) + 3(y-1)^2.$$