

A Tour of Theta Dualities on Moduli Spaces of Sheaves

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ABSTRACT. The purpose of this paper is twofold. First, we survey known results about theta dualities on moduli spaces of sheaves on curves and surfaces. Secondly, we establish new such dualities in the surface case. Among others, the case of elliptic K3 surfaces is studied in detail; we propose further conjectures which are shown to imply strange duality.

1. Introduction

The idea that sections of the determinant line bundles on moduli spaces of sheaves are subject to natural dualities was first formulated, and almost exclusively pursued, in the case of vector bundles on curves. The problem can however be generally stated as follows.

Let X be a smooth complex projective curve or surface with polarization H , let v be a class in the Grothendieck group $K(X)$ of coherent sheaves on X . Somewhat imprecisely, we denote by \mathfrak{M}_v the moduli space of Gieseker H -semistable sheaves on X of class v . Consider the bilinear form in K -theory

$$(1.1) \quad (v, w) \mapsto \chi(v \otimes w), \text{ for } v, w \in K(X),$$

and let $v^\perp \subset K(X)$ consist of the K -classes orthogonal to v relative to this form. There is a group homomorphism

$$\Theta : v^\perp \longrightarrow \text{Pic } \mathfrak{M}_v, \quad w \mapsto \Theta_w,$$

which was extensively considered in [DN] when X is a curve, and also in [LeP], [Li] in the case when X is a surface, as part of the authors' study of the Picard group of the moduli spaces of sheaves. When \mathfrak{M}_v admits a universal sheaf $\mathcal{E} \rightarrow \mathfrak{M}_v \times X$, Θ_w is given by

$$\Theta_w = \det \mathbf{R}p_!(\mathcal{E} \otimes q^*F)^{-1}.$$

Here F is any sheaf with K -type w , and p and q are the two projections from $\mathfrak{M}_v \times X$. The line bundle Θ_w is well defined also when the moduli space \mathfrak{M}_v is not fine, by descent from the Quot scheme.

Consider now two classes v and w in $K(X)$, orthogonal with respect to the bilinear form (1.1) *i.e.*, satisfying

$$\chi(v \otimes w) = 0.$$

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Assume that for points $[E] \in \mathfrak{M}_v$ and $[F] \in \mathfrak{M}_w$, the vanishing, vacuous when X is a curve,

$$(1.2) \quad H^2(E \otimes F) = 0 \text{ and } \text{Tor}^1(E, F) = \text{Tor}^2(E, F) = 0$$

occurs (away from codimension 2). Suppose further that the locus

$$(1.3) \quad \Theta = \{(E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w \text{ with } h^0(E \otimes F) \neq 0\}$$

gives rise to a divisor of the line bundle \mathcal{L} which splits as

$$(1.4) \quad \mathcal{L} = \Theta_w \boxtimes \Theta_v \text{ on } \mathfrak{M}_v \times \mathfrak{M}_w.$$

We then obtain a morphism, well-defined up to scalars,

$$(1.5) \quad \mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \longrightarrow H^0(\mathfrak{M}_w, \Theta_v).$$

The main questions of geometric duality in this context are simple to state and fundamentally naive.

Question 1. *What are the constraints on X , v , and w subject to which one has, possibly with suitable variations in the meaning of \mathfrak{M}_v ,*

$$(1.6) \quad h^0(\mathfrak{M}_v, \Theta_w) = h^0(\mathfrak{M}_w, \Theta_v)?$$

Question 2. *In the cases when the above equality holds, is the map D of equation (1.5) an isomorphism?*

This paper has two goals. One is to survey succinctly the existent results addressing Questions 1 and 2. The other one is to study the two questions in new geometric contexts, providing positive answers in some cases and support for further conjectures in other cases. Throughout we refer to the isomorphisms induced by jumping divisors of type (1.3) as *theta dualities*, or *strange dualities*. The latter term is in keeping with the terminology customary in the context of moduli spaces of bundles on curves.

We begin in fact by reviewing briefly the arguments that establish the duality in the case of vector bundles on curves. We point out that in this case the isomorphism can be regarded as a generalization of the classical Wirtinger duality on spaces of level 2 theta functions on abelian varieties. We moreover give a few low-rank/low-level examples. We end the section devoted to curves by touching on Beauville's proposal [Be2] concerning a strange duality on moduli spaces of symplectic bundles; we illustrate the symplectic duality by an example.

The rest of the paper deals with strange dualities for moduli spaces of sheaves on surfaces. In this context, Questions 1 and 2 were first posed by Le Potier. Note that in the curve case there are strong representation-theoretic reasons to expect affirmative answers to these questions. By contrast, no analogous reasons are known to us in the case of surfaces.

The first examples of theta dualities on surfaces are given in Section 3. There, we explain the theta isomorphism for pairs of rank 1 moduli spaces *i.e.*, for Hilbert schemes of points. We also give an example of strange duality for certain pairs of rank 0 moduli spaces.

Section 4 takes up the case of moduli spaces of sheaves on surfaces with trivial canonical bundle. The equality (1.6) of dimensions for dual spaces of sections has been noted [OG2][GNY] in the case of sheaves on $K3$ surfaces. Moreover, it was recently established in a few different flavors for sheaves on abelian surfaces

[MO3]. We review the numerical statements, which lead one to speculate that the duality map is an isomorphism in this context. We give a few known examples on $K3$ surfaces, involving cases when one of the two moduli spaces is itself a $K3$ surface or a Hilbert scheme of points. It is likely that more general instances of strange duality can be obtained, starting with the isomorphism on Hilbert schemes presented in Section 3 and applying Fourier-Mukai transformations. We hope to investigate this elsewhere. Finally, the last part of this section leaves the context of trivial canonical bundle, and surveys the known cases of strange duality on the projective plane, due to Dănilă [D1] [D2].

Section 5 is devoted to moduli spaces of sheaves on elliptically fibered $K3$ surfaces. We look at the case when the first Chern class of the sheaves in the moduli space has intersection number 1 with the class of the elliptic fiber. These moduli spaces have been explicitly shown birational to the Hilbert schemes of points on the same surface [OG1]. We conjecture that strange duality holds for many pairs of such moduli spaces, consisting of sheaves of ranks at least two. We support the conjecture by proving that if it is true for one pair of ranks, then it holds for any other, provided the sum of the ranks stays constant. Since any moduli space of sheaves on a $K3$ surface is deformation equivalent to a moduli space on an elliptic $K3$, the conjecture has implications for generic strange duality statements.

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2. Strange duality on curves

2.1. General setup. When X is a curve, the topological type of a vector bundle is given by its rank and degree, and its class in the Grothendieck group by its rank and determinant. We let $\mathfrak{M}(r, d)$ be the moduli space of semistable vector bundles with fixed numerical data given by the rank r and degree d . Similarly, we denote by $\mathfrak{M}(r, \Lambda)$ the moduli space of semistable bundles with rank r and fixed determinant Λ of degree d . For a vector bundle F which is orthogonal to the bundles in $\mathfrak{M}(r, d)$ *i.e.*,

$$\chi(E \otimes F) = 0, \text{ for all } E \in \mathfrak{M}(r, d),$$

we consider the jumping locus

$$\Theta_{r,F} = \{E \in \mathfrak{M}(r, d) \text{ such that } h^0(E \otimes F) \neq 0\}.$$

The additional numerical subscripts of thetas indicate the ranks of the bundles that make up the corresponding moduli space.

The well-understood structure of the Picard group of $\mathfrak{M}(r, d)$, given in [DN], reveals immediately that the line bundle associated to $\Theta_{r,F}$ depends only on the rank and determinant of F . Moreover, on the moduli space $\mathfrak{M}(r, \Lambda)$, $\Theta_{r,F}$ depends only on the rank and degree of F . In fact, the Picard group of $\mathfrak{M}(r, \Lambda)$ has a unique ample generator θ_r , and therefore

$$\Theta_{r,F} \cong \theta_r^l,$$

for some integer l .

For numerical choices (r, d) and (k, e) orthogonal to each other, the construction outlined in the Introduction gives a duality map

$$(2.1) \quad D : H^0(\mathfrak{M}(r, \Lambda), \Theta_{r,F})^\vee \longrightarrow H^0(\mathfrak{M}(k, e), \Theta_{k,E}), \text{ for } E \in \mathfrak{M}(r, \Lambda), F \in \mathfrak{M}(k, e).$$

The *strange duality* conjecture of Beauville [Be1] and Donagi-Tu [DT] predicted that the morphism D is an isomorphism. This was recently proved for a generic smooth curve in [Bel1], which inspired a subsequent argument of [MO1] for all smooth curves. The statement for all curves also follows from the generic-curve case in conjunction with the recent results of [Bel2]. We briefly review the arguments in the subsections below.

2.2. Degree zero. The duality is most simply formulated when $d = 0$, $e = k(g - 1)$ on the moduli space $\mathfrak{M}(r, \mathcal{O})$ of rank r bundles with trivial determinant. There is a canonical line bundle Θ_k on the moduli space $\mathfrak{M}(k, k(g - 1))$, associated with the divisor

$$\Theta_k = \{F \in \mathfrak{M}(k, k(g - 1)) \text{ such that } h^0(F) = h^1(F) \neq 0\}.$$

The map (2.1) becomes

$$(2.2) \quad D : H^0(\mathfrak{M}(r, \mathcal{O}), \theta_r^k)^\vee \longrightarrow H^0(\mathfrak{M}(k, k(g - 1)), \Theta_k^r),$$

Note that this interchanges the rank of the bundles that make up the moduli space and the level (tensor power) of the determinant line bundle on the moduli space.

To begin our outline of the arguments, let

$$(2.3) \quad v_{r,k} = \chi(\mathfrak{M}(r, \mathcal{O}), \theta_r^k)$$

be the *Verlinde number* of rank r and level k . As the theta bundles have no higher cohomology, the Verlinde number computes in fact the dimension $h^0(\theta_r^k)$ of the space of sections. The most elementary formula for $v_{r,k}$ reads

$$(2.4) \quad v_{r,k} = \frac{r^g}{(r+k)^g} \sum_{\substack{S \sqcup T = \{1, \dots, r+k\} \\ |S|=k}} \prod_{\substack{s \in S \\ t \in T}} \left| 2 \sin \pi \frac{s-t}{r+k} \right|^{g-1}.$$

This expression for $v_{r,k}$ was established through a concerted effort and variety of approaches that spanned almost a decade of work in the moduli theory of bundles on a curve. It is beyond the purpose of this note to give an overview of the Verlinde formula. Nonetheless, let us mention here that (2.4) implies the symmetry

$$h^0(\mathfrak{M}(r, \mathcal{O}), \Theta_r^k) = h^0(\mathfrak{M}(k, k(g - 1)), \Theta_k^r),$$

required by Question 1, setting the stage for the strange duality conjecture.

EXAMPLE 2.1. *Level 1 duality.* Equation (2.4) simplifies dramatically in level $k = 1$, yielding

$$v_{r,1} = \frac{r^g}{(r+1)^{g-1}} \prod_{p=1}^r \left| 2 \sin \frac{p\pi}{r+1} \right|^{g-1} = r^g.$$

This coincides with the dimension of the space of level r classical theta functions on the Jacobian. The corresponding isomorphism

$$(2.5) \quad D : H^0(\mathfrak{M}(r, \mathcal{O}), \theta_r)^\vee \rightarrow H^0(\text{Jac}^{g-1}(X), \Theta_1^r)$$

was originally established in [BNR], at that time in the absence of the Verlinde formula. Nonetheless, once the Verlinde formula is known, the isomorphism (2.5) may be proved by an easy argument which we learned from Mihnea Popa; see also [Bel2]. Let $\text{Jac}[r]$ denote the group of r -torsion points on the Jacobian, and let $\mathbf{H}[r]$ be the Heisenberg group of the line bundle Θ_1^r , exhibited as a central extension

$$0 \rightarrow \mathbb{C}^\star \rightarrow \mathbf{H}[r] \rightarrow \text{Jac}[r] \rightarrow 0.$$

Now $H^0(\text{Jac}^{g-1}(X), \Theta_1^r)$ is the Schrödinger representation of the Heisenberg group $\mathbf{H}[r]$ *i.e.*, the unique irreducible representation on which the center \mathbb{C}^\star acts by homotheties. We argue that the left hand side of (2.5) also carries such a representation of $\mathbf{H}[r]$. Indeed, the tensor product morphism

$$t : \mathfrak{M}(r, \mathcal{O}) \times \text{Jac}^{g-1}(X) \rightarrow \mathfrak{M}(r, r(g-1))$$

is invariant under the action of $\text{Jac}[r]$ on the source, given by

$$(\zeta, (E, L)) \mapsto (\zeta^{-1} \otimes E, \zeta \otimes L), \text{ for } \zeta \in \text{Jac}[r], (E, L) \in \mathfrak{M}(r, \mathcal{O}) \times \text{Jac}^{g-1}(X).$$

The pullback

$$t^* \Theta_r = \theta_r \boxtimes \Theta_1^r$$

carries an action of $\text{Jac}[r]$, while the theta bundle Θ_1^r on the right carries an action of $\mathbf{H}[r]$. This gives an induced action of the Heisenberg group on θ_r , covering the tensoring action of $\text{Jac}[r]$ on $\mathfrak{M}(r, \mathcal{O})$. The non-zero morphism \mathbf{D} is clearly $\mathbf{H}[r]$ -equivariant, hence it is an isomorphism.

It is more difficult to establish the isomorphism (2.2) for arbitrary levels. The argument in [Bel1] starts by noticing that since the dimensions of the two spaces of sections involved are the same, and since the map is induced by the divisor (1.3), \mathbf{D} is an isomorphism if one can generate pairs

$$(2.6) \quad (E_i, F_i) \in \mathfrak{M}(r, \mathcal{O}) \times \mathfrak{M}(k, k(g-1)), \quad 1 \leq i \leq v_{r,k}$$

such that

$$E_i \in \Theta_{F_j} \text{ if and only if } i \neq j.$$

Equivalently, one requires

$$(2.7) \quad h^0(E_i \otimes F_j) = 0 \text{ if and only if } i = j.$$

Finding the right number of such pairs relies on giving a suitable enumerative interpretation to the Verlinde formula (2.4), and this is the next step in [Bel1]. The requisite vector bundles (2.6) satisfying (2.7) are assembled first on a rational nodal curve of genus g . They are then deformed along with the curve, maintaining the same feature on neighboring smooth curves, and therefore establishing the duality for generic curves. To obtain the Verlinde count of bundles on the nodal curve, the author draws benefit from the *fusion rules* [TUY] of the Wess-Zumino-Witten theory, which express the Verlinde numbers in genus g in terms of Verlinde numbers in lower genera, and with point insertions. The latter are Euler characteristics of line bundles on moduli spaces of bundles with parabolic structures at the given points. Belkale realizes the count of vector bundles with property (2.7) on the nodal curve as an enumerative intersection in Grassmannians with recursive traits precisely matching those of the fusion rules.

Recently, the author extended the result from the generic curve case to that of arbitrary curves [Bel2]. He considers the spaces of sections $H^0(\mathfrak{M}(r, \mathcal{O}), \mathcal{L}^k)$

relatively over families of smooth curves. These spaces of sections give rise to vector bundles which come equipped with Hitchin's projectively flat connection [H]. Belkale proves, importantly, that

THEOREM 2.2. [Bel2] *The relative strange duality map is projectively flat with respect to the Hitchin connection. In particular, its rank is locally constant.*

This result moreover raises the question of extending the strange duality isomorphism to the boundary of the moduli space of curves.

The alternate proof of the duality in [MO1] is inspired by [Bel1], in particular by the interpretation of the isomorphism as solving a counting problem for bundles satisfying (2.6), (2.7). This count is carried out in [MO1] using the close intersection-theoretic rapport between the moduli space of bundles $\mathfrak{M}(r, d)$ and the Grothendieck Quot scheme $\text{Quot}_X(\mathcal{O}^N, r, d)$ which parametrizes rank r , degree d coherent sheaf quotients of the trivial sheaf \mathcal{O}^N on X . The latter is irreducible for large degree d , and compactifies the space of maps $\text{Mor}_d(X, G(r, N))$ from X to the Grassmannian of rank $N - r$ planes in \mathbb{C}^N . The N -asymptotics of tautological intersections on the Quot scheme encode the tautological intersection numbers on $\mathfrak{M}(r, d)$ [MO2]. This fact prompts one to take the viewpoint that the space of maps from the curve to the classifying space of GL_r is the right place to carry out the intersection theory of the moduli space of GL_r -bundles on the curve.

As a Riemann-Roch intersection on $\mathfrak{M}(r, \mathcal{O})$, the Verlinde number $v_{r,k}$ is also expressible on the Quot scheme, most simply as a tautological intersection on the scheme $\text{Quot}_X(\mathcal{O}^{k+r}, r, d)$ parametrizing rank r quotients of \mathcal{O}^{k+r} of suitably large degree d . Precisely, let \mathcal{E} be the rank k universal subsheaf of \mathcal{O}^{k+r} on $\text{Quot}_X(\mathcal{O}^{k+r}, r, d) \times X$, and set

$$a_k = c_k(\mathcal{E}^\vee|_{\text{Quot}_X(\mathcal{O}^{k+r}, r, d) \times \{\text{point}\}}).$$

Then

$$(2.8) \quad v_{r,k} = h^0(\mathfrak{M}(r, \mathcal{O}), \theta_r^k) = \frac{r^g}{(k+r)^g} \int_{\text{Quot}_X(\mathcal{O}^{k+r}, r, d)} a_k^{\text{top}}.$$

This formula was essentially written down by Witten [W] using physical arguments; it reflects the relationship between the GL_r WZW model and the sigma model of the Grassmannian, which he explores in [W]. Therefore, the modified Verlinde number

$$\tilde{v}_{r,k} = \frac{(k+r)^g}{r^g} v_{r,k} = \int_{\text{Quot}_X(\mathcal{O}^{k+r}, r, d)} a_k^{\text{top}}$$

becomes a count of Quot scheme points which obey incidence constraints imposed by the self intersections of the tautological class a_k . Geometrically interpreted, these constraints single out the finitely many subsheaves

$$E_i \hookrightarrow \mathcal{O}^{k+r}$$

which factor through a subsheaf S of \mathcal{O}^{k+r} of the same rank $k+r$ but of lower degree. Therefore, one obtains diagrams

$$(2.9) \quad \begin{array}{ccccccc} 0 & \longrightarrow & E_i & \longrightarrow & S & \longrightarrow & F_i \longrightarrow 0, \text{ for } 1 \leq i \leq \tilde{v}_{r,k}, \\ & & & \searrow & \downarrow & & \\ & & & & \mathcal{O}^{k+r} & & \end{array}$$

where the top sequence is exact and the triangle commutes. The sheaf S is obtained by a succession of elementary modifications of \mathcal{O}^{k+r} dictated by choices of representatives for the class a_k . For generic choices, the $\tilde{v}_{r,k}$ exact sequences (2.9) are the points of a *smooth* zero-dimensional Quot scheme on X associated with S , and therefore automatically satisfy

$$h^0(E_i^\vee \otimes F_j) = 0 \text{ iff } i = j.$$

Now the modified Verlinde number $\tilde{v}_{r,k}$ which gives the number of such pairs is the Euler characteristic of a *twisted* theta bundle on $\mathfrak{M}(r, 0)$. Precisely, for any reference line bundles L and M on X of degree $g-1$,

$$\tilde{v}_{r,k} = h^0(\mathfrak{M}(r, 0), \Theta_{r,M}^k \otimes \det^* \Theta_{1,L}),$$

where

$$\det : \mathfrak{M}(r, 0) \rightarrow \text{Jac}^0(X)$$

is the morphism which takes bundles to their determinants.

Along the lines of the remarks preceding equation (2.7), the pairs of bundles (E_i, F_i) coming from (2.9) easily give the following result:

THEOREM 2.3. **[MO1]** (*Generalized Wirtinger duality.*) *For any line bundles L and M of degree $g-1$, there is an isomorphism*

$$\tilde{\text{D}} : H^0(\mathfrak{M}(r, 0), \Theta_{r,M}^k \otimes \det^* \Theta_{1,L})^\vee \rightarrow H^0(\mathfrak{M}(k, 0), \Theta_{k,M}^r \otimes ((-1) \circ \det)^* \Theta_{1,L}).$$

Here, (-1) denotes the multiplication by -1 on the Jacobian $\text{Jac}^0(X)$.

EXAMPLE 2.4. *Wirtinger duality.* When $k=r=1$, the twisted duality map $\tilde{\text{D}}$ is the classical Wirtinger duality on the abelian variety $\text{Jac}(X)$ [M]. For simplicity let us take $L=M$, and let us assume that L is a theta characteristic. Then the line bundle

$$\Theta = \Theta_{1,L} = \Theta_{1,M}$$

is symmetric *i.e.*, $(-1)^* \Theta \cong \Theta$. Let δ be the morphism

$$\text{Jac}(X) \times \text{Jac}(X) \ni (A, B) \rightarrow (A^{-1} \otimes B, A \otimes B) \in \text{Jac}(X) \times \text{Jac}(X).$$

The see-saw theorem shows that

$$\delta^*(\Theta \boxtimes \Theta) = 2\Theta \boxtimes 2\Theta.$$

In turn, this induces the Wirtinger self-duality

$$\text{W} = \tilde{\text{D}} : |2\Theta|^\vee \rightarrow |2\Theta|.$$

One may establish the fact that W is an isomorphism by considering as before the action of the Heisenberg group $\text{H}[2]$. We refer the reader to Mumford's article [M] for such an argument. Note also that the study of the duality morphism W leads to various addition formulas for theta functions; these have consequences for the geometric understanding of the 2Θ -morphism mapping the abelian variety into projective space.

EXAMPLE 2.5. *Rank 2 self-duality.* The next case of self-duality occurs in rank 2 and level 2. The morphism $\widetilde{\mathbf{D}}$ leads to the study of the line bundles

$$\Theta_{L,M} = \Theta_{2,M}^2 \otimes \det^* \Theta_{1,L},$$

where L and M are line bundles of degree $g - 1$ on X . We assume as before that $\Theta_{1,L}$ is symmetric. We claim that the line bundles $\Theta_{L,M}$ are globally generated, giving rise to morphisms

$$f_{L,M} : \mathfrak{M}(2,0) \rightarrow |\Theta_{L,M}|^\vee \cong |\Theta_{L,M}|.$$

It would be interesting to understand the geometry of this self-duality in more detail.

To see that $\Theta_{L,M}$ is globally generated, observe that if A_1, A_2 are degree 0 line bundles on X , then, using the formulas in [DN], we have

$$\Theta_{L,M} = \Theta_{2,M \otimes A_1} \otimes \Theta_{2,M \otimes A_2} \otimes \det^* \Theta_{1,L \otimes A_1^\vee \otimes A_2^\vee}.$$

This rewriting of the line bundle $\Theta_{L,M}$ gives a section vanishing on the locus

$$\Delta_{A_1,A_2} = \{E \in \mathfrak{M}(2,0) \text{ such that } h^0(E \otimes M \otimes A_1) \neq 0 \text{ or } h^0(E \otimes M \otimes A_2) \neq 0 \text{ or } h^0(\det E \otimes L \otimes A_1^\vee \otimes A_2^\vee) \neq 0\}.$$

Fixing $E \in \mathfrak{M}(2,0)$, it suffices to show that one of the sections Δ_{A_1,A_2} does not vanish at E . This amounts to finding suitable line bundles A_1, A_2 such that $E \notin \Delta_{A_1,A_2}$, or equivalently

$$A_1, A_2 \notin \Theta_{1,E \otimes M} \text{ and } A_1^\vee \otimes A_2^\vee \notin \Theta_{1,\det E \otimes L}.$$

The existence of A_1, A_2 is guaranteed by Raynaud's observation [R] that for any vector bundle V of rank 2 and degree $2g - 2$, the locus

$$\Theta_{1,V} = \{A \in \text{Jac}(X), h^0(V \otimes A) = h^1(V \otimes A) \neq 0\}$$

has codimension 1 in $\text{Jac}(X)$.

Turning to the original formulation (2.2) of strange duality on curves, Theorem 2.3 and a restriction argument on the Jacobian imply that

THEOREM 2.6. [MO1] *The strange duality map \mathbf{D} of (2.2) is an isomorphism.*

As a consequence, we obtain the following

COROLLARY 2.7. *The theta divisors Θ_E generate the linear system $|\theta_r^k|$ on the moduli space $\mathfrak{M}(r, \mathcal{O})$, as E varies in the moduli space $\mathfrak{M}(k, k(g-1))$.*

EXAMPLE 2.8. *Verlinde bundles and Fourier-Mukai.* As observed by Popa [P], the strange duality isomorphism (2.2) can be elegantly interpreted by means of the Fourier-Mukai transform. Fixing a reference theta characteristic L , Popa defined the Verlinde bundles

$$E_{r,k} = \det_\star (\Theta_{r,L}^k)$$

obtained as the pushforwards of the k -pluritheta bundles via the morphism

$$\det : \mathfrak{M}(r,0) \rightarrow \text{Jac}(X).$$

He further showed that the Verlinde bundles satisfy IT_0 with respect to the normalized Poincaré bundle on the Jacobian, and studied the Fourier-Mukai transform

$\widehat{E}_{r,k}$. The construction of the duality morphism D globalizes as we let the determinant of the bundles in the moduli space vary in the Jacobian. The ensuing morphism

$$D : E_{r,k}^\vee \rightarrow \widehat{E}_{k,r}$$

collects all strange duality morphisms for various determinants, and it is therefore an isomorphism. We moreover note here that in level $k = 1$, Popa showed that both bundles are stable, therefore giving another proof of the fact that D is an isomorphism.

2.3. Arbitrary degrees. The proof outlined above effortlessly generalizes to arbitrary degrees. Specifically, let r and d be coprime integers, and h, k be any two non-negative integers, and fix S an auxiliary line bundle of degree d and rank r .

THEOREM 2.9. [MO1] *There is a level-rank duality isomorphism between*

$$H^0(\mathfrak{M}(hr, hd), \Theta_{hr, M \otimes S^\vee}^k \otimes \det^* \Theta_{1, L \otimes (\det S)^{-h}}) \text{ and} \\ H^0(\mathfrak{M}(kr, -kd), \Theta_{kr, M \otimes S}^h \otimes ((-1) \circ \det)^* \Theta_{1, L \otimes (\det S)^{-k}}).$$

Finally, just as in (2.2), the tensor product map induces the strange duality morphism for arbitrary degree:

$$D : H^0(\mathfrak{M}(hr, (\det S)^h), \theta_{hr}^k)^\vee \rightarrow H^0(\mathfrak{M}(kr, k(r(g-1) - d)), \Theta_{kr, S}^h).$$

THEOREM 2.10. [MO1] *The strange duality morphism D is an isomorphism.*

2.4. Symplectic strange duality. It is natural to inquire if the same duality occurs for moduli spaces of principal bundles with arbitrary structure groups. The symplectic group should be considered next, due to the fact that the moduli space of symplectic bundles is locally factorial. A recent conjecture of Beauville [Be2], which we now explain, focuses on this case.

Consider the moduli space \mathcal{M}_{Sp_r} of pairs (E, ϕ) , where E is a semistable bundle of rank $2r$, such that

$$\det E = \mathcal{O}_X \text{ and } \phi : \Lambda^2 E \rightarrow \mathcal{O}_X \text{ is a non-degenerate alternate form.}$$

Similarly, let $\widehat{\mathcal{M}}_{Sp_k}$ denote the cousin moduli space of pairs (F, ψ) as above, with the modified requirements

$$\det F = K_X^k, \text{ and } \psi : \Lambda^2 F \rightarrow K_X \text{ is a nondegenerate alternate form.}$$

We let \mathcal{L}_r and $\widehat{\mathcal{L}}_k$ be the determinant bundles on \mathcal{M}_{Sp_r} and $\widehat{\mathcal{M}}_{Sp_k}$ respectively. Now observe the tensor product map

$$t : \mathcal{M}_{Sp_r} \times \widehat{\mathcal{M}}_{Sp_k} \rightarrow \widehat{\mathcal{M}}^+$$

given by

$$(E, \phi) \times (F, \psi) \rightarrow (E \otimes F, \phi \otimes \psi).$$

Its image is contained in the moduli space $\widehat{\mathcal{M}}^+$ of even orthogonal pairs (G, q) , consisting of semistable bundles G of rank $4rk$, determinant K_X^{2rk} , with $h^0(G)$ even, and endowed with a quadratic form

$$q : \text{Sym}^2 G \rightarrow K_X.$$

Beauville showed that the pullback divisor $t^* \Delta$, with

$$\Delta = \left\{ (G, q) \in \widehat{\mathcal{M}}^+ \text{ such that } h^0(G) \neq 0 \right\},$$

determines a canonical section of $\mathcal{L}_r^k \boxtimes \widehat{\mathcal{L}}_k^r$. Hence, it gives rise to a morphism

$$D : H^0(\mathcal{M}_{Sp_r}, \mathcal{L}_r^k)^\vee \rightarrow H^0(\widehat{\mathcal{M}}_{Sp_k}, \widehat{\mathcal{L}}_k^r).$$

Beauville checked that the dimensions of the spaces of sections match,

$$h^0(\mathcal{M}_{Sp_r}, \mathcal{L}_r^k) = h^0(\widehat{\mathcal{M}}_{Sp_k}, \widehat{\mathcal{L}}_k^r),$$

and further conjectured that

CONJECTURE 2.11. [Be2] *The morphism D is an isomorphism.*

EXAMPLE 2.12. Level 1 symplectic duality. The particular case $k = 1$ of this conjecture is a consequence of the usual strange duality theorem. In this situation, there is an isomorphism

$$(\widehat{\mathcal{M}}_{Sp_1}, \widehat{\mathcal{L}}_1) \cong (\mathfrak{M}(2, K_X), \theta_2).$$

It suffices to explain that the linear system $|\theta_2^r|$ is spanned by the sections

$$\Delta_E = \{(F, \psi) \text{ such that } h^0(E \otimes F) \neq 0\},$$

as E varies in \mathcal{M}_{Sp_r} . In fact, we only need those E 's of the form

$$E = E' \oplus E'^\vee,$$

for $E' \in \mathfrak{M}(r, 0)$. In this case, Beauville observed that

$$\Delta_E = \Theta_{E'} = \{F \in \mathfrak{M}(2, K_X), \text{ such that } h^0(F \otimes E') \neq 0\}.$$

These generate the linear system $|\theta_2^r|$, by Corollary 2.7.

3. Duality on Hilbert schemes of points on a surface

In this section, we start investigating strange duality phenomena for surfaces. We begin with two examples involving the Hilbert scheme of points. In the next sections we will use these basic cases to obtain new examples of theta dualities for surfaces with trivial canonical bundles.

3.1. Notation. Denote by $X^{[k]}$ the Hilbert scheme of k points on the projective surface X , and let

$$0 \rightarrow \mathcal{I}_{\mathcal{Z}} \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\mathcal{Z}} \rightarrow 0$$

be the universal family on $X^{[k]} \times X$. Let p be the projection from the product $X^{[k]} \times X$ to the Hilbert scheme, and q be the projection to the surface X .

For any sheaf F on X , let $F^{[k]}$ be the line bundle

$$F^{[k]} = \det p_!(\mathcal{O}_{\mathcal{Z}} \otimes q^* F) = (\det(p_!(\mathcal{I}_{\mathcal{Z}} \otimes q^* F)))^{-1}.$$

(Note that in the literature on Hilbert schemes of points, $F^{[k]}$ often refers to the pushforward itself, not its determinant. It will be convenient for us not to comply with this practice, and single out, as above, the *determinant* line bundle by this notation.) Further, for any line bundle N on X , the S_k -equivariant line bundle $N^{\boxtimes k}$ on X^k descends to a line bundle on the symmetric product $X^{(k)}$. Let $N_{(k)}$ be the pullback of this descent bundle on the Hilbert scheme $X^{[k]}$, under the Hilbert-Chow morphism

$$f : X^{[k]} \rightarrow X^{(k)}.$$

We recall from [EGL] that

$$(3.1) \quad F^{[k]} = (\det F)_{(k)} \otimes M^{\text{rk} F},$$

where we set

$$M = \mathcal{O}^{[k]}.$$

Equation (3.1) implies in particular that

$$(3.2) \quad F^{[k]} \cong (F \otimes I_Z)^{[k]}$$

for any zero-cycle Z , since $F^{[k]}$ only depends on the determinant and rank of F .

3.2. The strange duality isomorphism. To set up the strange duality map, let L be a line bundle on X without higher cohomology. Let

$$n = \chi(L) = h^0(L),$$

and pick $1 \leq k \leq n$. Denote by $\theta_{L,k}$ the divisor on $X^{[k]} \times X^{[n-k]}$ which, away from the codimension 2 locus of pairs (Z, W) with overlapping support, is given by

$$\theta_{L,k} = \left\{ (I_Z, I_W) \in X^{[k]} \times X^{[n-k]} \text{ such that } h^0(I_Z \otimes I_W \otimes L) \neq 0 \right\}.$$

Using the seesaw theorem and equation (3.2), we find that

$$\mathcal{O}(\theta_{L,k}) \cong L^{[k]} \otimes L^{[n-k]} \text{ on } X^{[k]} \times X^{[n-k]}.$$

In particular,

$$\theta_{L,k} \in H^0(X^{[k]}, L^{[k]}) \otimes H^0(X^{[n-k]}, L^{[n-k]}).$$

In this subsection, we record the following

PROPOSITION 3.1. *Assume that L is a line bundle with $\chi(L) = n \geq k$, and no higher cohomology. The map*

$$D_L : H^0(X^{[k]}, L^{[k]})^\vee \longrightarrow H^0(X^{[n-k]}, L^{[n-k]})$$

induced by the divisor $\theta_{L,k}$ is an isomorphism.

Proof. The space of sections $H^0(X^{[k]}, L^{[k]})$ can be realized explicitly in terms of sections of L on X , cf. [EGL]. The Proposition follows from this identification. Specifically $H^0(X^{[k]}, L^{[k]})$ can be viewed as the invariant part of $H^0(X, L)^{\otimes k}$ under the antisymmetric action ϵ of the permutation group S_k ,

$$(3.3) \quad H^0(X^{[k]}, L^{[k]}) \cong H^0(X^k, L^{\boxtimes k})^{S_k, \epsilon} = \Lambda^k H^0(X, L).$$

To explain this isomorphism, consider the fiber diagram

$$\begin{array}{ccc} \widehat{X}_0^k & \xrightarrow{\hat{f}} & X_0^k \\ \downarrow \hat{g} & & \downarrow g \\ X_0^{[k]} & \xrightarrow{f} & X_0^{(k)}, \end{array}$$

where the bottom and right arrows f and g are the Hilbert-Chow and S_k -quotient morphisms respectively. The 0 subscripts indicate that we look everywhere at the open subschemes of zero cycles with at least $k - 1$ distinct points, which is enough to identify spaces of sections since the complements lie in codimension at least 2. The isomorphism (3.3) is obtained by pulling back $L^{[k]}$ via \hat{g} and pushing forward by \hat{f} .

In particular,

$$H^0(X^{[n]}, L^{[n]}) \cong H^0(X^n, L^{\boxtimes n})^{S_n, \epsilon} \cong \Lambda^n H^0(X, L)$$

is one-dimensional, and is spanned by the divisor

$$\theta_L = \left\{ I_V \in X^{[n]} \text{ such that } h^0(I_V \otimes L) \neq 0 \right\}.$$

Two remarks are now in order. First, under the rational map

$$\tau : X^{[k]} \times X^{[n-k]} \dashrightarrow X^{[n]}, \quad (I_Z, I_W) \mapsto I_Z \otimes I_W,$$

the rational pullback $\tau^*\theta_L$ corresponds unambiguously to $\theta_{L,k} \subset X^{[k]} \times X^{[n-k]}$, and thus gives an injective map

$$\tau^* : H^0(X^{[n]}, L^{[n]}) \rightarrow H^0(X^{[k]}, L^{[k]}) \otimes H^0(X^{[n-k]}, L^{[n-k]}).$$

Secondly, if s_1, \dots, s_n is a basis for $H^0(L)$, then the unique divisor θ_L of $L^{[n]}$ corresponds up to scalars to $s_1 \wedge \dots \wedge s_n$. Furthermore, I_V is a point in θ_L if and only if $g^{-1}(f(I_V))$ is in the vanishing locus of $s_1 \wedge \dots \wedge s_n$; the latter is regarded here as an element of $H^0(X^n, L^{\boxtimes n})$ *i.e.*, is viewed as the antisymmetrization of $s_1 \otimes \dots \otimes s_n$ in $H^0(X^n, L^{\boxtimes n})$. So the vanishing locus of $s_1 \wedge \dots \wedge s_n \in H^0(X^n, L^{\boxtimes n})$ agrees up to the quotient by the symmetric group with the vanishing locus θ_L .

We denote the above inclusion map by

$$\iota : H^0(X^{[n]}, L^{[n]}) \rightarrow H^0(X^n, L^{\boxtimes n}), \quad \theta_L \mapsto s_1 \wedge \dots \wedge s_n,$$

and further let

$$\begin{aligned} \lambda : H^0(X^{[k]}, L^{[k]}) \otimes H^0(X^{[n-k]}, L^{[n-k]}) &= \Lambda^k H^0(X, L) \otimes \Lambda^{n-k} H^0(X, L) \hookrightarrow \\ &\hookrightarrow H^0(X, L)^{\otimes k} \otimes H^0(X, L)^{\otimes (n-k)} = H^0(X^n, L^{\boxtimes n}) \end{aligned}$$

be the tautological inclusion map on each of the two spaces in the tensor product, identifying a k -form with the antisymmetrization of the corresponding tensor element. Note now that the diagram

$$\begin{array}{ccc} H^0(X^{[n]}, L^{[n]}) & \xrightarrow{\iota} & H^0(X^n, L^{\boxtimes n}) \\ \downarrow \tau^* & \nearrow \lambda & \\ H^0(X^{[k]}, L^{[k]}) \otimes H^0(X^{[n-k]}, L^{[n-k]}) & & \end{array}$$

commutes, since the vanishing loci of $\iota(\theta_L)$ and $\lambda(\theta_{L,k})$ on X^n coincide with $g^{-1}(f(\theta_L))$ on the open part of X^n consisting of tuples of distinct points.

The commutativity of the diagram implies now that under the isomorphism (3.3), the section $\theta_{L,k} \in H^0(X^{[k]}, L^{[k]}) \otimes H^0(X^{[n-k]}, L^{[n-k]})$ is identified (up to scalars) with the image of $s_1 \wedge \dots \wedge s_n$ under the natural algebraic inclusion

$$\Lambda^n H^0(L) \hookrightarrow \Lambda^k H^0(L) \otimes \Lambda^{n-k} H^0(L).$$

The latter induces an isomorphism

$$\Lambda^k H^0(L)^\vee \rightarrow \Lambda^{n-k} H^0(L),$$

therefore so does $\theta_{L,k}$.

3.3. A rank 0 example. After studying theta dualities for pairs of moduli spaces of rank 1 sheaves on a surface X , we give an example when the two dual vectors both have rank 0. Specifically we let $v = [\mathcal{O}_Z]$, with Z a punctual scheme of length n , and we let the orthogonal K -vector w be the class of a rank 0 sheaf on X supported on a primitive divisor D , and having rank 1 along D . Note that $\mathfrak{M}_v \cong X^{[n]}$, and consider the morphism

$$s : \mathfrak{M}_w \rightarrow |D|,$$

sending a sheaf to its schematic support, which lies in the linear system $|D|$.

The divisor

$$\Theta = \{(Z, F) \text{ such that } h^0(\mathcal{O}_Z \otimes F) \neq 0\} \hookrightarrow X^{[n]} \times \mathfrak{M}_w$$

is the pullback of the incidence divisor

$$\Delta = \{(Z, \Sigma) \in X^{(n)} \times |D| \text{ such that } Z \cap \Sigma \neq \emptyset\}$$

via the natural morphism

$$X^{[n]} \times \mathfrak{M}_w \rightarrow X^{(n)} \times |D|.$$

This implies that

$$\Theta_v = s^* \mathcal{O}(n), \text{ and } \Theta_w = D_{(n)}.$$

Therefore,

$$H^0(\mathfrak{M}_v, \Theta_w) = H^0(X^{[n]}, D_{(n)}) = H^0(X^{(n)}, D_{(n)}) = H^0(X^n, D^{\boxtimes n})^{S_n} = \text{Sym}^n H^0(D),$$

while

$$H^0(\mathfrak{M}_w, \Theta_v) = H^0(\mathfrak{M}_w, s^* \mathcal{O}(n)) = H^0(|D|, \mathcal{O}(n)) = \text{Sym}^n H^0(D)^\vee.$$

It is then clear that the two spaces of sections are naturally dual, with the duality induced by the divisor Θ .

4. Duality on $K3$ and abelian surfaces

4.1. Numerical evidence. We will collect evidence in favor of a strange duality theorem on surfaces X with trivial canonical bundle $K_X \cong \mathcal{O}_X$ *i.e.*, $K3$ or abelian surfaces.

We will change the notation slightly, writing as customary

$$v = \text{ch}(E) \sqrt{\text{Todd } X}$$

for the Mukai vector of the sheaves E in the moduli space \mathfrak{M}_v . We will endow the cohomology of X with the Mukai product, defined for two Mukai vectors $v = (v_0, v_2, v_4)$ and $w = (w_0, w_2, w_4)$ by

$$\langle v, w \rangle = \int_X v_2 w_2 - v_0 w_4 - v_4 w_0.$$

We will assume that the vector v is *primitive and positive*. The latter requirement means that v has positive rank, or otherwise, in rank 0, $c_1(v)$ is effective and $\langle v, v \rangle \neq 0, 4$. Moreover, we assume that the polarization H is *generic*. This ensures that the moduli space \mathfrak{M}_v consists only of stable sheaves. (It is likely that these assumptions can be relaxed.) We will give explicit expressions for the Euler characteristics

$$\chi(\mathfrak{M}_v, \Theta_w),$$

which will render obvious their symmetry in v and w .

For both $K3$ and abelian surfaces, the calculation of the Euler characteristics is facilitated by the presence of a *holomorphic symplectic structure* on the moduli spaces in question, as first established by Mukai [Muk2]. In fact, there are two basic examples of such holomorphic symplectic structures which are relevant for the discussion at hand. The first is provided by the Hilbert scheme of points $X^{[n]}$ on a $K3$ surface X . When X is an abelian surface, a small variation is required, in order to obtain an *irreducible* symplectic structure. To this end, one considers the addition map

$$a : X^{[n]} \rightarrow X^{(n)} \rightarrow X, [Z] \rightarrow l_1 z_1 + \dots + l_m z_m,$$

where Z is a punctual scheme supported on z_1, \dots, z_m , with lengths l_1, \dots, l_m respectively. The fibers of a are termed generalized Kummer varieties, and are irreducible holomorphic symplectic manifolds of dimension $2n - 2$.

The relevance of these two examples resides in the following observations due to O'Grady and Yoshioka [OG1] [Y1] [Y2]. First, when X is a $K3$ surface, the moduli space \mathfrak{M}_v is deformation equivalent to the Hilbert scheme of points $X^{[d_v]}$, with

$$d_v = \frac{1}{2} \langle v, v \rangle + 1.$$

The Euler characteristics of the theta line bundles Θ_w on \mathfrak{M}_v are deformation-invariant polynomials in the Beauville-Bogomolov form. These polynomials can therefore be calculated on the Hilbert scheme. The argument is presented in [OG2], [GNY], yielding the answer

$$(4.1) \quad \chi(\mathfrak{M}_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \binom{d_v + d_w}{d_v}.$$

The situation is slightly more involved when X is an abelian surface. In order to obtain an irreducible holomorphic symplectic structure, we need to look at the Albanese morphism α of \mathfrak{M}_v . To this end, write

$$\mathbf{RS} : \mathbf{D}(X) \rightarrow \mathbf{D}(\widehat{X})$$

for the Fourier-Mukai transform on X with respect to the normalized Poincaré sheaf \mathcal{P} on $X \times \widehat{X}$:

$$\mathbf{RS}(x) = \mathbf{Rpr}_{\widehat{X}!}(\mathcal{P} \otimes \mathbf{pr}_X^* x).$$

Following Yoshioka [Y1], we may define the following 'determinant' morphism

$$\alpha = (\alpha^+, \alpha^-) : \mathfrak{M}_v \rightarrow \widehat{X} \times X$$

with

$$\alpha^+(E) = \det E, \quad \alpha^-(E) = \det \mathbf{RS}(E).$$

(The identification of the target of α^+ with \widehat{X} requires the translation by a fixed reference line bundle Λ with $c_1(\Lambda) = -c_1(v)$. The same remark applies to the morphism α^- .) Yoshioka established that the fibers K_v of the Albanese morphism α are *irreducible* holomorphic symplectic manifolds, deformation equivalent to the generalized Kummer varieties of dimension $2d_v - 4$.

It is shown in [MO2] that

$$\chi(K_v, \Theta_w) = \frac{(d_v - 1)^2}{d_v + d_w - 2} \binom{d_v + d_w - 2}{d_v - 1}.$$

This formula is clearly not symmetric in v and w , and in fact it is not expected to be so. Instead, *three* symmetric formulas are obtained considering suitable variations

of the moduli spaces involved. More precisely, let us write \mathfrak{M}_v^+ , \mathfrak{M}_v^- for the fibers of the two morphisms α^+ , α^- . Then,

THEOREM 4.1. [MO3] *The following three symmetries are valid*

$$(4.2) \quad \chi(\mathfrak{M}_v^+, \Theta_w) = \chi(\mathfrak{M}_w^+, \Theta_v) = \frac{c_1(v \otimes w)^2}{2(d_v + d_w - 2)} \binom{d_v + d_w - 2}{d_v - 1}.$$

$$(4.3) \quad \chi(\mathfrak{M}_v^-, \Theta_w) = \chi(\mathfrak{M}_w^-, \Theta_v) = \frac{c_1(\widehat{v} \otimes \widehat{w})^2}{2(d_v + d_w - 2)} \binom{d_v + d_w - 2}{d_v - 1}.$$

$$(4.4) \quad \chi(K_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \frac{(d_v - 1)^2}{d_v + d_w - 2} \binom{d_v + d_w - 2}{d_v - 1}.$$

The last equation was derived under the assumption that $c_1(v)$ and $c_1(w)$ are proportional, which happens for instance if the Picard rank of X is 1; this assumption is likely unnecessary. In the second equation, the hats decorating v and w denote the cohomological Fourier-Mukai transform.

The numerical coincidences (4.1), (4.2), (4.3), (4.4) suggest strange duality phenomena on K3 and abelian surfaces. However, in order to make use of the numerics provided by these equations, one needs to assume that the Θ s have no higher cohomologies. This is true in many cases, but seems difficult to settle in general – see [MO3] for a discussion.

Nonetheless, in all four cases when the above numerical symmetries occur, the splitting (1.4) is easily established, cf. [MO3]. Moreover, the vanishing (1.2) is guaranteed if for instance $c_1(v \otimes w) \cdot H > 0$. This motivates the following

CONJECTURE 4.2. *Let X be a K3 or abelian surface. Assume that v and w are primitive, positive Mukai vectors, such that $\chi(v \otimes w) = 0$, and*

$$c_1(v \otimes w) \cdot H > 0.$$

Let $(\mathcal{M}_v, \mathcal{M}_w)$ denote the pair $(\mathfrak{M}_v, \mathfrak{M}_w)$ when X is a K3-surface, or any one of the pairs (K_v, \mathfrak{M}_w) , (K_w, \mathfrak{M}_v) , $(\mathfrak{M}_v^+, \mathfrak{M}_w^+)$ or $(\mathfrak{M}_v^-, \mathfrak{M}_w^-)$ when X is abelian. Then, the duality morphism

$$D : H^0(\mathcal{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathcal{M}_w, \Theta_v)$$

is either an isomorphism or zero.

4.2. Examples. Other than the numerical evidence provided by equations (4.1)-(4.4), the conjecture has not received much checking. We expect that several cases can be verified, starting with the statement for Hilbert schemes established in Section 3, by applying Fourier-Mukai transformations. We present here a handful of low-dimensional examples.

EXAMPLE 4.3. *Intersections of quadrics.* Let us begin with a classical example, due to Mukai [Muk2]. Assume that X is an intersection of three smooth quadrics in \mathbb{P}^5 ,

$$X = Q_0 \cap Q_1 \cap Q_2.$$

Let C be a hyperplane section, represented by a smooth curve of genus 5. Write ω for the class of a point on X . Then, the moduli space of sheaves on X with Mukai vector

$$v = 2 + C + 2\omega$$

is another $K3$ surface Y . In fact, Y may be realized as a double cover $Y \rightarrow \mathbb{P}^2$ branched along a sextic B as follows. For any $\lambda = [\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$, let

$$Q_\lambda = \lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 \hookrightarrow \mathbb{P}^5$$

be a quadric in the net generated by Q_0, Q_1, Q_2 . The sextic B corresponds to the singular quadrics Q_λ . For λ outside B , Q_λ may be identified with the Plücker embedding

$$G(2, 4) \hookrightarrow \mathbb{P}^5.$$

Therefore, the tautological sequence

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{O}^{\oplus 4} \rightarrow \mathcal{B} \rightarrow 0$$

on $G(2, 4) \cong Q_\lambda$ gives, by restriction to X , two natural rank 2 bundles $\mathcal{A}^\vee|_X$ and $\mathcal{B}|_X$, both belonging to the moduli space $\mathfrak{M}_v \cong Y$. The fiber of the double cover

$$f : Y \rightarrow \mathbb{P}^2$$

over $\lambda \in \mathbb{P}^2 \setminus B$ consists of these two sheaves on X .

Let

$$w = 1 - \omega$$

be a dual vector, so that $\mathfrak{M}_w \cong X^{[2]}$. There is a dual fibration

$$\widehat{f} : \mathfrak{M}_w \rightarrow (\mathbb{P}^2)^\vee,$$

defined by assigning

$$Z \in X^{[2]} \mapsto L_Z \in (\mathbb{P}^2)^\vee.$$

Here L_Z is a line in the net of quadrics, defined as

$$L_Z = \{\lambda \in \mathbb{P}^2 \text{ such that the quadric } Q_\lambda \text{ contains the line spanned by } Z\}.$$

Now the theta divisor

$$\Theta \hookrightarrow \mathfrak{M}_v \times \mathfrak{M}_w$$

is the pullback of the incidence divisor

$$\Delta \hookrightarrow \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$$

via the morphism

$$f \times \widehat{f} : \mathfrak{M}_v \times \mathfrak{M}_w \rightarrow \mathbb{P}^2 \times (\mathbb{P}^2)^\vee.$$

A direct argument, valid outside the branch locus, is easy to give. Any sheaf $F \in \mathfrak{M}_v \setminus f^{-1}(B)$ admits a surjective morphism $\mathcal{O}^4 \rightarrow F \rightarrow 0$ which determines a morphism $X \rightarrow G(2, 4) \hookrightarrow \mathbb{P}^5$. Then, one needs to check that the statement

$$h^0(F \otimes I_Z) > 0$$

is equivalent to the fact that the line spanned by Z is contained in the quadric $G(2, 4) \hookrightarrow \mathbb{P}^5$. A moment's thought shows this is the case, upon unravelling the definitions. We refer the reader to [OG2], Claim 5.16, for a more complete argument.

This implies that

$$H^0(\mathfrak{M}_v, \Theta_w) = H^0(Y, f^* \mathcal{O}(1)) \cong H^0(\mathbb{P}^2, \mathcal{O}(1))$$

is naturally dual to

$$H^0(\mathfrak{M}_w, \Theta_v) = H^0(X^{[2]}, \widehat{f}^* \mathcal{O}(1)) \cong H^0((\mathbb{P}^2)^\vee, \mathcal{O}(1)).$$

EXAMPLE 4.4. *O'Grady's generalization.* Let us now explain O'Grady's generalization of this example [OG2]. This covers the case when

$$v = 2 + C + 2\omega, \quad w = 1 - \omega,$$

where now C is any smooth genus $g \leq 8$ curve obtained as a hyperplane section of a *generic* K3 surface

$$X \hookrightarrow \mathbb{P}^g.$$

The situation is entirely similar to what we had before, namely the two moduli spaces come equipped with two dual fibrations f and \widehat{f} .

The previous discussion goes through for the vector w , setting

$$\widehat{f} : \mathfrak{M}_w \cong X^{[2]} \mapsto |I_X(2)|^\vee, Z \mapsto L_Z$$

where

$$L_Z = \{\text{quadrics } Q \text{ vanishing on } X, \text{ and which contain the line spanned by } Z\}.$$

Things are more involved for the vector v . For generic $F \in \mathfrak{M}_v$, O'Grady shows that F is locally free and globally generated, and $h^0(F) = 4$. Therefore, there is an exact sequence

$$0 \rightarrow E \rightarrow H^0(F) \otimes \mathcal{O}_X \rightarrow F \rightarrow 0,$$

inducing a morphism

$$X \rightarrow G(2, H^0(F)) \cong G(2, 4) \hookrightarrow \mathbb{P}(\Lambda^2 H^0(F)) \cong \mathbb{P}^5 \rightarrow \mathbb{P}(H^0(\mathcal{O}_X(1))) \cong \mathbb{P}^g,$$

where we used that $\Lambda^2 F = \mathcal{O}_X(1)$. In turn, this gives a quadric Q_F on \mathbb{P}^g of rank at most 6, vanishing on X . Phrased differently, we obtain

$$f : \mathfrak{M}_v \mapsto |I_X(2)| \cong \mathbb{P}^{\binom{g-2}{2}-1}, \quad F \mapsto Q_F.$$

O'Grady shows that f is a morphism, which double covers its image P . The image P is then shown to be a non-degenerate subvariety of the system of quadrics $|I_X(2)|$.

As before, the theta duality is established once it is checked that $\Theta \hookrightarrow \mathfrak{M}_v \times \mathfrak{M}_w$ is the pullback of the incidence divisor via $f \times \widehat{f}$; see Claim 5.16 in [OG2].

EXAMPLE 4.5. *Isotropic Mukai vectors.* Example 4.3 can be generalized in a slightly different direction. The exposition below is essentially lifted from Sawon [S]. The idea is to exploit the situation considered in Section 3.3, using Fourier-Mukai to obtain new numerics.

Assume that the Picard group of X has rank 1, and is generated by a smooth divisor C with self-intersection $C^2 = 2r^2(g-1)$. Sawon studies the case when

$$v = (r, C, r(g-1)), \quad w = (1, 0, 1-g).$$

Since the vector v is isotropic, \mathfrak{M}_v is a new K3 surface Y . Now, \mathfrak{M}_v may fail to be a fine moduli space, and therefore a universal sheaf may not exist on $X \times Y$. In fact, there is a gerbe $\alpha \in H^2(Y, \mathcal{O}^\star)$ which is the obstruction to the existence of a universal sheaf. In any case, there is an α -twisted universal sheaf \mathcal{U} on $X \times Y$. The Fourier-Mukai transform with kernel \mathcal{U} induces an isomorphism of moduli spaces

$$X^{[g]} \cong \mathfrak{M}_w \cong \mathfrak{M}_{\widehat{w}}(Y, \alpha).$$

The vector \widehat{w} is the cohomological Fourier-Mukai dual of w with kernel \mathcal{U} . Here, $\mathfrak{M}_{\widehat{w}}(Y, \alpha)$ denotes the moduli space of α -twisted sheaves on Y . Explicitly, each $F \in \mathfrak{M}_w$ satisfies WIT_1 with respect to \mathcal{U} , and the isomorphism is realized as

$$F \mapsto \widehat{F}$$

where \widehat{F} is the non-zero cohomology sheaf of the complex $\mathbf{R}p_!(\mathcal{U} \otimes q^*F)$, which occurs in degree 1. Note that since v and w are orthogonal, \widehat{w} has rank 0, and therefore there is a fibration given by taking supports

$$s : \mathfrak{M}_w \cong \mathfrak{M}_{\widehat{w}}(Y, \alpha) \rightarrow |D|.$$

In the above, D is a smooth curve on Y , whose class corresponds to w under the isomorphism [S]

$$H^2(Y) \cong v^\perp/v.$$

The theta divisor is then realized as

$$\Theta \hookrightarrow \mathfrak{M}_v \times \mathfrak{M}_w \cong Y \times \mathfrak{M}_{\widehat{w}}(Y, \alpha),$$

as the pullback of the incidence divisor

$$\Delta \hookrightarrow Y \times |D|$$

under the natural morphism

$$1 \times s : Y \times \mathfrak{M}_{\widehat{w}}(Y, \alpha) \rightarrow Y \times |D|.$$

Indeed, $h^1(E \otimes F) \neq 0$ iff the point represented by the sheaf $[E] \in \mathfrak{M}_v$ belongs to the support of the sheaf \widehat{F} . This follows by the definition of \widehat{F} and the base change theorem, after making use of the vanishing (1.2).

In this case, it is clear that on $\mathfrak{M}_w \cong \mathfrak{M}_{\widehat{w}}(Y, \alpha)$ we have

$$\Theta_v = s^*\mathcal{O}(1),$$

and on $\mathfrak{M}_v \cong Y$,

$$\Theta_w \cong D.$$

Moreover, observe that

$$H^0(\mathfrak{M}_w, \Theta_v) = H^0(\mathfrak{M}_{\widehat{w}}(Y, \alpha), s^*\mathcal{O}(1)) \cong H^0(|D|, \mathcal{O}(1)) = |D|^\vee,$$

while

$$H^0(\mathfrak{M}_v, \Theta_w) = H^0(Y, D) = |D|.$$

The spaces of sections are therefore naturally isomorphic, with the isomorphism induced by the divisor $\Theta = (1 \times s)^*\Delta$.

Finally, the above arguments should go through in the more general situation when v is any positive isotropic vector, and w is arbitrary; the last section of [S] contains a discussion of these matters.

EXAMPLE 4.6. *Strange duality on the projective plane.* In the light of the above discussion, one may wonder if examples of theta dualities can be established for other base surfaces. The first obstacle in this direction is the fact that in the case of arbitrary surfaces, the required symmetry of the Euler characteristics has not been proved yet, and there are no a priori reasons to expect it. We are however aware of sporadic examples of strange duality on \mathbb{P}^2 , due to Dănilă [D1][D2]. These examples concern the numerical classes

$$\text{rk}(v) = 2, c_1(v) = 0, -19 \leq \chi(v) \leq 2, \text{rk}(w) = 0, c_1(w) = 1, \chi(w) = 0.$$

In this case, the moduli space \mathfrak{M}_w is isomorphic to the dual projective space $(\mathbb{P}^2)^\vee$, via $L \mapsto \mathcal{O}_L(-1)$. Moreover, $\Theta_v \cong \mathcal{O}(n)$, for $n = c_2(v)$. For the dual moduli space, note the Barth morphism

$$J : \mathfrak{M}_v \rightarrow |\mathcal{O}_{(\mathbb{P}^2)^\vee}(n)|^\vee, \quad E \mapsto J_E,$$

mapping a sheaf E to its jumping set

$$J_E = \{\text{lines } L \text{ in } (\mathbb{P}^2)^\vee \text{ such that } E|_L \not\cong \mathcal{O}_L \oplus \mathcal{O}_L\}.$$

One shows that

$$\Theta_w = J^* \mathcal{O}(1),$$

and furthermore, the duality morphism coincides with the pullback by J ,

$$D = J^* : H^0((\mathbb{P}^2)^\vee, \mathcal{O}(n))^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_w).$$

The morphism D is equivariant for the action of SL_3 , and its source is an irreducible representation of the group. Therefore, to establish that D is an isomorphism, it is enough to check the equality of dimensions for the two spaces of sections involved.

The strategy for the dimension calculation is reminiscent of Thaddeus's work on the moduli space of rank 2 bundles over a curve. Dănilă considers the moduli space of coherent systems

$$0 \rightarrow \mathcal{O}_X \rightarrow E \otimes \mathcal{O}(1),$$

where E is a sheaf of rank 2 and $c_1(E) = 0$. Different stability conditions indexed by a real parameter are considered; the moduli spaces undergo birational changes in codimension 2 each time critical values of the stability parameter are crossed. The largest critical value corresponds to a simpler space, birational to a projective bundle over a suitable Hilbert scheme of points on \mathbb{P}^2 . The computation can therefore be carried out on the Hilbert scheme. We refer the reader to [D1] for details. Let us finally note that further numerics

$$\text{rk}(v) = 2, c_1(v) = 0, -3 \leq \chi(v) \leq 2, \text{rk}(w) = 0, c_1(w) = 2 \text{ or } 3, \chi(w) = 0$$

were established in [D2].

5. Theta divisors on elliptic K3s

We study here the natural theta divisor in a product of two numerically dual moduli spaces of sheaves on an elliptic $K3$ surface X , consisting of sheaves of ranks r and s respectively. When $r > 2, s \geq 2$, we show that this divisor is mapped to its counterpart on a new pair of moduli spaces, of sheaves with ranks $r - 1$ and $s + 1$, birational to the original one via O'Grady's transformations [OG1]. As the two moduli spaces are further identified birationally, using O'Grady's recipe, with Hilbert schemes of points $X^{[a]}$ and $X^{[b]}$ on X , the theta *bundles* on them are of the form $L^{[a]}$ respectively $L^{[b]}$ for a line bundle L on X , whenever $r, s \geq 2$. We conjecture that the theta *divisor* in the original product of moduli spaces of sheaves is accordingly identified with the theta divisor corresponding to L in the product $X^{[a]} \times X^{[b]}$. This divisor is the subject of Section 3.2. Therefore, Proposition 3.1 and the conjecture imply that theta duality holds for many pairs on elliptic K3s.

5.1. O'Grady's construction. To start, we recall O'Grady's construction. Let X be a smooth elliptic $K3$ surface with a section, and with Néron-Severi group

$$NS(X) = \mathbb{Z}\sigma + \mathbb{Z}f,$$

where σ and f are the classes of the section and of the fiber respectively. Note that

$$\sigma^2 = -2, f^2 = 0, \sigma f = 1.$$

Consider a Mukai vector v with

$$c_1(v) \cdot f = 1,$$

i.e.,

$$v = (r, \sigma + kf, p\omega) \in H^{2*}(X),$$

for some $k, p \in \mathbb{Z}$. Pick a *suitable* polarization $H = \sigma + mf$ *e.g.*, assume that m is sufficiently large. This choice of polarization ensures that \mathfrak{M}_v is a projective holomorphic symplectic manifold, consisting only of stable sheaves. We will denote by $2a$ the dimension of \mathfrak{M}_v ,

$$\langle v, v \rangle + 2 = 2a.$$

O'Grady [OG1] showed that \mathfrak{M}_v is birational to the Hilbert scheme $X^{[a]}$. We now describe this birational isomorphism which proceeds in steps successively modifying the rank of the sheaves in the moduli space. Since we are ultimately interested in theta divisors, we need to understand the birational transformations of the moduli space *away from codimension two*, so we will track the successive stages with some care.

Note first that twisting with $\mathcal{O}(f)$ gives an isomorphism

$$\mathfrak{M}_v \cong \mathfrak{M}_{\tilde{v}}, \quad \text{with } \tilde{v} = (r, \sigma + (k+r)f, (p+1)\omega).$$

This twist raises the Euler characteristic by 1. We normalize v by requiring that $p = 1 - r$, and we denote the moduli space in this case by \mathfrak{M}_r^a . Thus, points in \mathfrak{M}_r^a have the Mukai vector

$$(5.1) \quad v_{r,a} = (r, \sigma + (a - r(r-1))f, (1-r)\omega),$$

and geometrically this normalization amounts to imposing that

$$\chi(E_r) = 1 \text{ for } E_r \in \mathfrak{M}_r^a.$$

O'Grady shows that, as expected, the generic point E_r of \mathfrak{M}_r^a has exactly one section,

$$(5.2) \quad h^0(E_r) = 1, \text{ and moreover, } h^0(E_r(-f)) = 0.$$

Keeping track of codimensions, we further have, importantly,

$$(5.3) \quad h^0(E_r(-2f)) = 0 \text{ for } E_r \text{ outside a codimension 2 locus in } \mathfrak{M}_r^a.$$

Now stability forces the vanishing $h^2(E_r(-2f)) = 0$ for all sheaves in \mathfrak{M}_r^a . We conclude that

$$h^1(E_r(-2f)) = -\chi(E_r(-2f)) = 1$$

outside a codimension 2 locus in \mathfrak{M}_r^a . O'Grady singles out an open subscheme $U_r^a \subset \mathfrak{M}_r^a$ for which the vanishing (5.3) occurs. For sheaves E_r in U_r^a there is a unique nontrivial extension

$$(5.4) \quad 0 \rightarrow \mathcal{O} \rightarrow \tilde{E}_{r+1} \rightarrow E_r \otimes \mathcal{O}(-2f) \rightarrow 0.$$

The resulting middle term \tilde{E}_{r+1} is torsion-free with Mukai vector $v_{r+1,a}$.

However, \tilde{E}_{r+1} may not be stable. In fact, it fails to be stable if E_r belongs to a divisor D_r in U_r^a . For sheaves E_r away from D_r , we set

$$E_{r+1} \cong \tilde{E}_{r+1}.$$

For sheaves E_r in D_r , a stable sheaf E_{r+1} is obtained by modifying \tilde{E}_{r+1} . For $r \geq 2$, the corresponding extension \tilde{E}_{r+1} has a natural rank r subsheaf G_r such that

$$0 \rightarrow G_r \rightarrow \tilde{E}_{r+1} \rightarrow \mathcal{O}(f) \rightarrow 0.$$

The stabilization E_{r+1} of \tilde{E}_{r+1} then fits in an exact sequence

$$0 \rightarrow \mathcal{O}(f) \rightarrow E_{r+1} \rightarrow G_r \rightarrow 0.$$

Note that E_{r+1} has Mukai vector $v_{r+1,a}$ as well.

The assignment

$$E_r \mapsto E_{r+1}$$

identifies dense open sets $U_r^a \cong U_{r+1}^a$ (whose complements have codimension at least 2) in the moduli spaces with vectors $v_{r,a}$ and $v_{r+1,a}$. This gives rise to a birational map

$$\Phi_r : \mathfrak{M}_r^a \dashrightarrow \mathfrak{M}_{r+1}^a.$$

The rank 1 moduli space \mathfrak{M}_1^a is isomorphic to the Hilbert scheme $X^{[a]}$ via

$$Z \mapsto I_Z(\sigma + af).$$

Additional requirements on the scheme Z single out the open set U_1^a . For each rank, one gets therefore a birational isomorphism of \mathfrak{M}_r^a with the Hilbert scheme $X^{[a]}$.

A good understanding of the morphisms Φ_r hinges crucially on identifying the divisors D_r along which the semistable reduction needs to be performed, and this is the most difficult part of O'Grady's work. Since the U_r s are isomorphic, the D_r s can be identified with divisors on the Hilbert scheme $X^{[a]}$. Let S be the divisor of cycles in $X^{[a]}$ which intersect the section σ of the elliptic fibration. In the notation of Section 3,

$$\mathcal{O}(S) = \mathcal{O}(\sigma)_{(a)}.$$

Let T be the divisor consisting of points I_Z such that

$$h^0(I_Z((a-1)f)) \neq 0.$$

O'Grady proves that

$$D_1 = S \cup T, \text{ and } D_r = S \text{ for } r \geq 2.$$

5.2. Theta divisors. With these preliminaries understood, consider now two normalized moduli spaces \mathfrak{M}_r^a and \mathfrak{M}_s^b of stable sheaves on X , identified birationally, away from codimension 2, with Hilbert schemes. The tensor product of two points $E_r \in \mathfrak{M}_r^a$ and $F_s \in \mathfrak{M}_s^b$ has Euler characteristic

$$\chi(E_r \otimes F_s) = a + b - 2 - (r + s)(r + s - 2).$$

Since

$$c_1(E_r \otimes F_s) \cdot f = r + s,$$

tensorization by $\mathcal{O}(f)$ raises the Euler characteristic by $r + s$. We will assume from now on that

$$r + s \mid a + b - 2.$$

In fact, we will furthermore assume that

$$-\nu = \frac{a + b - 2}{r + s} - (r + s - 2) > 1 \Leftrightarrow a + b \geq (r + s)^2 + 2.$$

The definition of ν is so that

$$\chi(E_r \otimes F_s \otimes \mathcal{O}(\nu f)) = 0, \text{ for } E_r \in \mathfrak{M}_r^a, F_s \in \mathfrak{M}_s^b.$$

Any semistable sheaf E on X whose first Chern class has positive intersection with the fiber class *i.e.*,

$$c_1(E) = \alpha\sigma + \beta f, \text{ for } \alpha > 0,$$

satisfies $H^2(E) = 0$ forced by the stability condition. Therefore the locus

$$(5.5) \quad \Theta_{r,s} = \{(E_r, F_s) \in \mathfrak{M}_r^a \times \mathfrak{M}_s^b \text{ such that } h^0(E_r \otimes F_s \otimes \mathcal{O}(\nu f)) \neq 0\}$$

should indeed correspond to a divisor. We further let Θ_{F_s} be the locus

$$\Theta_{F_s} = \{E_r \in \mathfrak{M}_r^a \text{ such that } h^0(E_r \otimes F_s(\nu f)) \neq 0\}$$

in \mathfrak{M}_r^a , and denote by Θ_{E_r} its analogue in \mathfrak{M}_s^b .

Now Lemma I.6.19 of [OG1] gives an explicit description of the morphism

$$\Theta : \nu_{r,a}^\perp \rightarrow \text{Pic}(\mathfrak{M}_r^a) \cong \text{Pic}(X^{[a]}).$$

In particular, the class of the theta line bundle $\mathcal{O}(\Theta_{F_s})$ in the Picard group of $X^{[a]}$ is

$$(5.6) \quad \mathcal{O}(\Theta_{F_s}) = \mathcal{O}(\sigma)_{(a)}^{(r+s)} \otimes \mathcal{O}(f)_{(a)}^{2(r+s)-2-\nu} \otimes M, \text{ for } r \geq 2, s \geq 1.$$

As in Section 3.1, $\mathcal{O}(f)_{(a)}$ and $\mathcal{O}(\sigma)_{(a)}$ denote the line bundles on $X^{[a]}$ induced by the generators $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ of the Picard group of X , and $-2M$ is the exceptional divisor in $X^{[a]}$. Letting

$$(5.7) \quad L = \mathcal{O}((r+s)\sigma + (2(r+s) - 2 - \nu)f) \text{ on } X,$$

equation (5.6) reads

$$(5.8) \quad \mathcal{O}(\Theta_{F_s}) = L^{[a]} \text{ for } r \geq 2, s \geq 1.$$

We conclude that

$$(5.9) \quad \mathcal{O}(\Theta_{r,s}) = \mathcal{O}(\Theta_{r-1,s+1}) = L^{[a]} \boxtimes L^{[b]}, \text{ for } r > 2, s \geq 2.$$

The line bundle on the right comes equipped with the theta divisor $\theta_{L,a}$ discussed in Section 3.2. Away from codimension 2, this divisor is supported on the locus

$$\{(I_Z, I_W) \in X^{[a]} \times X^{[b]} \text{ such that } h^0(I_Z \otimes I_W \otimes L) \neq 0\}.$$

It seems now reasonable to expect that

CONJECTURE 5.1. *The locus $\Theta_{r,s}$ is a divisor. Moreover, under O'Grady's birational identification $\mathfrak{M}_r^a \times \mathfrak{M}_s^b \dashrightarrow X^{[a]} \times X^{[b]}$, we have*

$$\Theta_{r,s} = \theta_{L,a} \text{ for } r, s \geq 2.$$

Note that when $\nu < -1$, L has no higher cohomology. This follows by a simple induction on $r+s$. Therefore, $\theta_{L,a}$ induces an isomorphism

$$D_L : H^0(X^{[a]}, L^{[a]})^\vee \rightarrow H^0(X^{[b]}, L^{[b]}).$$

With

$$\Theta_{r,a} = \mathcal{O}(\Theta_{E_r}) \text{ and } \Theta_{s,b} = \mathcal{O}(\Theta_{F_s}),$$

the conjecture implies that the theta duality map

$$H^0(\mathfrak{M}_r^a, \Theta_{s,b})^\vee \rightarrow H^0(\mathfrak{M}_s^b, \Theta_{r,a})$$

is an isomorphism. This consequence of the conjecture can be rephrased in a more intrinsic form as follows.

COROLLARY 5.2. *Let v and w be Mukai vectors of ranks $r \geq 2$ and $s \geq 2$. Assume that*

- (i) $\chi(v \otimes w) = 0$,
- (ii) $c_1(v) \cdot f = c_1(w) \cdot f = 1$,
- (iii) $\langle v, v \rangle + \langle w, w \rangle \geq 2(r + s)^2$.

Then, the duality morphism

$$D : H^0(\mathfrak{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_v)$$

is an isomorphism.

As a first piece of evidence for Conjecture 5.1, we record the natural

PROPOSITION 5.3. *Letting Φ be the birational map*

$$\Phi = (\Phi_{r-1}^{-1}, \Phi_s) : \mathfrak{M}_r^a \times \mathfrak{M}_s^b \dashrightarrow \mathfrak{M}_{r-1}^a \times \mathfrak{M}_{s+1}^b,$$

and assuming that

$$-\nu = \frac{a + b - 2}{r + s} - (r + s - 2) > 1,$$

we have

$$\Phi(\Theta_{r,s}) = \Theta_{r-1,s+1}, \text{ for } r > 2, s \geq 2.$$

Proof. We will only check the set-theoretic equality, leaving the necessary modifications for the argument in families to the interested reader. The precise description of the morphisms Φ_{r-1} and Φ_s in the previous subsection will be crucial for establishing this fact, via a somewhat involved diagram chase.

To begin, recall the basic exact sequence (5.4) of O'Grady's birational isomorphism, giving rise to two nontrivial extensions

$$(5.10) \quad 0 \rightarrow \mathcal{O} \rightarrow \tilde{E}_r \rightarrow E_{r-1}(-2f) \rightarrow 0$$

and

$$(5.11) \quad 0 \rightarrow \mathcal{O} \rightarrow \tilde{F}_{s+1} \rightarrow F_s(-2f) \rightarrow 0.$$

Then E_r and F_{s+1} are obtained by stabilizing \tilde{E}_r and \tilde{F}_{s+1} if necessary; this process is required for sheaves in the divisorial locus S .

We will first consider the situation when both E_{r-1} and F_s are chosen outside S , so that

$$E_r = \tilde{E}_r, \quad F_{s+1} = \tilde{F}_{s+1}.$$

Note moreover that we may assume that either E_{r-1} or F_s is locally free, as this happens outside a set of codimension 2 in the product of moduli spaces. To establish the Proposition in this case, it suffices to show that

$$(5.12) \quad h^0(E_r \otimes F_s \otimes \mathcal{O}(\nu f)) = 0 \text{ iff } h^0(E_{r-1} \otimes F_{s+1} \otimes \mathcal{O}(\nu f)) = 0.$$

Tensoring (5.10) by $F_s(\nu f)$ we get the following sequence in cohomology

$$\begin{aligned} H^0(F_s(\nu f)) \rightarrow H^0(E_r \otimes F_s(\nu f)) \rightarrow H^0(E_{r-1} \otimes F_s((\nu - 2)f)) \xrightarrow{g} H^1(F_s(\nu f)) \\ \rightarrow H^1(E_r \otimes F_s(\nu f)) \rightarrow H^1(E_{r-1} \otimes F_s((\nu - 2)f)) \rightarrow 0. \end{aligned}$$

Similarly, twisting (5.11) by $E_{r-1}(\nu f)$ we obtain

$$\begin{aligned} H^0(E_{r-1}(\nu f)) &\rightarrow H^0(E_{r-1} \otimes F_{s+1}(\nu f)) \rightarrow H^0(E_{r-1} \otimes F_s((\nu-2)f)) \xrightarrow{h} H^1(E_{r-1}(\nu f)) \\ &\rightarrow H^1(E_{r-1} \otimes F_{s+1}(\nu f)) \rightarrow H^1(E_{r-1} \otimes F_s((\nu-2)f)) \rightarrow 0. \end{aligned}$$

Since $\nu < -1$, (5.3) implies that

$$H^0(F_s(\nu f)) = H^0(E_{r-1}(\nu f)) = 0.$$

We conclude then from the two cohomology sequences that the statement (5.12):

$$h^0(E_r \otimes F_s \otimes \mathcal{O}(\nu f)) = 0 \text{ iff } h^0(E_{r-1} \otimes F_{s+1} \otimes \mathcal{O}(\nu f)) = 0$$

can be rephrased as

$$g \text{ is an isomorphism iff } h \text{ is an isomorphism.}$$

This last equivalence is evident when we consider the following cohomology diagram:

$$\begin{array}{ccc} H^0(E_{r-1} \otimes F_s((\nu-2)f)) & \xrightarrow{h} & H^1(E_{r-1}(\nu f)) \\ \downarrow g & & \downarrow \cong \\ H^1(F_s(\nu f)) & \xrightarrow{\cong} & H^2(\mathcal{O}((\nu+2)f)) \end{array}$$

The right and bottom maps come from (5.10) and (5.11); these morphisms are surjective. Further, the dimensions are a priori the same in all but the top left corner. Indeed,

$$h^1(E_{r-1}(\nu f)) = -\chi(E_{r-1}(\nu f)) = -\nu - 1, \quad h^1(F_s(\nu f)) = -\chi(F_s(\nu f)) = -\nu - 1.$$

The above equalities hold since $\chi(E_{r-1}) = \chi(F_s) = 1$ and twisting by f raises the Euler characteristic by 1. Also,

$$h^2(\mathcal{O}((\nu+2)f)) = h^0(\mathcal{O}((-\nu-2)f)) = -\nu - 1.$$

Thus, the right vertical and bottom arrows are isomorphisms. Therefore, g is an isomorphism if and only if h is one. This establishes (5.12) in the case when E_r and F_s arise as stable extensions.

Next, we need to examine the case when E_{r-1} is in the divisorial locus S on $X^{[a]}$, but F_{s+1} is not in the divisorial locus S of $X^{[b]}$, so in particular (5.2) holds for F_{s+1} . The situation when both E_{r-1} and F_{s+1} are in the special locus has codimension 2 in the product $\mathfrak{M}_r^a \times \mathfrak{M}_s^b$, therefore we ignore it. For the same reason, in the arguments below we assume that F_{s+1} is locally free.

We thus have $F_{s+1} = \tilde{F}_{s+1}$. More delicately, to each $E_{r-1} \in D_{r-1}$, the birational map Φ_{r-1} associates a vector bundle $E_r \in D_r$ as follows. According to O'Grady's argument, $E_{r-1}(-2f)$ has a unique stable locally free subsheaf G_{r-1} satisfying

$$(5.13) \quad 0 \rightarrow G_{r-1} \rightarrow E_{r-1}(-2f) \rightarrow \mathcal{O}_{f_0} \rightarrow 0.$$

In this exact sequence, the fiber f_0 is the unique elliptic fiber such that

$$\dim \text{Hom}(E_{r-1}, \mathcal{O}_{f_0}) = 1,$$

whereas the Hom groups with values in the structure sheaves of all other elliptic fibers are zero. Now the extension group $\text{Ext}^1(G_{r-1}, \mathcal{O}(f))$ is two-dimensional.

Among these extensions there is a unique one whose middle term has the *same* jumping fiber f_0 *i.e.*, we have

$$(5.14) \quad 0 \rightarrow \mathcal{O}(f) \rightarrow E_r \xrightarrow{\pi} G_{r-1} \rightarrow 0,$$

and

$$(5.15) \quad \dim \operatorname{Hom}(E_r, \mathcal{O}_{f_0}) = 1.$$

The emerging sheaf E_r is locally free and stable. The assignment

$$E_{r-1} \mapsto E_r$$

induces a birational isomorphism $D_{r-1} \dashrightarrow D_r$.

We now show that in this case also,

$$(5.16) \quad h^0(E_r \otimes F_s \otimes \mathcal{O}(\nu f)) = 0 \text{ iff } h^0(E_{r-1} \otimes F_{s+1} \otimes \mathcal{O}(\nu f)) = 0,$$

which will conclude the proof of the Proposition.

Tensoring (5.14) by $F_s(\nu f)$ and taking cohomology, we have

$$\begin{aligned} H^0(F_s((\nu+1)f)) &\rightarrow H^0(E_r \otimes F_s(\nu f)) \rightarrow H^0(G_{r-1} \otimes F_s(\nu f)) \xrightarrow{j} H^1(F_s((\nu+1)f)) \\ &\rightarrow H^1(E_r \otimes F_s(\nu f)) \rightarrow H^1(G_{r-1} \otimes F_s(\nu f)) \rightarrow 0. \end{aligned}$$

The first H^0 group is zero by (5.2). We conclude that

$$(5.17) \quad h^0(E_r \otimes F_s(\nu f)) = 0 \text{ iff } j \text{ is an isomorphism.}$$

As earlier, there is a cohomology commutative diagram

$$\begin{array}{ccc} H^0(G_{r-1} \otimes F_s(\nu f)) & \xrightarrow{j} & H^1(F_s((\nu+1)f)) \\ \downarrow \beta & & \downarrow \cong \\ H^1(E_r(\nu+2)f) & \xrightarrow{\alpha} & H^1(G_{r-1}((\nu+2)f)) \twoheadrightarrow H^2(\mathcal{O}((\nu+3)f)) \end{array}$$

where the left and right vertical maps are obtained from the sequence (5.11), and the top and bottom ones from (5.14). The right onto vertical morphism is an isomorphism for dimension reasons. The bottom morphism is surjective. Therefore, (5.18)

$$j \text{ is an isomorphism iff } \bar{\beta} : H^0(G_{r-1} \otimes F_s(\nu f)) \rightarrow \operatorname{Coker} \alpha \text{ is an isomorphism.}$$

On the other hand, as in the above argument for stable extensions E_r and F_s ,

$$(5.19) \quad h^0(E_{r-1} \otimes F_{s+1}(\nu f)) = 0 \text{ iff } h : H^0(E_{r-1} \otimes F_s((\nu-2)f)) \cong H^1(E_{r-1}(\nu f))$$

Now (5.17), (5.18), and (5.19) imply assertion (5.16) once we establish that

$$\bar{\beta} \text{ is an isomorphism iff } h \text{ is an isomorphism.}$$

This follows by chasing the commutative diagram

$$\begin{array}{ccccc} H^0(G_{r-1} \otimes F_s(\nu f)) & \hookrightarrow & H^0(E_{r-1} \otimes F_s((\nu-2)f)) & \longrightarrow & H^0(F_s|_{f_0}) \\ \downarrow \beta & & \downarrow h & & \downarrow \cong \\ H^0(\mathcal{O}_{f_0}) & \xrightarrow{\gamma} & H^1(G_{r-1}((\nu+2)f)) & \longrightarrow & H^1(\mathcal{O}_{f_0}). \end{array}$$

Here the rows are obtained from the exact sequence (5.13) after appropriate tensorizations, and the columns are obtained from the exact sequence (5.11). Note that the vertical arrow on the right is an isomorphism between one-dimensional spaces. If we assume that the middle vertical arrow h is an isomorphism, then the left vertical map β gives an injection

$$\bar{\beta} : H^0(G_{r-1} \otimes F_s(\nu f)) \rightarrow \text{Coker } \gamma,$$

and the dimension count forces it to be an isomorphism. Conversely, if $\bar{\beta}$ is an isomorphism, then the dimension count shows that

$$H^1(G_{r-1} \otimes F_s(\nu f)) = 0.$$

Thus, the last map in the top row of the diagram is surjective. This implies that the middle vertical map h is an isomorphism by the five lemma.

To finish the proof of Proposition 5.3, it remains to explain that $\bar{\beta}$ and $\bar{\beta}$ coincide. This comes down to showing that

$$\text{Coker } \alpha \cong \text{Coker } \gamma.$$

We claim that there is a commutative diagram

$$\begin{array}{ccc} & & H^1(E_r((\nu+2)f)) \\ & \nearrow \epsilon & \downarrow \alpha \\ H^0(\mathcal{O}_{f_0}) & \xrightarrow{\gamma} & H^1(G_{r-1}((\nu+2)f)). \end{array}$$

The map γ is injective, and it is easy to see that the image of α has dimension 1. Once shown to exist, ϵ induces therefore an isomorphism between the images of α and γ in $H^1(G_{r-1}((\nu+2)f))$. To define ϵ , we first explain that the morphism

$$\pi : \text{Ext}^1(\mathcal{O}_{f_0}, E_r) \rightarrow \text{Ext}^1(\mathcal{O}_{f_0}, G_{r-1})$$

is surjective, with π being the second map in (5.14). Indeed, using the exact sequence (5.14), we have

$$\text{Ext}^1(\mathcal{O}_{f_0}, E_r) \xrightarrow{\pi} \text{Ext}^1(\mathcal{O}_{f_0}, G_{r-1}) \rightarrow \text{Ext}^2(\mathcal{O}_{f_0}, \mathcal{O}(f)) \xrightarrow{\tau} \text{Ext}^2(\mathcal{O}_{f_0}, E_r) \rightarrow 0.$$

It suffices to show that τ is an isomorphism. This follows by counting dimensions. Indeed, using equation (5.15), we have

$$\text{Ext}^2(\mathcal{O}_{f_0}, E_r) = \text{Ext}^0(E_r, \mathcal{O}_{f_0}) = 1, \quad \text{Ext}^2(\mathcal{O}_{f_0}, \mathcal{O}(f)) = 1.$$

Finally, let us denote by $e \in \text{Ext}^1(\mathcal{O}_{f_0}, G_{r-1})$ the extension class of the exact sequence (5.13). Pick $\bar{e} \in \text{Ext}^1(\mathcal{O}_{f_0}, E_r)$ such that $\pi(\bar{e}) = e$, and define ϵ to be the multiplication by \bar{e} . This choice makes the above triangular diagram commutative, completing the proof of the Proposition.

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