ON VERLINDE SHEAVES AND STRANGE DUALITY OVER ELLIPTIC NOETHER-LEFSCHETZ DIVISORS

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Abstract. We extend results on generic strange duality for $K3$ surfaces by showing that the proposed isomorphism holds over an entire Noether-Lefschetz divisor in the moduli space of quasipolarized $K3$s. We interpret the statement globally as an isomorphism of sheaves over this divisor, and also describe the global construction over the space of polarized $K3$s.

1. **Introduction.**

1.1. *Setup.* For a fixed polarized complex $K3$ surface (X, H), let $v, w \in H^\star(X, \mathbb{Z})$ be two primitive elements which are orthogonal in the sense that

$$\int_X v \cup w = 0.$$

Consider the moduli space $\mathfrak{M}_v$ of Gieseker H-stable sheaves E on X of Mukai vector v:

$$\mathrm{ch}(E)\sqrt{\mathrm{Todd}\ (X)} = v.$$

The Mukai vector w induces a determinant line bundle

$$\Theta_w \to \mathfrak{M}_v,$$

constructed in [LP2][Li]. Specifically, if a universal family $\mathcal{E} \to \mathfrak{M}_v \times X$ is available, we set

$$\Theta_w = \det \mathbf{R}p_!(\mathcal{E} \otimes^{\mathbf{L}} q^\star F)^{-1},$$

for a complex $F \to X$ of Mukai vector w. Similarly we obtain the line bundle $\Theta_v \to \mathfrak{M}_w$.

If $c_1(v \cdot w) \cdot H > 0$, as explained in [Sc2], the set

$$\Theta = \{(E, F) : \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \hookrightarrow \mathfrak{M}_v \times \mathfrak{M}_w$$

is the zero locus of a section of the line bundle

$$\Theta_w \boxtimes \Theta_v \to \mathfrak{M}_v \times \mathfrak{M}_w,$$

and induces a map

$$\mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v). \tag{1}$$

According to Le Potier's strange duality conjecture [LP1], D is expected to be an isomorphism.

1.2. *Results.* In [MOY] we established the conjecture for generic surfaces (X, H) in the moduli space $\mathcal{K}_\ell$ of primitively *quasipolarized* $K3$ surfaces of degree 2ℓ, and for many pairs of Mukai vectors (v, w) which satisfy

$$c_1(v) = c_1(w) = H.$$

The proof involves degeneration to the locus of elliptic $K3$ surfaces with section and irreducible at worst nodal fibers.

In the present paper, we study the problem for elliptic $K3$s with arbitrary singular fibers. In other words, we consider the entire Noether–Lefschetz divisor

$$\mathcal{P}_1 \hookrightarrow \mathcal{K}_\ell$$

consisting of pairs (X, H) of elliptically fibered $K3$s which are quasipolarized by means of a numerical section H. We show

Theorem 1. *For any surface (X, H) in $\mathcal{P}_1$, fix two orthogonal Mukai vectors v and w of ranks $r, s \geq 3$ with*

$$c_1(v) = c_1(w) = H,$$

and satisfying further

$$\langle v, v\rangle + \langle w, w\rangle \geq 2(r+s)^2.$$

Then the duality morphism D is an isomorphism.

In Section 2 we record basic properties of the Noether-Lefschetz divisor $\mathcal{P}_1$. In Section 3, we prove the theorem above. In Section 4, the duality is stated globally as an isomorphism of sheaves, the *Verlinde* sheaves, over the entire divisor $\mathcal{P}_1$. The Verlinde sheaves are also constructed more generally over the locus $\mathcal{K}_\ell^\circ \hookrightarrow \mathcal{K}_\ell$ of polarized $K3$s. It would be interesting to extend this construction to $\mathcal{K}_\ell$ in a suitable manner.

2. **The Noether-Lefschetz divisor $\mathcal{P}_1$.** Let $(\mathcal{X}, \mathcal{H}) \to \mathcal{K}_\ell$ be the moduli stack of quasipolarized $K3$ surfaces (X, H) of degree $H^2 = 2\ell$ with $\ell \neq 1$.

We consider the Noether-Lefschetz loci of quasipolarized elliptically fibered $K3$ surfaces in $\mathcal{K}_\ell$. Specifically, for each $k > 0$, we denote by $\mathcal{P}_k$ the Noether-Lefschetz stack parametrizing triples (X, H, F) consisting of quasipolarized $K3$'s of degree 2ℓ, and divisor classes F over X satisfying

$$F^2 = 0, \quad F \cdot H = k.$$

We claim that

$$\mathcal{P}_1 \hookrightarrow \mathcal{K}_\ell$$

is a substack of $\mathcal{K}_\ell$ parametrizing exactly the quasipolarized $K3$s which can be elliptically fibered with section, and with the quasipolarization a numerical section. This is expressed by the lemma below. The statement is standard, but a reference seemed difficult to find.

Lemma 1. *Let (X, H) be a quasipolarized $K3$ surface of degree 2ℓ with $\ell \neq 1$, and let F be a divisor class on X satisfying*

$$F^2 = 0, \quad F \cdot H = 1.$$

Then

(i) *F is effective and $\mathcal{O}(F)$ is globally generated;*
(ii) *the induced map $\pi : X \to \mathbb{P}^1$ is an elliptic fibration with section σ, having F as the fiber class;*
(iii) *the quasipolarization equals $H = \sigma + (\ell + 1)F$;*
(iv) *the class F satisfying the two numerical assumptions above is unique.*

Proof. Note first that $\chi(\mathcal{O}(F)) = 2$. Since $-F \cdot H = -1$, and H is nef, $-F$ cannot be effective, so

$$h^2(\mathcal{O}(F)) = h^0(\mathcal{O}(-F)) = 0, \text{ and } \quad h^0(\mathcal{O}(F)) \geq \chi(\mathcal{O}(F)) = 2.$$

Thus F is effective.

We treat separately the two possibilities that $\mathcal{O}(F)$ be nef or not. First, if $\mathcal{O}(F)$ is nef, by the theorem of Piatetski-Shapiro and Shafarevich [PS] there exists an elliptic fibration

$$\pi : X \to \mathbb{P}^1$$

such that $F = mf$, where f is the class of a fiber. In fact,

$$F \cdot H = 1 \implies m = 1, \quad F = f, \quad H \cdot f = 1.$$

We next show that the fibration has a section. It is easy to check that the class

$$\Sigma = H - (\ell + 1)f$$

has self-intersection -2. Since $\chi(\mathcal{O}(\Sigma)) = 1$, Σ is either effective or anti-effective. In fact, Σ is effective, since $\Sigma \cdot H > 0$. Let C be a curve in the linear series $\mathcal{O}(\Sigma)$. Now, for any component R of a fiber we have $R \cdot f = 0$ by Zariski's lemma, cf. III.8.2 [BPV]. Since $C \cdot f = 1$, C must have a component which intersects each fiber with multiplicity 1. The other components of C must be supported on components of the fibers. The transversal component gives a section σ of the elliptic fibration π.

We now argue that $H = \sigma + (\ell + 1)f$. From the above discussion, we already know that

$$H = \sigma + mf + \sum m_i R_i$$

where R_i are components of fibers and $m = \ell+1$. In fact, by absorbing other fiber classes into the constant m, we may assume R_i are supported on fibers with two components or more. We have the following possibilities:

(i) fibers of type I_n, consisting in a polygon of rational curves $C_1, \ldots, C_n$;
(ii) fibers of type III, consisting of 2 rational curves C_1, C_2 meeting tangentially;
(iii) fibers of type IV consisting of 3 concurrent rational curves C_1, C_2, C_3;
(iv) fibers of type $I_n^\star$ which can be written as

$$C_1 + C_2 + C_3 + C_4 + 2(D_1 + \ldots + D_n)$$

where

$$C_1 \cdot D_1 = C_2 \cdot D_1 = C_3 \cdot D_n = C_4 \cdot D_n = 1$$

and $D_i \cdot D_{i+1} = 1$ for $1 \leq i \leq n-1$;
(v) fibers of type $II^\star, III^\star, IV^\star$ corresponding to the graphs E_6, E_7, E_8.

Consider a fiber of type (i) and its contribution $\sum m_i C_i$ to the divisor H. We claim this contribution is a multiple of the fiber. Indeed, label the components so that C_1 intersects the section σ. Since $H \cdot C_i \geq 0$ for all i, we obtain the inequalities

$$-2m_1 + m_2 + m_n \geq -1,\ -2m_2 + m_1 + m_3 \geq 0,\ \ldots, -2m_n + m_1 + m_{n-1} \geq 0.$$

If $-2m_1 + m_2 + m_n \geq 0$, then after adding the above inequalities, we conclude that we must have equality throughout. Thus $m_1 = \ldots = m_n = m$ which shows that $\sum m_i C_i = mf$ as claimed. The case

$$-2m_1 + m_2 + m_n = -1$$

is impossible. Indeed, since

$$\sum_{k \neq 1} (-2m_k + m_{k-1} + m_{k+1}) = -(-2m_1 + m_2 + m_n) = 1$$

we conclude that for some index k_0

$$-2m_k + m_{k-1} + m_{k+1} = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq 1, k_0. \end{cases}$$

This system is easily seen not to have any solutions. The remaining fiber types (ii)-(v) are entirely similar, and we will not verify them explicitly. In all cases, we find that $\sum m_i C_i$ must contribute a multiple of the fiber, hence

$$H = \sigma + mf$$

for some integer m. In fact, $m = \ell + 1$ by computing $H^2 = 2\ell$. This completes the proof when $\mathcal{O}(F)$ is nef.

We assume now that $\mathcal{O}(F)$ is not nef and we will reach a contradiction. Then there exists an irreducible curve Γ_1 such that

$$F \cdot \Gamma_1 < 0.$$

The curve Γ_1 is a component of an effective curve of class F and furthermore $\Gamma_1^2 < 0$. Thus Γ_1 is a smooth rational curve on X. Let H' be an ample class, and set $F_0 = F$. The reflection of F along Γ_1 then yields an effective class, cf. proof of Theorem 2.2 in [S]:

$$F_1 = F_0 + (F_0 \cdot \Gamma_1)\Gamma_1$$

which has the property that

$$F_1^2 = F_0^2 = 0, \;\; F_1 \cdot H' < F_0 \cdot H'.$$

If F_1 is not nef, then we continue the process reflecting along a smooth rational curve Γ_2. The process will eventually stop since $F_i \cdot H'$ is a decreasing sequence of non-negative integers. At the end, we find a nef line bundle $\mathcal{O}(F_k)$ of zero self-intersection, where

$$F_k = F + (F_0 \cdot \Gamma_1)\Gamma_1 + (F_1 \cdot \Gamma_2)\Gamma_2 + \ldots + (F_{k-1} \cdot \Gamma_k)\Gamma_k.$$

Therefore $F_k = mf$, where $m \geq 0$ by nefness. In particular,

$$F = mf + \sum n_i \Gamma_i$$

where $n_i = -F_{i-1} \cdot \Gamma_i > 0$. Using $F \cdot H = 1$ we conclude

$$m(H \cdot f) + \sum n_i (H \cdot \Gamma_i) = 1.$$

Since H is nef, the intersection numbers above are nonnegative. If $H \cdot f = 0$, since $H^2 > 0$, by the Hodge index theorem we find $f^2 \leq 0$. Since equality occurs, f must be numerically trivial which is not the case since it intersects H' nontrivially. Therefore

$$H \cdot f = 1, \;\; m = 1, \;\;\; H \cdot \Gamma_i = 0 \text{ for all } i.$$

The argument given in the nef case then shows that the elliptic fibration π has a section σ, and

$$H = \sigma + (\ell + 1)f.$$

We conclude

$$H \cdot \Gamma_i = \sigma \cdot \Gamma_i + (\ell + 1) f \cdot \Gamma_i = 0.$$

Thus either $\sigma \cdot \Gamma_i \leq 0$ or $f \cdot \Gamma_i \leq 0$. This means Γ_i is contained in σ or in the fiber f. The first case cannot occur since then

$$\Gamma_i = \sigma \text{ and } \sigma \cdot \Gamma_i + (\ell + 1) f \cdot \Gamma_i = 0 \text{ shows } \ell = 1$$

which is not allowed. Thus Γ_i is a component of the fiber of f. However, in this case $f \cdot \Gamma_i = 0$ by Zariski's lemma. Since

$$F = f + \sum n_i \Gamma_i$$

has zero self intersection, we find

$$(\sum n_i \Gamma_i)^2 = 0,$$

where Γ_i are components of the fiber. This yields $\sum n_i \Gamma_i = nf$ for some integer n, again by Zariski's lemma. Thus $F = (n+1)f$, and since $F \cdot H = 1$ then F is the fiber class.

Finally, we establish the uniqueness of F as claimed in (iv). If F' is another class with

$$F'^2 = 0, \ F' \cdot H = 1$$

then we can write

$$F' = a\sigma + R$$

where R is supported on components of fibers. We have $R \cdot f = 0$ and

$$F' \cdot H = (a\sigma + R) \cdot (\sigma + (\ell + 1)f) = 1 \implies R \cdot \sigma = 1 - a(\ell - 1).$$

In addition

$$F'^2 = 0 \implies -2a^2 + 2a(R \cdot \sigma) + R^2 = 0.$$

This yields

$$R^2 = -2a + 2a^2(\ell + 1).$$

By Zariski's lemma, $R^2 \leq 0$, which implies $a = 0$. Furthermore, we obtain $R^2 = 0$, showing that $R = mf$, again by Zariski's lemma. Moreover, $R \cdot \sigma = 1$ hence $m = 1$. Therefore $F' = f$, proving uniqueness. □

3. **Strange duality along $\mathcal{P}_1$.** We now show Theorem 1 of the Introduction. For $(X, H) \in \mathcal{P}_1$, we consider the orthogonal Mukai vectors

$$v = r + H + a\,[\mathrm{pt}], \ \ w = s + H + b\,[\mathrm{pt}] \tag{2}$$

with $r, s \geq 3$, satisfying further

$$\langle v, v \rangle + \langle w, w \rangle \geq 2(r+s)^2. \tag{3}$$

We form the moduli spaces of stable sheaves $\mathfrak{M}_v$ and $\mathfrak{M}_w$ together with the corresponding theta line bundles. Stability of the sheaves in $\mathfrak{M}_v$ and $\mathfrak{M}_w$ is with respect to a polarization which is suitable in the sense of Friedman. For such polarizations, and sheaves of fiber degree 1, stability on the surface is equivalent to stability of the restriction to a generic fiber, cf. Theorem 5, chapter 6 of [F]. [1] Both moduli spaces are smooth and projective.

[1]As shown in the appendix of [MOY], this choice of polarization is in fact irrelevant under the stronger assumptions that

$$\langle v, v \rangle \geq 2(r-1)(r^2+1), \ \ \langle w, w \rangle \geq 2(s-1)(s^2+1).$$

Under these conditions, in [MOY], the strange duality map

$$\mathsf{D}:\ H^0(\mathfrak{M}_v,\Theta_w)^\vee \to H^0(\mathfrak{M}_w,\Theta_v)$$

was proven to be an isomorphism over the open sublocus of $\mathcal{P}_1$ consisting of surfaces with Picard rank 2.

We now assume that X has Picard rank larger than 2. The elliptic fibration has finitely many reducible fibers. Fourier-Mukai functors were studied in this setting in [HMS]. Specifically, let

$$\pi : X \to \mathbb{P}^1$$

be any quasipolarized elliptically fibered $K3$ surface with section class σ and fiber class f. Consider the product $Y = X \times_{\mathbb{P}^1} X$ with projections p and q to the two factors, and let

$$\Delta \subset X \times_{\mathbb{P}^1} X$$

be the diagonal. The π-relative Fourier-Mukai functor

$$\mathsf{S} : \mathbf{D}(X) \longrightarrow \mathbf{D}(X)$$

with kernel

$$\mathcal{P} = \mathcal{I}_\Delta \otimes \mathcal{O}(p^\star\sigma + q^\star\sigma)$$

is an equivalence of bounded derived categories of coherent sheaves by Proposition 2.16 of [HMS]. As (X,H) is in $\mathcal{P}_1$, by Lemma 1

$$c_1(v) = c_1(w) = \sigma + (\ell+1)f.$$

Along the lines of [B], we shall prove shortly that the Fourier-Mukai transform S induces a birational morphism, regular in codimension 1, between the moduli spaces $\mathfrak{M}_v$ and $\mathfrak{M}_w$ on the one hand, and the Hilbert schemes of d_v respectively d_w points on X on the other:

$$\Psi_v : \mathfrak{M}_v \dashrightarrow X^{[d_v]}, \quad \Psi_w : \mathfrak{M}_w \dashrightarrow X^{[d_w]}.$$

Assuming this for the moment, we explain how to complete the proof of Theorem 1, much as in [MOY]. We determine first the exact numerics of the transformation S by a cohomological Fourier-Mukai calculation. Let $V \in \mathbf{D}(X)$ be any complex of rank r, Euler characteristic χ, and first Chern class

$$c_1(V) = k\sigma + mf,$$

Indeed, in this case, the different moduli spaces are birational in codimension 1.

for integers k and m. Recalling p and q are the projections from $Y = X \times_{\mathbb{P}^1} X$, we have

$$\begin{aligned}
\det \mathsf{S}(V) &= \det \mathbf{R}q_\star(\mathcal{P} \otimes p^\star V) = \det \mathbf{R}q_\star(I_\Delta \otimes p^\star V(\sigma) \otimes q^\star \mathcal{O}(\sigma)) \\
&= \det \mathbf{R}q_\star(I_\Delta \otimes p^\star V(\sigma)) \otimes \mathcal{O}(\sigma)^{\chi(V|_f)} \\
&= \det \mathbf{R}q_\star(p^\star V(\sigma)) \otimes \det \mathbf{R}q_\star(\mathcal{O}_\Delta \otimes p^\star V(\sigma))^{-1} \otimes \mathcal{O}(k\sigma) \\
&= \det \mathbf{R}q_\star(p^\star V(\sigma)) \otimes \det V(\sigma)^{-1} \otimes \mathcal{O}(k\sigma) \\
&= \det \mathbf{R}q_\star(p^\star V(\sigma)) \otimes \mathcal{O}(-r\sigma - mf).
\end{aligned}$$

To calculate the first term, it is more convenient to work on the product

$$j : Y \hookrightarrow X \times X.$$

Let $\bar{p}, \bar{q}$ denote the two projections from $X \times X$, and let $\mathrm{pr} = \pi \times \pi : X \times X \to \mathbb{P}^1 \times \mathbb{P}^1$. Observing that

$$j_\star \mathcal{O}_Y = \mathrm{pr}^\star \mathcal{O}_{\Delta/\mathbb{P}^1 \times \mathbb{P}^1} = \mathrm{pr}^\star(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} - \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathcal{O}_{X \times X} - \bar{p}^\star \mathcal{O}(-f) \otimes \bar{q}^\star \mathcal{O}(-f),$$

we calculate

$$\begin{aligned}
\det \mathbf{R}q_\star(p^\star V(\sigma)) &= \det \mathbf{R}\bar{q}_\star\ (\bar{p}^\star V(\sigma) \otimes j_\star \mathcal{O}_Y) \\
&= \det \mathbf{R}\bar{q}_\star\ (\bar{p}^\star V(\sigma)) \otimes \det \mathbf{R}\bar{q}_\star\ (\bar{p}^\star V(\sigma) \otimes \bar{p}^\star \mathcal{O}(-f) \otimes \bar{q}^\star \mathcal{O}(-f))^{-1} \\
&= \det(\mathbf{R}\bar{q}_\star\ (\bar{p}^\star V(\sigma - f)) \otimes \mathcal{O}(-f))^{-1} \\
&= \mathcal{O}(-f)^{-\chi(V(\sigma - f))} = \mathcal{O}((\chi - 2r + m - 3k)f).
\end{aligned}$$

To summarize, we obtained

$$\det \mathsf{S}(V) = \mathcal{O}(-r\,\sigma + (\chi - 2r - 3k)\,f).$$

Now let E and F be stable sheaves whose Mukai vectors v and w are given by (2). By the preceding calculation

$$\det \mathsf{S}(E^\vee) = \mathcal{O}(-r\sigma + (a - r + 3)f),$$

$$\det \mathsf{S}(F) = \mathcal{O}(-s\sigma + (b - s - 3)f).$$

Assuming the birational isomorphism with the Hilbert scheme, for generic E and F we therefore have that

$$\mathsf{S}(E^\vee) = I_Z \otimes \mathcal{O}(r\sigma - (a - r + 3)f)[-1], \tag{4}$$

$$\mathsf{S}(F) = I_W^\vee \otimes \mathcal{O}(-s\sigma + (b - s - 3)f), \tag{5}$$

where Z and W are zero dimensional subschemes of lengths d_v and d_w respectively. In fact, we will only explain the first equality below; the second can be deduced from the first by Grothendieck duality as in Proposition 2 of [MOY].

We finally calculate

$$\begin{aligned}\mathbb{H}^0(E\otimes^{\mathbf{L}}F) &= \mathrm{Hom}_{\mathbf{D}(X)}(E^\vee,\,F)=\mathrm{Hom}_{\mathbf{D}(X)}\big(\mathsf{S}(E^\vee),\,\mathsf{S}(F)\big)\\ &= \mathrm{Ext}^1(I_Z\otimes L,\,I_W^\vee)=\mathrm{Ext}^1(I_W^\vee,\,I_Z\otimes L)^\vee\\ &= \mathbb{H}^1(I_W\otimes^{\mathbf{L}}I_Z\otimes L)^\vee.\end{aligned}$$

On the third line, using (4) and (5), we have set

$$L=\mathcal{O}\left((r+s)\sigma+(r+s-a-b)f\right).$$

The orthogonality condition

$$H^2=-rb-sa$$

for the Mukai vectors v and w together with the bound (3) on the dimensions d_v and d_w ensure that $-a-b>r+s$, so the line bundle L is big and nef, without higher cohomology on X.

Thus, under the birational map

$$\Psi_v\times\Psi_w:\ \mathfrak{M}_v\times\mathfrak{M}_w\dashrightarrow X^{[d_v]}\times X^{[d_w]}$$

the two theta divisors

$$\Theta=\{(E,\,F):\ \mathbb{H}^0(E\otimes^{\mathbf{L}}F)\neq 0\}\subset\mathfrak{M}_v\times\mathfrak{M}_w,$$

and

$$\theta_L=\{(I_Z,\,I_W):\ \mathbb{H}^0(I_Z\otimes^{\mathbf{L}}I_W\otimes L)\neq 0\}\subset X^{[d_v]}\times X^{[d_w]}$$

coincide. The line bundles Θ_w, Θ_v on the two higher-rank moduli spaces and $L^{[d_v]}$, $L^{[d_w]}$ on the two Hilbert schemes are also identified. As explained in Section 3 of [MO], for line bundles L without higher cohomology, θ_L is known to induce an isomorphism

$$H^0(X^{[d_v]},\,L^{[d_v]})^\vee\longrightarrow H^0(X^{[d_w]},\,L^{[d_w]}). \tag{6}$$

Therefore, under the identifications above, Θ also induces the isomorphism of equation (1):

$$\mathsf{D}:H^0(\mathfrak{M}_v,\,\Theta_w)^\vee\longrightarrow H^0(\mathfrak{M}_w,\,\Theta_v).$$

We turn now to the proof that Ψ_v is an isomorphism in codimension 1, which was given for a surface $\pi:X\to\mathbb{P}^1$ with irreducible fibers in [BH], [MOY]. We thus take up the case when the fibration has at least one reducible fiber. We shall explain that the *inverse*

$$\Psi_v^{-1}:X^{[d_v]}\dashrightarrow\mathfrak{M}_v$$

is a regular embedding defined on a subscheme $U\subset X^{[d_v]}$ with $\operatorname{codim}(X^{[d_v]}\setminus U)\geq 2$. The same is then true about Ψ_v on $\mathfrak{M}_v$. Indeed, if this were not the case, as the two moduli spaces are holomorphic symplectic, Ψ_v would at least admit by [H], Section 2.2, an extension $\overline{\Psi}_v$ to a regular embedding defined away from codimension 2 on $\mathfrak{M}_v$. Thus $\overline{\Psi}_v$ would extend over a divisorial locus $D \subset \mathfrak{M}_v$ where the original map Ψ_v is assumed undefined. But then

$$\overline{\Psi}_v(D) \subset X^{[d_v]} \setminus U,$$

a contradiction as the latter has codimension 2 in $X^{[d_v]}$.

We are thus left to analyze the domain of Ψ_v^{-1}. The inverse is a Fourier-Mukai transform whose kernel is a complex $\mathcal{Q}[1]$ over $X \times_{\mathbb{P}^1} X$. We write T for the Fourier-Mukai transform with kernel $\mathcal{Q}$ so that

$$\mathsf{S} \circ \mathsf{T} = [-1], \ \mathsf{T} \circ \mathsf{S} = [-1].$$

We claim that for generic Z, the sheaf

$$M = I_Z \otimes \mathcal{O}(r\sigma - (a - r + 3)f)$$

is WIT_0 for the kernel $\mathcal{Q}$. Its transform is then a stable torsion free sheaf in $\mathfrak{M}_v$, cf. Section 7 of [B]. To prove the claim, we adapt arguments of [B], as follows. On general grounds, cf. Lemma 6.1 in [B], there is a short exact sequence

$$0 \to A \to M \to B \to 0$$

where A is T-WIT_0, while B is T-WIT_1. We prove that $B = 0$, following Lemma 6.4 in [B]. Assuming otherwise, we have $\mathsf{T}(B) \neq 0$, and therefore there exists $x \in X$ and a non-zero morphism

$$\mathsf{T}^1(B) \to \mathbb{C}_x.$$

Note however that

$$\mathbb{C}_x = \mathsf{T}^1(I_x(o)),$$

where I_x is the ideal sheaf of the point x in its fiber, and o denotes the intersection of the fiber through x with the section. In fact, $I_x(o) = \mathsf{S}^0(\mathbb{C}_x)$, by Lemma 6.3.7 of [C]. By Parseval, we now obtain a non-zero morphism

$$M \to B \to I_x(o).$$

This morphism must factor through the restriction of M to the fiber C through x, yielding a non-zero map

$$I_Z|_C \otimes \mathcal{O}(ro) \to I_x(o).$$

Thus it suffices to show

$$\mathrm{Hom}_C(I_Z|_C \otimes \mathcal{O}((r-1)o), I_x) = 0.$$

We prove this is the case for $r \geq 3$ and subschemes Z such that

(i) Z intersects any smooth fiber in at most two points;

(ii) Z intersects any singular fiber in at most one point which is not a node or a cusp (if the fiber is irreducible) or does not lie on at least two irreducible components.

This locus has complement of codimension 2 in the Hilbert scheme of X.

When C is a smooth fiber, $\zeta = Z \cap C$ has length at most equal to 2, by (i). Then

$$I_Z|_C = I_{\zeta/C} \oplus T$$

where T is a torsion sheaf supported at ζ. This can be seen by restricting the ideal sequence of Z to the curve C. In fact, the same statement also holds when C is singular, as Z is subject to (ii). When C is smooth, it suffices therefore to prove

$$\mathrm{Hom}_C(I_{\zeta/C}((r-1)o), I_x) = 0 \iff H^0(\mathcal{O}_C(-(r-1)o + \zeta - x)) = 0.$$

Since for $r \geq 3$ the degree is negative, the conclusion follows. When C is a singular fiber, the scheme $\zeta = Z \cap C$ has length 1. We show

$$\mathrm{Hom}_C(I_{\zeta/C}((r-1)o), \mathcal{O}_C) = 0 \text{ which gives } \mathrm{Hom}_C(I_{\zeta/C}((r-1)o), I_x) = 0.$$

Indeed, by duality, this is the same as proving

$$H^1(I_{\zeta/C}((r-1)o))) = 0.$$

Here we used that the dualizing sheaf of C is trivial. Assume first $\zeta \neq o$. From the exact sequence

$$0 \to I_{\zeta/C}(o) \to I_{\zeta/C}((r-1)o) \to \mathbb{C}_o^{r-2} \to 0$$

we see it suffices to show

$$H^1(I_{\zeta/C}(o)) = 0.$$

Next, from the exact sequence

$$0 \to \mathcal{O}(-o) \to \mathcal{O} \to \mathbb{C}_o \to 0$$

we conclude

$$H^0(\mathcal{O}(-o)) = 0,\ H^1(\mathcal{O}(-o)) = \mathbb{C} \implies H^0(\mathcal{O}(o)) = \mathbb{C},\ H^1(\mathcal{O}(o)) = 0.$$

The exact sequence

$$0 \to I_{\zeta/C}(o) \to \mathcal{O}_C(o) \to \mathbb{C}_\zeta \to 0$$

and the fact that

$$H^0(\mathcal{O}_C(o)) \to \mathbb{C}_\zeta$$

is an isomorphism for $\zeta \neq o$ yield $H^1(I_{\zeta/C}(o)) = 0$, as claimed. The vanishing of higher cohomology also holds for $\zeta = o$ since $H^1(\mathcal{O}((r-2)o)) = 0$. This completes the proof. □

4. **The Verlinde sheaves.** We will reinterpret Theorem 1 as giving an isomorphism of sheaves defined over the divisor $\mathcal{P}_1$ in the moduli space of quasipolarized $K3$s.

4.1. *Construction.* For a fixed integer n, we may consider over $\mathcal{K}_\ell$ the relative Hilbert scheme of n points

$$\pi : \mathcal{X}^{[n]} \to \mathcal{K}_\ell,$$

viewed as the relative moduli stack of rank 1 torsion free sheaves of trivial determinant and second Chern number $-n$.

More generally, to consider spaces of higher rank sheaves as the $K3$ surface varies in moduli, we restrict attention to the open substack

$$\mathcal{K}_\ell^\circ \hookrightarrow \mathcal{K}_\ell$$

where the line bundle $\mathcal{H}$ over the universal surface

$$\pi : \mathcal{X} \to \mathcal{K}_\ell$$

is ample. We construct

$$M[v] \to \mathcal{K}_\ell^\circ,$$

the moduli space of $\mathcal{H}$-semistable sheaves with rank r, determinant $d\mathcal{H}$ and Euler characteristic $a - r$ over the fibers of $\pi : \mathcal{X}^\circ \to \mathcal{K}_\ell^\circ$.

The construction of the theta bundles over $M[v]$ is subtler. To start, let

$$\pi : \mathcal{X}_1^\circ \to \mathcal{K}_{\ell,1}^\circ$$

be the universal family over the moduli stack $\mathcal{K}_{\ell,1}^\circ$ of polarized $K3$s with a marked point. It has a canonical section

$$\sigma : \mathcal{K}_{\ell,1}^\circ \to \mathcal{X}_1^\circ.$$

Let

$$\begin{aligned} \mathcal{V} &= (r-d)\,\mathcal{O} + d\,\mathcal{H} + \alpha\,\mathcal{O}_\sigma, \\ \mathcal{W} &= (s-e)\,\mathcal{O} + e\,\mathcal{H} + \beta\,\mathcal{O}_\sigma, \end{aligned}$$

be classes in the K-theory of $\mathcal{X}_1^\circ$. Over a fixed marked polarized $K3$ surface (X, H, p), they have the Mukai vectors

$$v = r + dH + a[\mathrm{pt}], \ \ w = s + eH + b[\mathrm{pt}],$$

for

$$\alpha = a - r - \frac{dH^2}{2},$$

$$\beta = b - s - \frac{eH^2}{2}.$$

We further denote as

$$\pi_v : M[v]_1 \longrightarrow \mathcal{K}_{\ell,1}^\circ$$

the relative moduli space of semistable sheaves of type v over the fibers of $\pi : \mathcal{X}_1^\circ \to \mathcal{K}_{\ell,1}^\circ$. The class $\mathcal{W}$ induces standardly a determinant line bundle

$$\overline{\Theta}_w \to M[v]_1,$$

via descent from

$$\mathcal{Q} \to M[v]_1,$$

where $\mathcal{Q}$ is an open subscheme of a suitable quot scheme. Explicitly, over $\mathcal{Q}$, we have

$$\overline{\Theta}_w = \det \mathbf{R}p_!(\mathcal{E} \otimes q^\star \mathcal{W})^{-1}$$

for the universal quotient sheaf $\mathcal{E} \to \mathcal{Q} \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{X}_1^\circ$. The fiber of the forgetful map

$$M[v]_1 \;\to\; M[v]$$

over a point $(X, H, E \to X) \in M[v]$ is the surface X. To describe the restriction of $\overline{\Theta}_w$ to this fiber, we let $\Delta \subset X \times X$ be the diagonal and denote by p, q the projections from $X \times X$ to the two factors. Then

$$\overline{\Theta}_w\big|_X = \det \mathbf{R}p_\star\left(q^\star E \otimes ((s-e)\mathcal{O} \oplus q^\star(eH) \oplus \beta\, \mathcal{O}_\Delta)\right)^{-1} = \det E^{-\beta} = H^{-\beta d}.$$

We conclude that the product line bundle

$$\overline{\Theta}_w \otimes \pi_v^\star \mathcal{H}^{\beta d} \text{ on } M[v]_1 \tag{7}$$

restricts trivially to the fibers of the map

$$M[v]_1 \longrightarrow M[v]$$

forgetting the marking. By the seesaw lemma, the product (7) is in fact the pullback to $M[v]_1$ of a line bundle $\Theta_w \to M[v]$:

$$\overline{\Theta}_w \otimes \pi_v^\star \mathcal{H}^{\beta d} = \mathrm{pr}^\star \Theta_w.$$

While the determinant line bundle Θ_w is uniquely defined for a fixed $K3$ surface, over the relative moduli space $M[v]$, Θ_w depends on choice of $\mathcal{H}$, and therefore can be canonically defined only up to tensoring by line bundles pulled back from $\mathcal{K}_\ell^\circ$.

Remark. The same construction gives the theta line bundle on the relative moduli space $\mathcal{SU}_g(r) \longrightarrow M_g$ of semistable rank r bundles with trivial determinant over smooth curves of genus g. They are naturally defined on the basechanged moduli space

$$\mathcal{SU}_{g,1}(r) = \mathcal{SU}_g(r) \times_{M_g} M_{g,1} \longrightarrow M_{g,1},$$

relative to the K-theory class

$$\mathcal{O} + (g-1)\mathcal{O}_\sigma$$

on the universal curve $\mathcal{C} \to M_{g,1}$, and are then seen to be pulled back under the forgetful map

$$\mathcal{SU}_{g,1}(r) \to \mathcal{SU}_g(r).$$

Pushing forward the k-tensor powers of the theta line bundles to M_g, we obtain the Verlinde bundles

$$\mathcal{V}_{r,k} \to M_g.$$

Their first Chern classes remain unknown in general.

4.2. *Global strange duality.* Over $\mathcal{K}_\ell^\circ$ we define now the Verlinde complexes

$$\mathbf{W} = \mathbf{R}\pi_{v\star}\Theta_w, \quad \mathbf{V} = \mathbf{R}\pi_{w\star}\Theta_v. \tag{8}$$

Consider the fiber product

$$\pi : M[v] \times_{\mathcal{K}_\ell^\circ} M[w] \to \mathcal{K}_\ell^\circ,$$

endowed with the canonical Brill-Noether locus,

$$\Theta = \{(X, H, E, F) \text{ so that } \mathbb{H}^0(X, E \otimes^{\mathbf{L}} F) \neq 0\} \subset M[v] \times_{\mathcal{K}_\ell^\circ} M[w]. \tag{9}$$

One expects Θ to be a divisor. This was established in [MOY] when v and w satisfy

$$c_1(v) = c_1(w) = \mathcal{H}.$$

The corresponding line bundle, also denoted for simplicity as Θ, is in any case always defined on the product space, and splits by the seesaw lemma as

$$\Theta \simeq \Theta_w \boxtimes \Theta_v. \tag{10}$$

The above equation is correct up to a line bundle twist

$$\mathcal{T} \to \mathcal{K}_\ell^\circ$$

which will be found explicitly below, and which for now we absorb into any one of the theta bundles. The two line bundles Θ_w and Θ_v are ambiguous up to reverse twistings by a line bundle from $\mathcal{K}_\ell^\circ$,

$$(\Theta_v, \Theta_w) \sim \left(\Theta_v \otimes \pi_w^\star \mathcal{L}, \Theta_w \otimes \pi_v^\star \mathcal{L}^{-1}\right), \text{ for } \mathcal{L} \in \operatorname{Pic} \mathcal{K}_\ell^\circ,$$

while Θ is canonical. Pushing forward the canonical theta line bundle via π, we get

$$\mathbf{R}\pi_\star \Theta \simeq \mathbf{W} \otimes^{\mathbf{L}} \mathbf{V}, \tag{11}$$

and the above ambiguity carries over to the Verlinde complexes $\mathbf{W}$ and $\mathbf{V}$. The divisor (9) then induces a morphism

$$\mathsf{D} : \mathbf{W}^\vee \to \mathbf{V}.$$

In [MOY], also having assumed that

$$\chi(v), \chi(w) \leq 0,$$

we showed that over a Zariski open subset of $\mathcal{K}_\ell^\circ$, the higher cohomology sheaves vanish while $\mathcal{H}^0(\mathsf{D})$ induces an isomorphism between the zeroth cohomology sheaves.

Remark. Even though not necessary for our argument, let us determine the twist $\mathcal{T} \to \mathcal{K}_\ell^\circ$ in the decomposition

$$\Theta = \Theta_w \boxtimes \Theta_v \otimes \mathrm{pr}^\star \mathcal{T} \tag{12}$$

over $M[v] \times_{\mathcal{K}_\ell^\circ} M[w]$, where pr is the projection to $\mathcal{K}_\ell^\circ$. Above, we absorbed this twist into the Verlinde complexes, for the ease of exposition.

First, we may pass to the moduli stack $\mathcal{M}[v]$ and $\mathcal{M}[w]$ of all sheaves over X, without changing the above equations. We let

$$\mathcal{E} \to \mathcal{M}[v]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{X}_1^\circ, \quad \mathcal{F} \to \mathcal{M}[w]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{X}_1^\circ$$

be the universal families of sheaves, and further set, on the same product spaces,

$$\overline{\mathcal{E}} = \mathcal{E} - \mathrm{pr}_2^\star \mathcal{V}, \quad \overline{\mathcal{F}} = \mathcal{F} - \mathrm{pr}_2^\star \mathcal{W}.$$

Considering now the triple product

$$\mathcal{M}[v]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{M}[w]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{X}_1^\circ,$$

we calculate

$$\Theta \otimes \Theta_v^{-1} \otimes \Theta_w^{-1}$$

as the pushforward

$$\left(\det \mathbf{R} p_{12\star}\left(p_{13}^\star \mathcal{E} \otimes^{\mathbf{L}} p_{23}^\star \mathcal{F} - p_{13}^\star \mathcal{E} \otimes^{\mathbf{L}} p_3^\star \mathcal{W} - p_{23}^\star \mathcal{F} \otimes^{\mathbf{L}} p_3^\star \mathcal{V}\right)\right)^{-1} \otimes \mathrm{pr}^\star \mathcal{H}^{-d\beta - e\alpha}$$

$$= \left(\det \mathbf{R} p_{12\star}\left(p_{13}^\star \overline{\mathcal{E}} \otimes^{\mathbf{L}} p_{23}^\star \overline{\mathcal{F}} - p_3^\star (\mathcal{V} \otimes^{\mathbf{L}} \mathcal{W})\right)\right)^{-1} \otimes \mathrm{pr}^\star \mathcal{H}^{-d\beta - e\alpha},$$

where $\mathcal{H} \to \mathcal{K}_{\ell,1}^\circ$ is viewed on $\mathcal{M}[v]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{M}[w]_1$ via pullback by the natural projection

$$\mathrm{pr} : \mathcal{M}[v]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{M}[w]_1 \to \mathcal{K}_{\ell,1}^\circ.$$

We apply Grothendieck-Riemann-Roch to compute

$$\mathrm{ch}\, \mathbf{R} p_{12\star}\left(p_{13}^\star \overline{\mathcal{E}} \otimes^{\mathbf{L}} p_{23}^\star \overline{\mathcal{F}}\right).$$

By construction, $\mathrm{ch}\, \overline{\mathcal{E}}$ and $\mathrm{ch}\, \overline{\mathcal{F}}$ restrict trivially over the fibers of

$$p_{12} : \mathcal{M}[v]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{M}[w]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{X}_1^\circ \to \mathcal{M}[v]_1 \times_{\mathcal{K}_{\ell,1}^\circ} \mathcal{M}[w]_1.$$

The Chern character of the pushforward above is thus supported in codimension 2 or higher, and therefore gives

$$\det \mathbf{R} p_{12\star}\left(p_{13}^\star \overline{\mathcal{E}} \otimes^{\mathbf{L}} p_{23}^\star \overline{\mathcal{F}}\right) = \mathcal{O}.$$

Recalling the morphism $\pi : \mathcal{X}_1^\circ \to \mathcal{K}_{\ell,1}^\circ$ which describes the universal surface, we find that

$$\begin{aligned}\Theta \quad \otimes \quad & \Theta_v^{-1} \otimes \Theta_w^{-1} = \det \mathbf{R} p_{12\star} \left[p_3^\star (\mathcal{V} \otimes^{\mathbf{L}} \mathcal{W}) \right] \otimes \mathrm{pr}^\star \mathcal{H}^{-d\beta - e\alpha} \\ = \quad & \mathrm{pr}^\star \left(\det \mathbf{R}\pi_\star \left(\mathcal{V} \otimes^{\mathbf{L}} \mathcal{W} \right) \otimes \mathcal{H}^{-d\beta - e\alpha} \right) \\ = \quad & \mathrm{pr}^\star \left(\det \mathbf{R}\pi_\star \left[((r-d)\mathcal{O} + d\mathcal{H} + \alpha \mathcal{O}_\sigma) \otimes^{\mathbf{L}} ((s-e)\mathcal{O} + e\mathcal{H} + \beta \mathcal{O}_\sigma) \right] \otimes \mathcal{H}^{-d\beta - e\alpha} \right) \\ = \quad & \mathrm{pr}^\star \left(\lambda^{-(r-d)(s-e)} \otimes (\det \pi_\star \mathcal{H})^{e(r-d)+d(s-e)} \otimes \left(\det \pi_\star \mathcal{H}^2 \right)^{de} \right).\end{aligned}$$

Here, we wrote

$$\lambda = (\det \mathbf{R}\pi_\star \mathcal{O}_{\mathcal{X}})^{-1} \to \mathcal{K}_\ell$$

for the Hodge bundle. This yields the following

Proposition 1. *The twist $\mathcal{T}$ defined by equation* (12) *is given by*

$$\mathcal{T} = \lambda^{-(r-d)(s-e)} \otimes (\det \pi_\star \mathcal{H})^{e(r-d)+d(s-e)} \otimes \left(\det \pi_\star \mathcal{H}^2 \right)^{de}.$$

4.3. *Extensions of the Verlinde sheaves and desiderata.* We now turn our attention to the locus of elliptic $K3$ with section, where the Verlinde sheaves and the isomorphism D can be extended from

$$\mathcal{P}_1^\circ = \mathcal{P}_1 \cap \mathcal{K}_\ell^\circ$$

to all of $\mathcal{P}_1$ by the results of Section 3, as we now explain.

The universal data over $\mathcal{P}_1$ consists of the triple

$$(\mathcal{X}, \mathcal{H}, \mathcal{F}) \to \mathcal{P}_1,$$

where $\mathcal{F}$ denotes the universal fiber class of the elliptic fibration. We consider the line bundle

$$\mathcal{L} = \mathcal{H}^{r+s} \otimes \mathcal{O}(\mathcal{F})^{-(r+s)\,\ell - a - b},$$

which restricts over each (X, H, F) to

$$L = \mathcal{O}((r+s)\sigma + (r+s-a-b)f).$$

In the product of Hilbert schemes we have the universal theta divisor

$$\theta = \{(X, Z, W) : \mathbb{H}^0(X,\ I_Z \otimes^{\mathbf{L}} I_W \otimes \mathcal{L}|_X) \neq 0\} \subset \mathcal{X}^{[d_v]} \times_{\mathcal{P}_1} \mathcal{X}^{[d_w]}.$$

To write the corresponding line bundle, we denote by

$$\mathcal{Z} \subset \mathcal{X}^{[d_v]} \times_{\mathcal{K}_\ell} \mathcal{X}, \quad \mathcal{W} \subset \mathcal{X}^{[d_w]} \times_{\mathcal{K}_\ell} \mathcal{X},$$

the universal subschemes, and set standardly

$$\mathcal{L}^{[d_v]} = \det \mathbf{R} p_\star \left(\mathcal{O}_{\mathcal{Z}} \otimes q^\star \mathcal{L} \right), \ \ \mathcal{L}^{[d_w]} = \det \mathbf{R} p_\star \left(\mathcal{O}_{\mathcal{W}} \otimes q^\star \mathcal{L} \right).$$

From the product

$$\mathcal{X}^{[d_v]} \times_{\mathcal{K}_\ell} \mathcal{X}^{[d_w]} \times_{\mathcal{K}_\ell} \mathcal{X},$$

we calculate

$$\begin{aligned}\theta &= \det\left(\mathbf{R}p_{12\star}\left(p_{13}^\star\mathcal{I}_{\mathcal{Z}}\otimes^{\mathbf{L}}p_{23}^\star\mathcal{I}_{\mathcal{W}}\otimes p_3^\star\mathcal{L}\right)\right)^{-1}\\ &= \det\left(\mathbf{R}p_{12\star}\left(p_{13}^\star(\mathcal{O}-\mathcal{O}_{\mathcal{Z}})\otimes^{\mathbf{L}}p_{23}^\star(\mathcal{O}-\mathcal{O}_{\mathcal{W}})\otimes p_3^\star\mathcal{L}\right)\right)^{-1}\\ &= \mathcal{L}^{[d_v]}\boxtimes\mathcal{L}^{[d_w]}\otimes\pi^\star(\det\pi_\star\mathcal{L})^{-1}\otimes\det\mathbf{R}p_{12\star}\left(p_{13}^\star\mathcal{O}_{\mathcal{Z}}\otimes^{\mathbf{L}}p_{23}^\star\mathcal{O}_{\mathcal{W}}\otimes p_3^\star\mathcal{L}\right)\\ &= \mathcal{L}^{[d_v]}\boxtimes\mathcal{L}^{[d_w]}\otimes\pi^\star(\det\pi_\star\mathcal{L})^{-1}.\end{aligned}$$

On the third line, the last bundle is the determinant of a complex of sheaves supported on the codimension 2 locus of intersecting subschemes in $\mathcal{X}^{[d_v]}\times_{\mathcal{K}_\ell}\mathcal{X}^{[d_w]}$ – thus it is trivial. Lemma 5.1 of [EGL] implies that

$$\pi_\star\mathcal{L}^{[d_v]}=\Lambda^{[d_v]}\pi_\star\mathcal{L},\ \ \pi_\star\mathcal{L}^{[d_w]}=\Lambda^{[d_w]}\pi_\star\mathcal{L}.$$

The higher direct images of the line bundles $\mathcal{L}^{[d_v]},\mathcal{L}^{[d_w]}$ vanish by Theorem 5.2.1 of [Sc1]. We therefore finally have

$$\pi_\star\theta\simeq\Lambda^{d_v}(\pi_\star\mathcal{L})\otimes\Lambda^{d_w}(\pi_\star\mathcal{L})\otimes(\det\pi_\star\mathcal{L})^{-1}\cong\mathbf{W}'\otimes\mathbf{V}'.$$

We set

$$\mathbf{W}'=\pi_\star\mathcal{L}^{[d_v]},\ \mathbf{V}'=\pi_\star\mathcal{L}^{[d_w]}\otimes(\det\pi_\star\mathcal{L})^\vee.$$

As before these sheaves are only defined up to reverse twistings by a line bundle from $\mathcal{P}_1$. The divisor θ induces the duality isomorphism

$$\mathsf{D}':\mathbf{W}'^\vee\to\mathbf{V}'$$

over $\mathcal{P}_1$, which is a global version of (6).

Section 3 shows that the universal relative Fourier-Mukai transform induces a birational map

$$\mathcal{X}^{[d_v]}\times_{\mathcal{P}_1^\circ}\mathcal{X}^{[d_w]}\dashrightarrow M[v]\times_{\mathcal{P}_1^\circ}M[w]$$

regular in codimension 1 over each fiber, such that the divisors θ and Θ are precisely matched. Because of regularity in codimension 1, the pushforward sheaves $\pi_\star\theta$ and $R^0\pi_\star\Theta$ coincide. Therefore

$$\mathbf{W}'\otimes\mathbf{V}'\cong\mathcal{H}^0(\mathbf{W})\otimes\mathcal{H}^0(\mathbf{V})$$

over $\mathcal{P}_1^\circ$. We can furthermore align the line bundle twists inherent in the definition of $\mathbf{W},\mathbf{V},\mathbf{W}',\mathbf{V}'$ so that

$$\mathcal{H}^0(\mathsf{D})=\mathsf{D}'$$

over this locus. We thus extended the Verlinde sheaves from $\mathcal{P}_1^\circ\hookrightarrow\mathcal{P}_1$.

The resolution of the following query will however be of much greater interest.

Question 1. *Is it possible to extend* $\mathbf{W}, \mathbf{V}$ *from*

$$\mathcal{K}_\ell^\circ \hookrightarrow \mathcal{K}_\ell$$

in such a fashion that

$$c_1(\mathbf{W}) = -c_1(\mathbf{V})?$$

Combined with the results of [MOY], this would establish the strange duality conjecture over the entire locus where there is no higher cohomology, since the Baily-Borel compactification of $\mathcal{K}_\ell$ has one dimensional boundary. It would be interesting to investigate whether D is in fact a quasi-isomorphism between the complexes $\mathbf{W}^\vee$ and $\mathbf{V}$.

Regarding the canonical line bundle Θ, it is also natural to wonder

Question 2. *Is the Chern character* $ch(\mathbf{R}\pi_\star\Theta)$ *in the ring generated by the Hodge class* $\lambda = -c_1(R^2\pi_\star\mathcal{O}_{\mathcal{X}^\circ})$ *studied in* [GK]*?*

5. **Acknowledgements.** The authors were supported by NSF grants DMS 1001604, DMS 1001486, DMS 1150675 and by the Sloan Foundation. Correspondence with G. van der Geer on related topics is gratefully acknowledged.

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