## 1. Change of Variables Theorem

**Theorem 1.1** (2 Dimensional Change of Variables Theorem)**.** *Let $V$ and $W$ be bounded open sets in $\mathbb{R}^2$ and $\mathbf{R}: V \to W$ be a $C^1$ map which is one to one and onto. To be more specific, letting $(u,v)$ denote a point in $V$, $\mathbf{R}(u,v) = (X(u,v), Y(u,v))$ where $X$ and $Y$ are functions on $V$. Suppose that $f: W \to \mathbb{R}$ is a continuous bounded function, then*

$$\int\int_W f(x,y)\,dxdy = \int\int_V f(\mathbf{R}(u,v))J(u,v)\,dudv,$$

*where*

$$J(u,v) = \left|\det\begin{bmatrix} \mathbf{R}_u(u,v) \\ \mathbf{R}_v(u,v)\end{bmatrix}\right| = \left|\det\begin{bmatrix} X_u(u,v) & Y_u(u,v) \\ X_v(u,v) & Y_v(u,v)\end{bmatrix}\right|$$

*is called the Jacobian of the transformation* $\mathbf{R}$.

**Proof.** We may view $\mathbf{R}$ as parameterizing a surface in $\mathbb{R}^3$ by setting the $z-$component equal to zero. That is we interpret $\mathbf{R}$ as $\mathbf{R}(u,v) = (X(u,v), Y(u,v), 0)$. With this interpretation, the integral $\int\int_W f(x,y)\,dxdy$ becomes the surface integral $\int\int_W f\,dS$ which may be computed as:

$$\begin{aligned}\int\int_W f\,dS &= \int\int_V f(\mathbf{R}(u,v))|\mathbf{R}_u(u,v)\times\mathbf{R}_v(u,v)|\,dudv \\ &= \int\int_V f(\mathbf{R}(u,v))J(u,v)\,dudv.\end{aligned}$$

This is because

$$\begin{aligned}dS &= |\mathbf{R}_u(u,v)\times\mathbf{R}_v(u,v)|\,dudv \\ &= \left|(0,0,\det\begin{bmatrix} X_u(u,v) & Y_u(u,v) \\ X_v(u,v) & Y_v(u,v)\end{bmatrix}\right|\,dudv = J(u,v)\,dudv.\end{aligned}$$

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**Theorem 1.2** (3 Dimensional Change of Variables Theorem)**.** *Let $V$ and $W$ be bounded open sets in $\mathbb{R}^3$ and $\mathbf{R}: V \to W$ $(\mathbf{R}(u,v,w) = (X(u,v,w), Y(u,v,w), Z(u,v,w))$ be a $C^1$ map which is one to one and onto. Suppose that $f: W \to \mathbb{R}$ is a continuous bounded function, then*

$$\int\int\int_W f(x,y,z)\,dxdydz = \int\int\int_V f(\mathbf{R}(u,v,w))J(u,v,w)\,dudvdw,$$

*where*

$$\begin{aligned}J(u,v,w) &= \left|\det\begin{bmatrix} \mathbf{R}_u(u,v,w) \\ \mathbf{R}_v(u,v,w) \\ \mathbf{R}_w(u,v,w)\end{bmatrix}\right| \\ &= \left|\det\begin{bmatrix} X_u(u,v,w) & Y_u(u,v,w) & Z_u(u,v,w) \\ X_v(u,v,w) & Y_v(u,v,w) & Z_v(u,v,w) \\ X_w(u,v,w) & Y_w(u,v,w) & Z_w(u,v,w)\end{bmatrix}\right|.\end{aligned}$$

*Remark* 1.3. In these theorems we are making the change of variables: $(x,y) = \mathbf{R}(u,v)$ and $(x,y,z) = \mathbf{R}(u,v,w)$ respectively. The content of the theorems may be summarized by the equations

$$dxdy = |\det\mathbf{R}'(u,v)|dudv \text{ and}$$
$$dxdydz = |\det\mathbf{R}'(u,v,w)|dudvdw$$

where

$$\mathbf{R}'(u,v) \equiv \begin{bmatrix} \mathbf{R}_u(u,v) \\ \mathbf{R}_v(u,v) \end{bmatrix} \text{ and } \mathbf{R}'(u,v,w) \equiv \begin{bmatrix} \mathbf{R}_u(u,v,w) \\ \mathbf{R}_v(u,v,w) \\ \mathbf{R}_w(u,v,w) \end{bmatrix}.$$

**Example 1.4** (Polar Coordinates)**.** In polar coordinates we have $x = r\cos\theta$ and $y = r\sin\theta$, that is $(x,y) = \mathbf{R}(r,\theta) \equiv (r\cos\theta, r\sin\theta)$. Therefore

$$\mathbf{R}'(r,\theta) = \begin{bmatrix} \mathbf{R}_r(r,\theta) \\ \mathbf{R}_\theta(r,\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}.$$

So $J(r,\theta) = |r\cos^2\theta - (-r\sin^2\theta)| = r$ and thus

$$dxdy = rdrd\theta.$$

**Example 1.5** (Spherical Coordinates)**.** In spherical coordinates we have $x = r\sin\phi\cos\theta$ and $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, i.e.

$$(x,y,z) = \mathbf{R}(r,\theta,\phi) \equiv (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi).$$

Therefore

$$\mathbf{R}'(r,\theta,\phi) = \begin{bmatrix} \mathbf{R}_r(r,\theta,\phi) \\ \mathbf{R}_\theta(r,\theta,\phi) \\ \mathbf{R}_\phi(r,\theta,\phi) \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \end{bmatrix}.$$

So

$$\begin{aligned} J(r,\theta,\phi) &= \left| \det \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \end{bmatrix} \right| \\ &= r^2 \left| \det \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{bmatrix} \right| \\ &= r^2 \left| \cos\phi \det \begin{bmatrix} -\sin\phi\sin\theta & \sin\phi\cos\theta \\ \cos\phi\cos\theta & \cos\phi\sin\theta \end{bmatrix} - \sin\phi \det \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta \\ -\sin\phi\sin\theta & \sin\phi\cos\theta \end{bmatrix} \right| \\ &= r^2 \left| \cos\phi\,(-\sin\phi\cos\phi) - \sin\phi(\sin^2\phi) \right| = r^2\sin\phi \end{aligned}$$

and hence

$$dxdyz = r^2\sin\phi drd\theta d\phi.$$

## 2. Change of Variables Problems

In the next exercise, evaluate the given double or triple integrals. In these problems $dA = dxdy$ and $dV = dxdydz$.

(1) $\iint_D 3x dxdy$, where $D = \{(x,y) : 1 \leq 2x - y \leq 3,\ -2 \leq x + y \leq 0\}$

(2) $\iint_D x dA$ where $D = \{(x,y) : 1 \leq x(1-y) \leq 2,\ 1 \leq xy \leq 3\}$

(3) $\iint_D \left(x^4 - y^4\right) dA$, where $D = \left\{(x,y) : 1 \leq x^2 - y^2 \leq 2,\ 0 \leq xy \leq 1\right\}$

(4) $\iint_D \left(x^2 + y^2\right)^{3/2} dA$, where $D = \left\{(x,y) : 1 \leq x^2 + y^2 \leq 4\right\}$

(5) $\iiint_D (y - z) dV$, where

$$D = \{(x,y,z) : -1 \leq x - z \leq 1,\ 0 \leq y + z \leq 2,\ 1 \leq x + z \leq 3\}.$$

(6) Evaluate

$$\iiint_D \frac{1}{x^2 + y^2 + z^2} dV,$$

where $D$ is the region between the spheres

$$x^2 + y^2 + z^2 = 1 \text{ and } x^2 + y^2 + z^2 = 9.$$

(7) Establish the formula $V = \frac{4\pi}{3} abc$ for the volume of the region enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(8) Evaluate $\iiint_D z^2 dV$ where $D$ is the interior of the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{4} = 1.$$