MATH 20E HANDOUT: DERIVATIVE TESTS

Theorem 1 (First Derivative Test)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a "nice" function. Suppose f has a local maximum or minimum at $R_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, then $\nabla f(R_0) = 0$.*

Proof. Let $R(t) = R_0 + tv$ and set $g_v(t) = f(R_0 + tv)$.Then g has a local maximum (minimum) at $t = 0$ then

$$0 = \dot{g}_v(0) = \nabla f(R_0) \cdot v$$

wherein the last equality we have used the chain rule. Since $v \in \mathbb{R}^3$ is arbitrary, it follows that $\nabla f(R_0) = 0$. ■

Question: How do we decide if a critical point R_0 of f is in fact a local minimum or maximum?

One way to answer this question is to use the single variable second derivative test. For example, suppose that $\ddot{g}_v(0) = \frac{d^2}{dt^2} g(t)|_{t=0}$ is positive, (i.e. $\ddot{g}_v(0) > 0$) for all $v \in \mathbb{R}^3$ then g_v has a local minimum $t = 0$ for all v. That is $f(R_0 + tv) > f(R_0)$ for all $v \in \mathbb{R}^3$ and t small so f has a local minimum at R_0. Similarly if $\ddot{g}_v(0) < 0$ for all $v \in \mathbb{R}^3$ then f has a local maximum. Another possibility is there exists $u, v \in \mathbb{R}^3$ such that $\ddot{g}_v(0) > 0$ while $\ddot{g}_u(0) < 0$. In this case f has a saddle point at R_0.

To make the above tests effective we need to compute $\ddot{g}_v(0)$ explicitly. The results are

$$\dot{g}_v(t) = \nabla f(R_0 + tv) \cdot v = \sum v_i \frac{\partial f}{\partial x_i} (R_0 + tv) \text{ and}$$

$$\ddot{g}_v(t) = \sum_i v_i \left(\nabla \frac{\partial f}{\partial x^i} (R_0 + tv) \right) \cdot v = \sum_{i,j} v_i v_j \frac{\partial^2 f}{\partial x^i \partial x^j} (R_0 + tv).$$

This may be written as

$$\ddot{g}_v(0) = \sum_{i,j} v_i v_j \frac{\partial^2 f}{\partial x_i \partial x_j} (R_0) = v \cdot H v^t$$

where H is the 3×3 matrix of second partial derivatives of f at (x_0, y_0, z_0):

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} (\nabla f)_x \\ (\nabla f)_y \\ (\nabla f)_z \end{pmatrix}.$$

The matrix H is called the *Hessian* matrix of f at (x_0, y_0, z_0). Using some facts from linear algebra we may express the second derivative test described above in terms of three numbers:

$$D_1 = f_{xx},\ D_2 = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx} f_{yy} - f_{xy}^2 \text{ and } D_3 = \det(H).$$

Notice that D_1 is the determinant of the upper left 1×1 submatrix of H, while D_2 is the determinant of the upper left 2×2 submatrix of H, and D_3 is the determinant of all of H. With this notation we may reformulate the second derivative test as follows.

Theorem 2 (Second Derivative Test)**.** *Let H be the Hessian of f at R_0. The test* ***fails*** *(i.e. gives no information) if $D_3 = \det(H) = 0$. So we now assume that $D_3 \neq 0$.*

(1) *If $D_1 > 0$, $D_2 > 0$ and $D_3 > 0$, then f has a local minimum at (x_0, y_0, z_0), i.e. $(+,+,+) \Longrightarrow$ local minimum.*
(2) *If $D_1 < 0$, $D_2 > 0$ and $D_3 < 0$, then f has a local maximum at (x_0, y_0, z_0), i.e. $(-,+,-) \Longrightarrow$ local maximum.*
(3) *In all other cases f has a saddle point at (x_0, y_0, z_0).*

Example 1. Suppose that the Hessian matrix at a critical point of f turns out to be

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $D_1 = 0$, $D_2 = -1$, and $D_3 = -1 = \det H \neq 0$. The second derivative test applies because $\det H \neq 0$. Since we are not in the $(+,+,+)$ or the $(-,+,-)$ cases, the second derivative test tell us we are at a **saddle point**.

Remark 1. Explicitly one may show if $e_1 = \frac{1}{\sqrt{2}}(1,1,0)$, $e_2 = \frac{1}{\sqrt{2}}(1,-1,0)$ and $e_3 = (0,0,1)$ and $v = \sum_{i=1}^{3} v_i e_i$, then $vHv^t = 1v_1^2 - 1v_2^2 + v_3^2$. Indeed if $v = \sum_{i=1}^{3} v_i e_i$, then $v = \left(\frac{1}{\sqrt{2}}(v_1+v_2), \frac{1}{\sqrt{2}}(v_1-v_2), v_3\right)$ and hence

$$v \cdot Hv^t = v \cdot H \begin{pmatrix} \frac{1}{\sqrt{2}}(v_1+v_2) \\ \frac{1}{\sqrt{2}}(v_1+v_2) \\ v_3 \end{pmatrix} = v \cdot \begin{bmatrix} \frac{1}{\sqrt{2}}(v_1+v_2) \\ \frac{1}{\sqrt{2}}(v_1+v_2) \\ v_3 \end{bmatrix}$$
$$= \frac{1}{2}(v_1^2 - v_2^2) + \frac{1}{2}(v_1^2 - v_2^2) + v_3^2 = v_1^2 - v_2^2 + v_3^2$$

So again we see we are at a **saddle point**.

Remark 2 (Important). Keep in mind that the Second Derivative Test can be applied only at a point where the first partial derivatives are all 0.

2. Homework Problems

(1) Let $f(x,y,z) = -2x^2 + 3xy - 3y^2 - x + 12y - z^2$. Determine the critical points of f (i.e., where $\text{grad}(f) = 0$), and at each critical point use the second derivative test to determine whether f has a local minimum, maximum, or saddle point there.

(2) Repeat problem 1 for $f(z,y,z) = x^2 + y^2 + z^2 - xy - 3yz$.