WINTER 2002, DRIVER, MATH 20E PRACTICE FINAL
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1. Let $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and $\mathbf{G} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.

a. Compute **curl F** and **curl G**.

Ans **curl F** $= \mathbf{i} + y\mathbf{j} + (1-z)\mathbf{k}$ and **curl G** $= 0$.

b. Is **F** conservative? If **G** conservative? In your answer, did you make any assumptions about the domain of definition of either of these vector fields?

Ans **F** is not conservative, **G** is conservative. We needed to make sure that the domains of definition are simply connected. Both **F** and **G** are defined everywhere so this isn't a problem.

2. Let C be the circle defined by $y^2 + z^2 = 1, x = 0$.

a. Compute $\int\int_C \mathbf{F} \cdot d\mathbf{R}$ for the **F** in problem 1.

Ans $\pm\pi$. One could use Stokes' Theorem here to simplify the calculations.

b. Compute $\int\int_C \mathbf{G} \cdot d\mathbf{R}$ for the **G** in problem 1.

Ans 0.

3. Let S be the part of the plane $z = 3 - x - y$ which lies over $0 \le x \le \pi/2,\ 0 \le y \le \pi/3$.

a. Compute both $d\mathbf{S}$ and dS, with $d\mathbf{S}$ pointing upward (i.e. $\mathbf{k} \cdot d\mathbf{S} > 0$).

Ans $d\mathbf{S} = \mathbf{i} + \mathbf{j} + \mathbf{k}\ dx\ dy$, $dS = \sqrt{3}\ dx\ dy$.

b. Let $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$. Compute $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

Ans π^2.

c. Why you wouldn't use Stokes' Theorem to evaluate the integral from part b?

Ans The boundary of S is really difficult to describe due to various factors of π appearing in the limits. In order to properly apply Stokes' Theorem, you need to integrate over the boundary of S (written as C below):

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{R}.$$

4. Now let V be the solid tetrahedron bounded by $z = 3 - x - y$ and the coordinate axes $x = 0, y = 0, z = 0$.

a. Graph the shadow this tetrahedron makes in the xy-plane.

Ans You should have graphed the region between $y = 3 - x$, $y = 0$, $x = 3$, and $x = 0$.

b. Write the limits of integration for the triple integral

$$\int\int\int_V dz\ dy\ dx.$$

Ans

$$\int_0^3 \int_0^{3-x} \int_0^{3-y-x} dz\ dy\ dx.$$

c. Evaluate the integral above, i.e. find the volume of V.

Ans 9/2.

d. Now let S be the top face of V, i.e. the part where $z = 3 - x - y$. Set $\mathbf{F} = x\mathbf{i}$. Compute $\int\int_S \mathbf{F} \cdot d\mathbf{S}$. (Hint: Recall that you already computed $d\mathbf{S}$ in Problem 2.)

Ans 9/2.

e. What can you conclude about $\int\int \mathbf{F} \cdot d\mathbf{S}$ over the rest of the faces of V? (Hint: Use the Divergence Theorem.)

Ans Let T be the boundary of the tetrahedron. By the Divergence Theorem, $\int\int_T \mathbf{F} \cdot d\mathbf{S} = \int\int\int_V (\nabla \cdot \mathbf{F})\ dV$. But $\nabla \cdot \mathbf{F} = 1$, so this last quantity was computed to be 9/2 in part c. On the left side, we saw that the integral over the top face gave us 9/2. Thus, the other faces must contribute 0. In other words, $\int\int \mathbf{F} \cdot d\mathbf{S} = 0$, when the integral is taken over the remaining faces.

5. Let V be the set $\{(x, y, z) : 0 \le 3x + 3y + 3z \le 9, 0 \le y + z \le 3, 0 \le 2z \le 6$. We wish to compute

$$\int\int\int_V z^5\ dx\ dy\ dz.$$

a. Give a suitable three-dimensional change of variables which would simplify the integral.

Ans Take $u = x + y + z, v = y + z, w = z$. After computing the Jacobian, we see $du\ dv\ dw = 6dx\ dy\ dz$.

b. Compute the integral with your choice of change of variables.

Ans 9/2.

6. Let $f(x, y, z) = x^2y + 4x + y^2z - z$.

a. Find all critical points for f.

Ans When we set **grad** $f = 0$, we get the points $(-2, 1, -2)$ and $(2, -1, 2)$.

b. Characterize each point found in part a as either a local maximum, a local minimum, or a saddle point.

Ans First, the Hessian is:

$$H = \begin{pmatrix} 2y\ 2x\ 0 \\ 2x\ 2z\ 2y \\ 0\ 2y\ 0 \end{pmatrix}$$

We see that both $(-2, 1, -2)$ and $(2, -1, 2)$ are saddle points from the second derivative test: (+, -, -) and (-, -, +) respectively.

7. Set

$$\mathbf{F} = ln(y^2 + z^2)\mathbf{i} + \left(\frac{2xy}{y^2 + z^2}\right)\mathbf{j} + \left(\frac{2xz}{y^2 + z^2}\right)\mathbf{k}.$$

a. What is the domain of definition of **F**? Is this domain simply connected?

Ans The domain is all (x, y, z) such that either y or z is non-zero. If both y and z are zero then $ln(y^2 + z^2)$ is undefined. This domain is not simply connected. We are missing the entire x-axis.

b. Is **F** conservative? If so, find a potential function for **F**.

Ans **F** is conservative. In fact, we may use the technique on p.206 of the text to obtain the potential function $\varphi = x ln(y^2 + z^2)$. You can check quickly that $\mathbf{F} = \textbf{grad}\ \varphi$.

8. Let $\varphi = e^{xyz}$ and $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

a. Compute $\nabla \cdot \varphi\mathbf{F}$.

Ans $3e^{xyz} + 3xyze^{xyz}$

b. Compute $\nabla \times \varphi\mathbf{F}$.

Ans $\big(x(z^2 - y^2)e^{xyz}\big)\mathbf{i} + \big(y(x^2 - z^2)e^{xyz}\big)\mathbf{j} + \big(z(y^2 - x^2)e^{xyz}\big)\mathbf{k}$.