29. Unbounded operators and quadratic forms

29.1. **Unbounded operator basics.**

Definition 29.1. If X and Y are Banach spaces and D is a subspace of X, then a linear transformation T from D into Y is called a linear transformation (or operator) from X to Y with domain D. We will sometimes wr If D is dense in X, T is said to be *densely defined*.

Notation 29.2. If S and T are operators from X to Y with domains $D(S)$ and $D(T)$ and if $D(S) \subset D(T)$ and $Sx = Tx$ for $x \in D(S)$, then we say T is an *extension* of S and write $S \subset T$.

We note that $X \times Y$ is a Banach space in the norm

$$\|\langle x, y\rangle\| = \sqrt{\|x\|^2 + \|y\|^2}.$$

If H and K are Hilbert spaces, then $H \times K$ and $K \times H$ become Hilbert spaces by defining

$$(\langle x, y\rangle, \langle x', y'\rangle)_{H\times K} := (x, x')_H + (y, y')_K$$

and

$$(\langle y, x\rangle, \langle y', x'\rangle)_{K\times H} := (x, x')_H + (y, y')_K.$$

Definition 29.3. If T is an operator from X to Y with domain D, the *graph*of T is

$$\Gamma(T) := \{\langle x, Dx\rangle : x \in D(T)\} \subset H \times K.$$

Note that $\Gamma(T)$ is a subspace of $X \times Y$.

Definition 29.4. An operator $T : X \to Y$ is *closed*if $\Gamma(T)$ is closed in $X \times Y$.

Remark 29.5. It is easy to see that T is closed iff for all sequences $x_n \in D$ such that there exists $x \in X$ and $y \in Y$ such that $x_n \to x$ and $Tx_n \to y$ implies that $x \in \mathcal{D}$ and $Tx = y$.

Let H be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|v\| := \sqrt{(v, v)}$. As usual we will write H^* for the continuous dual of H and $\overline{H^*}$ for the continuous conjugate linear functionals on H. Our convention will be that $(\cdot, v) \in H^*$ is linear while $(v, \cdot) \in \overline{H^*}$ is conjugate linear for all $v \in H$.

Lemma 29.6. *Suppose that $T : H \to K$ is a densely defined operator between two Hilbert spaces H and K. Then*

1. *T^* is always a closed but not necessarily densely defined operator.*
2. *If T is closable, then $\bar{T}^* = T^*$.*
3. *T is closable iff $T^* : K \to H$ is densely defined.*
4. *If T is closable then $\bar{T} = T^{**}$.*

Proof. Suppose $\{v_n\} \subset D(T)$ is a sequence such that $v_n \to 0$ in H and $Tv_n \to k$ in K as $n \to \infty$. Then for $l \in D(T^*)$, by passing to the limit in the equality, $(Tv_n, l) = (v_n, T^*l)$ we learn $(k, l) = (0, T^*l) = 0$. Hence if T^* is densely defined, this implies $k = 0$ and hence T is closable. This proves one direction in item 3. To prove the other direction and the remaining items of the Lemma it will be useful to express the graph of T^* in terms of the graph of T. We do this now.

Recall that $k \in D(T^*)$ and $T^*k = h$ iff $(k, Tx)_K = (h, x)_H$ for all $x \in D(T)$. This last condition may be written as $(k, y)_K - (h, x)_H = 0$ for all $\langle x, y\rangle \in \Gamma(T)$.

Let $V : H \times K \to K \times H$ be the unitary map defined by $V\langle x, y\rangle = \langle -y, x\rangle$. With this notation, we have $\langle k, h\rangle \in \Gamma(T^*)$ iff $\langle k, h\rangle \perp V\Gamma(T)$, i.e.

$$\Gamma(T^*) = (V\Gamma(T))^\perp = V(\Gamma(T)^\perp), \tag{29.1}$$

where the last equality is a consequence of V being unitary. As a consequence of Eq. (29.1), $\Gamma(T^*)$ is always closed and hence T^* is always a closed operator, and this proves item 1. Moreover if T is closable, then

$$\Gamma(T^*) = V\Gamma(T)^\perp = V\overline{\Gamma(T)}^\perp = V\Gamma(\bar{T})^\perp = \Gamma(\bar{T}^*)$$

which proves item 2.

Now suppose T is closable and $k \perp \mathcal{D}(T^*)$. Then

$$\langle k, 0\rangle \in \Gamma(T^*)^\perp = V\Gamma(T)^{\perp\perp} = V\overline{\Gamma(T)} = V\Gamma(\bar{T}),$$

where $\bar{T}$ denotes the closure of T. This implies that $\langle 0, k\rangle \in \Gamma(\bar{T})$. But $\bar{T}$ is a well defined operator (by the assumption that T is closable) and hence $k = \bar{T}0 = 0$. Hence we have shown $\mathcal{D}(T^*)^\perp = \{0\}$ which implies $\mathcal{D}(T^*)$ is dense in K. This completes the proof of item 3.

4. Now assume T is closable so that T^* is densely defined. Using the obvious analogue of Eq. (29.1) for T^* we learn $\Gamma(T^{**}) = U\Gamma(T^*)^\perp$ where $U\langle y, x\rangle = \langle -x, y\rangle = -V^{-1}\langle y, x\rangle$. Therefore,

$$\Gamma(T^{**}) = UV(\Gamma(T)^\perp)^\perp = -\overline{\Gamma(T)} = \overline{\Gamma(T)} = \Gamma(\bar{T})$$

and hence $\bar{T} = T^{**}$. ■

Lemma 29.7. *Suppose that H and K are Hilbert spaces, $T : H \to K$ is a densely defined operator which has a densely defined adjoint T^*. Then* $\mathrm{Nul}(T^*) = Ran(T)^\perp$ *and* $\mathrm{Nul}(\bar{T}) = Ran(T^*)^\perp$ *where $\bar{T}$ denotes the closure of T.*

Proof. Suppose that $k \in \mathrm{Nul}(T^*)$ and $h \in \mathcal{D}(T)$, then $(k, Th) = (T^*k, h) = 0$. Since $h \in \mathcal{D}(T)$ is arbitrary, this proves that $\mathrm{Nul}(T^*) \subset \mathrm{Ran}(T)^\perp$. Now suppose that $k \in \mathrm{Ran}(T)^\perp$. Then $0 = (k, Th)$ for all $h \in D(T)$. This shows that $k \in \mathcal{D}(T^*)$ and that $T^*k = 0$. The assertion $\mathrm{Nul}(\bar{T}) = \mathrm{Ran}(T^*)^\perp$ follows by replacing T by T^* in the equality, $\mathrm{Nul}(T^*) = \mathrm{Ran}(T)^\perp$. ■

Definition 29.8. A quadratic form q on H is a dense subspace $\mathcal{D}(q) \subset H$ called the domain of q and a sesquilinear form $q : \mathcal{D}(q) \times \mathcal{D}(q) \to \mathbb{C}$. (**Sesquilinear** means that $q(\cdot, v)$ is linear while $q(v, \cdot)$ is conjugate linear on $\mathcal{D}(q)$ for all $v \in \mathcal{D}(q)$.) The form q is **symmetric** if $q(v, w) = \overline{q(w, v)}$ for all $v, w \in \mathcal{D}(q)$, q is **positive** if $q(v) \geq 0$ (here $q(v) = q(v, v)$) for all $v \in \mathcal{D}(q)$, and q is **semi-bounded** if there exists $M_0 \in (0, \infty)$ such that $q(v, v) \geq -M_0\|v\|^2$ for all $v \in \mathcal{D}(q)$.

29.2. **Lax-Milgram Methods.** For the rest of this section q will be a sesquilinear form on H and to simplify notation we will write X for $\mathcal{D}(q)$.

Theorem 29.9 (Lax-Milgram)**.** *Let $q : X \times X \to \mathbb{C}$ be a sesquilinear form and suppose the following added assumptions hold.*

1. *X is equipped with a Hilbertian inner product $(\cdot, \cdot)_X$.*
2. *The form q is **bounded** on X, i.e. there exists a constant $C < \infty$ such that $|q(v, w)| \leq C\|v\|_X \cdot \|w\|_X$ for all $v, w \in X$.*
3. *The form q is **coercive,** i.e. there exists $\epsilon > 0$ such that $|q(v, v)| \geq \epsilon\|v\|_X^2$ for all $v \in X$.*

Then the maps $\mathcal{L} : X \to \overline{X^*}$ and $\mathcal{L}^\dagger : X \to X^*$ defined by $\mathcal{L}v := q(v, \cdot)$ and $\mathcal{L}^\dagger v := q(\cdot, v)$ are linear and (respectively) conjugate linear isomorphisms of Hilbert spaces. Moreover

$$\|\mathcal{L}^{-1}\| \le \epsilon^{-1} \text{ and } \|(\mathcal{L}^\dagger)^{-1}\| \le \epsilon^{-1}.$$

Proof. The operator $\mathcal{L}$ is bounded because

$$\|\mathcal{L}v\|_{X^*} = \sup_{w \neq 0} \frac{|q(v,w)|}{\|w\|_X} \le C\|v\|_X. \tag{29.2}$$

Similarly $\mathcal{L}^\dagger$ is bounded with $\left\|\mathcal{L}^\dagger\right\| \le C$.

Let $\beta : X \to \overline{X^*}$ denote the linear Riesz isomorphism defined by $\beta(x) = (x, \cdot)_X$ for $x \in X$. Define $R := \beta^{-1}\mathcal{L} : X \to X$ so that $\mathcal{L} = \beta R$, i.e.

$$\mathcal{L}v = q(v, \cdot) = (Rv, \cdot)_X \text{ for all } v \in X.$$

Notice that R is a bounded **linear** map with operator bound less than C by Eq. (29.2). Since

$$\left(\mathcal{L}^\dagger v\right)(w) = q(w,v) = (Rw, v)_X = (w, R^*v)_X \text{ for all } v, w \in X,$$

we see that $\mathcal{L}^\dagger v = (\cdot, R^*v)_X$, i.e. $R^* = \bar{\beta}^{-1}\mathcal{L}^\dagger$, where $\bar{\beta}(x) := \overline{(x, \cdot)_X} = (\cdot, x)_X$. Since β and $\bar{\beta}$ are linear and conjugate linear isometric isomorphisms, to finish the proof it suffices to show R is invertible and that $\|R^{-1}\|_X \le \epsilon^{-1}$.

Since

$$|(v, R^*v)_X| = |(Rv, v)_X| = |q(v,v)| \ge \epsilon\|v\|_X^2, \tag{29.3}$$

one easily concludes that $\mathrm{Nul}(R) = \{0\} = \mathrm{Nul}(R^*)$. By Lemma 29.7, $\overline{\mathrm{Ran}(R)} = \mathrm{Nul}(R^*)^\perp = \{0\}^\perp = X$ and so we have shown $R : X \to X$ is injective and has a dense range. From Eq. (29.3) and the Schwarz inequality, $\epsilon\|v\|_X^2 \le \|Rv\|_X\|v\|_X$, i.e.

$$\|Rv\|_X \ge \epsilon\|v\|_X \text{ for all } v \in X. \tag{29.4}$$

This inequality proves the range of R is closed. Indeed if $\{v_n\}$ is a sequence in X such that $Rv_n \to w \in X$ as $n \to \infty$ then Eq. (29.4) implies

$$\epsilon\|v_n - v_m\|_X \le \|Rv_n - Rv_m\|_X \to 0 \text{ as } m, n \to \infty.$$

Thus $v := \lim_{n\to\infty} v_n$ exists in X and hence $w = Rv \in \mathrm{Ran}(R)$ and so $\mathrm{Ran}(R) = \overline{\mathrm{Ran}(R)}^X = X$. So $R : X \to X$ is a bijective map and hence invertible. By replacing v by $R^{-1}v$ in Eq. (29.4) we learn R^{-1} is bounded with operator norm no larger than ϵ^{-1}. ∎

Theorem 29.10. *Let q be a bounded coercive sesquilinear form on X as in Theorem 29.9. Further assume that the inclusion map $i : X \to H$ is bounded and let L and $L^\dagger$ be the unbounded linear operators on H defined by:*

$$\mathcal{D}(L) := \{v \in X : w \in X \to q(v,w) \text{ is } H \text{ - continuous}\},$$
$$\mathcal{D}(L^\dagger) := \{w \in X : v \in X \to q(v,w) \text{ is } H \text{ - continuous}\}$$

and for $v \in \mathcal{D}(L)$ and $w \in \mathcal{D}(L^\dagger)$ define $Lv \in H$ and $L^\dagger w \in H$ by requiring

$$q(v, \cdot) = (Lv, \cdot) \text{ and } q(\cdot, w) = (\cdot, L^\dagger w).$$

Then $\mathcal{D}(L)$ and $\mathcal{D}(L^\dagger)$ are dense subspaces of X and hence of H. The operators $L^{-1} : H \to \mathcal{D}(L) \subset H$ and $(L^\dagger)^{-1} : H \to \mathcal{D}(L^\dagger) \subset H$ are bounded when viewed as *operators from H to H with norms less than or equal to $\epsilon^{-1}\left\|i\right\|^2_{L(X,H)}$. Furthermore, $L^*=L^\dagger$ and $(L^\dagger)^*=L$ and in particular both L and $L^\dagger=L^*$ are closed operators.*

Proof. Let $\alpha:H\to\overline{X^*}$ be defined by $\alpha(v)=(v,\cdot)|_X$. If $(v,\cdot)_X$ is perpendicular to $\alpha(H)=\overline{i^*\left(H^*\right)}\subset\overline{X^*}$, then

$$0=((v,\cdot)_X,\alpha(w))_{\overline{X^*}}=((v,\cdot)_X,(w,\cdot))_{\overline{X^*}}=(v,w) \text{ for all } w\in H.$$

Taking $w=v$ in this equation shows $v=0$ and hence the orthogonal complement of $\alpha(H)$ in $\overline{X^*}$ is $\{0\}$ which implies $\alpha(H)=\overline{i^*\left(H^*\right)}$ is dense in $\overline{X^*}$.

Using the notation in Theorem 29.9, we have $v\in\mathcal{D}(L)$ iff $\mathcal{L}v\in\overline{i^*\left(H^*\right)}=\alpha(H)$ iff $v\in\mathcal{L}^{-1}\left(\alpha(H)\right)$ and for $v\in\mathcal{D}(L)$, $\mathcal{L}v=(Lv,\cdot)|_X=\alpha(Lv)$. This and a similar computation shows

$$\mathcal{D}(L)=\mathcal{L}^{-1}(\overline{i^*\left(H^*\right)})=\mathcal{L}^{-1}\left(\alpha(H)\right) \text{ and } \mathcal{D}(L^\dagger):=(\mathcal{L}^\dagger)^{-1}(i^*\left(H^*\right))=(\mathcal{L}^\dagger)^{-1}(\bar{\alpha}(H))$$

and for $v\in\mathcal{D}(L)$ and $w\in\mathcal{D}(L^\dagger)$ we have $\mathcal{L}v=(Lv,\cdot)|_X=\alpha(Lv)$ and $\mathcal{L}^\dagger w=(\cdot,L^\dagger w)|_X=\bar{\alpha}(L^\dagger w)$. The following commutative diagrams summarizes the relationships of L and $\mathcal{L}$ and $L^\dagger$ and $\mathcal{L}^\dagger$,

$$\begin{array}{ccccc} & X & \overset{\mathcal{L}}{\to} & \overline{X^*} & \\ i & \uparrow & & \uparrow & \alpha \\ & D(L) & \overset{L}{\to} & H & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & X & \overset{\mathcal{L}^\dagger}{\to} & X^* & \\ i & \uparrow & & \uparrow & \bar{\alpha} \\ & D(L^\dagger) & \overset{L^\dagger}{\to} & H & \end{array}$$

where in each diagram i denotes an inclusion map. Because $\mathcal{L}$ and $\mathcal{L}^\dagger$ are invertible, $L:D(L)\to H$ and $L^\dagger:D(L^\dagger)\to H$ are invertible as well. Because both $\mathcal{L}$ and $\mathcal{L}^\dagger$ are isomorphisms of X onto $\overline{X^*}$ and X^* respectively and $\alpha(H)$ is dense in $\overline{X^*}$ and $\bar{\alpha}(H)$ is dense in X^*, the spaces $\mathcal{D}(L)$ and $\mathcal{D}(L^\dagger)$ are dense subspaces of X, and hence also of H.

For the norm bound assertions let $v\in\mathcal{D}(L)\subset X$ and use the coercivity estimate on q to find

$$\begin{aligned}\epsilon\|v\|_H^2 &\le \epsilon\left\|i\right\|^2_{L(X,H)}\|v\|_X^2\le\left\|i\right\|^2_{L(X,H)}|q(v,v)|=\left\|i\right\|^2_{L(X,H)}|(Lv,v)_H|\\ &\le\left\|i\right\|^2_{L(X,H)}\|Lv\|_H\|v\|_H.\end{aligned}$$

Hence $\epsilon\|v\|_H\le\left\|i\right\|^2_{L(X,H)}\|Lv\|_H$ for all $v\in\mathcal{D}(L)$. By replacing v by $L^{-1}v$ (for $v\in H$) in this last inequality, we find

$$\|L^{-1}v\|_H\le\frac{\|i\|^2_{L(X,H)}}{\epsilon}\|v\|_H, \text{ i..e } \|L^{-1}\|_{B(H)}\le\epsilon^{-1}\left\|i\right\|^2_{L(X,H)}.$$

Similarly one shows that $\|(L^\dagger)^{-1}\|_{B(H)}\le\epsilon^{-1}\left\|i\right\|^2_{L(X,H)}$ as well.

For $v\in\mathcal{D}(L)$ and $w\in\mathcal{D}(L^\dagger)$,

$$(Lv,w)=q(v,w)=(v,L^\dagger w) \tag{29.5}$$

which shows $L^\dagger\subset L^*$. Now suppose that $w\in\mathcal{D}(L^*)$, then

$$q(v,w)=(Lv,w)=(v,L^*w) \text{ for all } v\in\mathcal{D}(L).$$

By continuity if follows that

$$q(v,w)=(v,L^*w) \text{ for all } v\in X$$

and therefore by the definition of $L^\dagger$, $w\in\mathcal{D}(L^\dagger)$ and $L^\dagger w=L^*w$, i.e. $L^*\subset L^\dagger$. Since we have shown $L^\dagger\subset L^*$ and $L^*\subset L^\dagger$, $L^\dagger=L^*$. A similar argument shows that

$\left(L^{\dagger}\right)^{*}=L$. Because the adjoints of operators are always closed, both $L=\left(L^{\dagger}\right)^{*}$ and $L^{\dagger}=L^{*}$ are closed operators. ■

Corollary 29.11. *If q in Theorem 29.10 is further assumed to be symmetric then L is self-adjoint, i.e. $L^{*}=L$.*

Proof. This simply follows from Theorem 29.10 upon observing that $L=L^{\dagger}$ when q is symmetric. ■

29.3. **Close, symmetric, semi-bounded quadratic forms and self-adjoint operators.**

Definition 29.12. A symmetric, sesquilinear quadratic form $q: X\times X\to\mathbb{C}$ is **closed** if whenever $\{v_n\}_{n=1}^{\infty}\subset X$ is a sequence such that $v_n\to v$ in H and

$$q(v_n-v_m):=q(v_n-v_m,v_n-v_m)\to 0 \text{ as } m,n\to\infty$$

implies that $v\in X$ and $\lim_{n\to\infty}q(v-v_n)=0$. The form q is said to be **closable** iff for all $\{v_n\}\subset X$ such that $v_n\to 0\in H$ and $q(v_n-v_m)\to 0$ as $m,n\to\infty$ implies that $q(v_n)\to 0$ as $n\to\infty$.

Example 29.13. Let H and K be Hilbert spaces and $T: H\to K$ be a densely defined operator. Set $q(v,w):=(Tv,Tw)_K$ for $v,w\in X:=\mathcal{D}(q):=\mathcal{D}(T)$. Then q is a positive symmetric quadratic form on H which is closed iff T is closed and is closable iff T is closable.

For the remainder of this section let $q: X\times X\to\mathbb{C}$ be a symmetric, sesquilinear quadratic form which is semi-bounded and satisfies $q(v)\geq -M_0\|v\|^2$ for all $v\in X$ and some $M_0<\infty$.

Notation 29.14. For $v,w\in X$ and $M>M_0$ let $(v,w)_M:=q(v,w)+M(v,w)$. Notice that

$$\begin{aligned}\|v\|_M^2&=q(v)+M\|v\|^2=q(v)+M_0\|v\|^2+(M-M_0)\|v\|^2\\&\geq (M-M_0)\|v\|^2,\end{aligned}\tag{29.6}$$

from which it follows that $(\cdot,\cdot)_M$ is an inner product on X and $i: X\to H$ is bounded by $(M-M_0)^{-1/2}$. Let H_M denote the Hilbert space completion of $(X,(\cdot,\cdot)_M)$.

Formally, $H_M=\mathcal{C}/\sim$, where $\mathcal{C}$ denotes the collection of $\|\cdot\|_M$–Cauchy sequences in X and $\sim$ is the equivalence relation, $\{v_n\}\sim\{u_n\}$ iff $\lim_{n\to\infty}\|v_n-u_n\|_M=0$. For $v\in X$, let $i(v)$ be the equivalence class of the constant sequence with elements v. Notice that if $\{v_n\}$ and $\{u_n\}$ are in $\mathcal{C}$, then $\lim_{m,n\to\infty}(v_n,u_m)_M$ exists. Indeed, let C be a finite upper bound for $\|u_n\|_M$ and $\|v_n\|_M$. (Why does this bound exist?) Then

$$\begin{aligned}|(v_n,u_m)_M-(v_k,u_l)_M|&=|(v_n-v_k,u_m)_M+(v_k,u_m-u_l)_M|\\&\leq C\{\|v_n-v_k\|_M+\|u_m-u_l\|_M\}\end{aligned}\tag{29.7}$$

and this last expression tends to zero as $m,n,k,l\to\infty$. Therefore, if $\bar{v}$ and $\bar{u}$ denote the equivalence class of $\{v_n\}$ and $\{u_n\}$ in $\mathcal{C}$ respectively, we may define $(\bar{v},\bar{u})_M:=\lim_{m,n\to\infty}(v_n,u_m)_M$. It is easily checked that H_M with this inner product is a Hilbert space and that $i: X\to H_M$ is an isometry.

Remark 29.15. The reader should verify that all of the norms, $\{\|\cdot\|_M : M>M_0\}$, on X are equivalent so that H_M is independent of $M>M_0$.

Lemma 29.16. *The inclusion map $i : X \to H$ extends by continuity to a continuous linear map $\hat{\imath}$ from H_M into H. Similarly, the quadratic form $q : X \times X \to \mathbb{C}$ extends by continuity to a continuous quadratic form $\hat{q} : H_M \times H_M \to \mathbb{C}$. Explicitly, if $\bar{v}$ and $\bar{u}$ denote the equivalence class of $\{v_n\}$ and $\{u_n\}$ in $\mathcal{C}$ respectively, then $\hat{\imath}(\bar{v}) = H - \lim_{n\to\infty} v_n$ and $\hat{q}(\bar{v}, \bar{u}) = \lim_{m,n\to\infty} q(v_n, u_n)$.*

Proof. This routine verification is left to the reader. ■

Lemma 29.17. *Let q be as above and $M > M_0$ be given.*

(1) *The quadratic form q is closed iff $(X, (\cdot, \cdot)_M)$ is a Hilbert space.*
(2) *The quadratic form q is closable iff the map $\hat{\imath} : H_M \to H$ is injective. In this case we identify H_M with $\hat{\imath}(H_M) \subset H$ and therefore we may view $\hat{q}$ as a quadratic form on H. The form $\hat{q}$ is called the* ***closure*** *of q and as the notation suggests is a closed quadratic form on H.*

A more explicit description of $\hat{q}$ is as follows. The domain $\mathcal{D}(\hat{q})$ consists of those $v \in H$ such that there exists $\{v_n\} \subset X$ such that $v_n \to v$ in H and $q(v_n - v_m) \to 0$ as $m, n \to \infty$. If $v, w \in \mathcal{D}(\hat{q})$ and $v_n \to v$ and $w_n \to w$ as just described, then $\hat{q}(v, w) := \lim_{n\to\infty} q(v_n, w_n)$.

Proof. 1. Suppose q is closed and $\{v_n\}_{n=1}^{\infty} \subset X$ is a $\|\cdot\|_M$ – Cauchy sequence. By the inequality in Eq. (29.6), $\{v_n\}_{n=1}^{\infty}$ is $\|\cdot\|_H$ – Cauchy and hence $v := \lim_{n\to\infty} v_n$ exists in H. Moreover,

$$q(v_n - v_m) = \|v_n - v_m\|_M^2 - M\|v_n - v_m\|_H^2 \to 0$$

and therefore, because q is closed, $v \in \mathcal{D}(q) = X$ and $\lim_{n\to\infty} q(v - v_n) = 0$ and hence $\lim_{n\to\infty} \|v_n - v\|_M^2 = 0$. The converse direction is simpler and will be left to the reader.

2. The proof that q is closable iff the map $\hat{\imath} : H_M \to H$ is injective will be complete once the reader verifies that the following assertions are equivalent. 1) $\hat{\imath} : H_1 \to H$ is injective, 2) $\hat{\imath}(\bar{v}) = 0$ implies $\bar{v} = 0$, 3) if $v_n \overset{H}{\to} 0$ and $q(v_n - v_m) \to 0$ as $m, n \to \infty$ implies that $q(v_n) \to 0$ as $n \to \infty$.

By construction H_M equipped with the inner product $(\cdot, \cdot)_M := \hat{q}(\cdot, \cdot) + M(\cdot, \cdot)$ is complete. So by item 1. it follows that $\hat{q}$ is a closed quadratic form on H if q is closable. ■

Example 29.18. Suppose $H = L^2([-1, 1])$, $\mathcal{D}(q) = C([-1, 1])$ and $q(f, g) := f(0)\bar{g}(0)$ for all $f, g \in \mathcal{D}(q)$. The form q is not closable. Indeed, let $f_n(x) = (1 + x^2)^{-n}$, then $f_n \to 0 \in L^2$ as $n \to \infty$ and $q(f_n - f_m) = 0$ for all m, n while $q(f_n - 0) = q(f_n) = 1 \nrightarrow 0$ as $n \to \infty$. This example also shows the operator $T : H \to \mathbb{C}$ defined by $\mathcal{D}(T) = C([-1, 1])$ with $Tf = f(0)$ is not closable.

Let us also compute T^* for this example. By definition $\lambda \in D(T^*)$ and $T^*\lambda = f$ iff $(f, g) = \lambda\overline{Tg} = \lambda\overline{g(0)}$ for all $g \in C([-1, 1])$. In particular this implies $(f, g) = 0$ for all $g \in C([-1, 1])$ such that $g(0) = 0$. However these functions are dense in H and therefore we conclude that $f = 0$ and hence $\mathcal{D}(T^*) = \{0\}$!!

Exercise 29.1. Keeping the notation in Example 29.18, show $\overline{\Gamma(T)} = H \times \mathbb{C}$ which is clearly not the graph of a linear operator $S : H \to \mathbb{C}$.

Proposition 29.19. *Suppose that $A : H \to H$ is a densely defined positive symmetric operator, i.e. $(Av, w) = (v, Aw)$ for all $v, w \in \mathcal{D}(A)$ and $(v, Av) \geq 0$ for all $v \in \mathcal{D}(A)$. Define $q_A(v, w) := (v, Aw)$ for $v, w \in \mathcal{D}(A)$. Then q_A is closable and the closure $\hat{q}_A$ is a non-negative, symmetric closed quadratic form on H.*

Proof. Let $(\cdot,\cdot)_1 = (\cdot,\cdot) + q_A(\cdot,\cdot)$ on $\mathcal{D}(A)\times\mathcal{D}(A)$, $v_n \in \mathcal{D}(A)$ such that $H\text{-}\lim_{n\to\infty} v_n = 0$ and

$$q_A(v_n - v_m) = (A(v_n - v_m), (v_n - v_m)) \to 0 \text{ as } m, n \to \infty.$$

Then

$$\limsup_{n\to\infty} q_A(v_n) \le \lim_{n\to\infty} \|v_n\|_1^2 = \lim_{m,n\to\infty} (v_m, v_n)_1 = \lim_{m,n\to\infty} \{(v_m, v_n) + (v_m, Av_n)\} = 0,$$

where the last equality follows by first letting $m \to \infty$ and then $n \to \infty$. Notice that the above limits exist because of Eq. (29.7). ■

Lemma 29.20. *Let A be a positive self-adjoint operator on H and define $q_A(v,w) := (v, Aw)$ for $v, w \in \mathcal{D}(A) = \mathcal{D}(q_A)$. Then q_A is closable and the closure of q_A is*

$$\hat{q}_A(v,w) = (\sqrt{A}v, \sqrt{A}w) \text{ for } v, w \in X := \mathcal{D}(\hat{q}_A) = \mathcal{D}(\sqrt{A}).$$

Proof. Let $\hat{q}(v,w) = (\sqrt{A}v, \sqrt{A}w)$ for $v, w \in X = \mathcal{D}(\sqrt{A})$. Since $\sqrt{A}$ is self-adjoint and hence closed, it follows from Example 29.13 that $\hat{q}$ is closed. Moreover, $\hat{q}$ extends q_A because if $v, w \in \mathcal{D}(A)$, then $v, w \in \mathcal{D}(A) = \mathcal{D}((\sqrt{A})^2)$ and $\hat{q}(v,w) = (\sqrt{A}v, \sqrt{A}w) = (v, Aw) = q_A(v,w)$. Thus to show $\hat{q}$ is the closure of q_A it suffices to show $\mathcal{D}(A)$ is dense in $X = \mathcal{D}(\sqrt{A})$ when equipped with the Hilbertian norm, $\|w\|_1^2 = \|w\|^2 + \hat{q}(w)$.

Let $v \in \mathcal{D}(\sqrt{A})$ and define $v_m := 1_{[0,m]}(A)v$. Then using the spectral theorem along with the dominated convergence theorem one easily shows that $v_m \in X = \mathcal{D}(A)$, $\lim_{m\to\infty} v_m = v$ and $\lim_{m\to\infty} \sqrt{A}v_m = \sqrt{A}v$. But this is equivalent to showing that $\lim_{m\to\infty} \|v - v_m\|_1 = 0$. ■

Theorem 29.21. *Suppose $q : X \times X \to \mathbb{C}$ is a symmetric, closed, semi-bounded (say $q(v,v) \ge -M_0\|v\|^2$) sesquilinear form. Let $L : H \to H$ be the possibly unbounded operator defined by*

$$D(L) := \{v \in X : q(v,\cdot) \text{ is } H - \text{continuous}\}$$

and for $v \in D(L)$ let $Lv \in H$ be the unique element such that $q(v,\cdot) = (Lv,\cdot)|_X$. Then

1. *L is a densely defined self-adjoint operator on H and $L \ge -M_0 I$.*
2. *$D(L)$ is a **form core** for q, i.e. the closure of $D(L)$ is a dense subspace in $(X, \|\cdot\|_M)$. More explicitly, for all $v \in X$ there exists $v_n \in D(L)$ such that $v_n \to v$ in H and $q(v - v_n) \to 0$ as $n \to \infty$.*
3. *For and $M \ge M_0$, $D(q) = D\left(\sqrt{L + MI}\right)$.*
4. *Letting $q_L(v,w) := (Lv, w)$ for all $v, w \in D(L)$, we have q_L is closable and $\hat{q}_L = q$.*

Proof. 1. From Lemma 29.17, $(X, (\cdot,\cdot)_X := (\cdot,\cdot)_M)$ is a Hilbert space for any $M > M_0$. Applying Theorem 29.10 and Corollary 29.11 with q being $(\cdot,\cdot)_X$ gives a self-adjoint operator $L_M : H \to H$ such that

$$D(L_M) := \{v \in X : (v,\cdot)_X \text{ is } H - \text{continuous}\}$$

and for $v \in D(L_M)$,

$$(L_M v, w)_H = (v, w)_X = q(v,w) + M(v,w) \text{ for all } w \in X. \tag{29.8}$$

Since $(v,\cdot)_X$ is $H-$continuous iff $q(v,\cdot)$ is $H-$continuous it follows that $D(L_M) = D(L)$ and moreover Eq. (29.8) is equivalent to

$$((L_M - MI)\,v,w)_H = q(v,w) \text{ for all } w \in X.$$

Hence it follows that $L := L_M - MI$ and so L is self-adjoint. Since $(Lv,v) = q(v,v) \geq -M_0 \|v\|^2$, we see that $L \geq -M_0 I$.

2. The density of $\mathcal{D}(L) = \mathcal{D}(L_M)$ in $(X, (\cdot,\cdot)_M)$ is a direct consequence of Theorem 29.10.

3. For

$$v,w \in \mathcal{D}(Q) := \mathcal{D}\left(\sqrt{L_M}\right) = \mathcal{D}\left(\sqrt{L+MI}\right) = \mathcal{D}\left(\sqrt{L+M_0 I}\right)$$

let $Q(v,w) := \left(\sqrt{L_M}v, \sqrt{L_M}w\right)$. For $v,w \in D(L)$ we have

$$Q(v,w) = (L_M v, w) = (Lv,w) + M\,(v,w) = q(v,w) + M\,(v,w) = (v,w)_M\,.$$

By Lemma 29.20, Q is a closed, non-negative symmetric form on H and $\mathcal{D}(L) = \mathcal{D}\,(L_M)$ is dense in $(\mathcal{D}(Q), Q)$. Hence if $v \in \mathcal{D}(Q)$ there exists $v_n \in \mathcal{D}(L)$ such that $Q(v - v_n) \to 0$ and this implies $q(v_m - v_n) \to 0$ as $m,n \to \infty$. Since q is closed, this implies $v \in \mathcal{D}(q)$ and furthermore that $Q(v,w) = (v,w)_M$ for all $v,w \in \mathcal{D}(Q)$.

Conversely, by item 2., if $v \in X = \mathcal{D}(q)$, there exists $v_n \in \mathcal{D}(L)$ such that $\|v - v_m\|_M \to 0$. From this it follows that $Q\,(v_m - v_n) \to 0$ as $m,n \to \infty$ and therefore since Q is closed, $v \in \mathcal{D}(Q)$ and again $Q(v,w) = (v,w)_M$ for all $v,w \in \mathcal{D}(q)$. This proves item 3. and also shows that

$$q(v,w) = \left(\sqrt{L+MI}v, \sqrt{L+MI}w\right) - M(v,w) \text{ for all } v,w \in X = \mathcal{D}\left(\sqrt{L_M}\right).$$

4. Since $q_L \subset q$, q_L is closable and the closure of q_L is still contained in q. Since $q_L = Q - L\,(\cdot,\cdot)$ on $D(L)$ and the closure of $Q|_{D(L)} = (\cdot,\cdot)_M$, it is easy to conclude that the closure of q_L is q as well. ■

Notation 29.22. Let $\mathcal{P}$ denote the collection of positive self-adjoint operators on H and $\mathcal{Q}$ denote the collection of positive and closed symmetric forms on H.

Theorem 29.23. *The map $A \in \mathcal{P} \to \hat{q}_A \in \mathcal{Q}$ is bijective, where $\hat{q}_A(v,w) := (\sqrt{A}v, \sqrt{A}w)$ with $\mathcal{D}(\hat{q}_A) = \mathcal{D}(\sqrt{A})$ is the closure of the quadratic form $q_A(v,w) := (Av,w)$ for $v,w \in \mathcal{D}\,(q) := \mathcal{D}(A)$. The inverse map is given by $q \in \mathcal{Q} \to A_q \in \mathcal{P}$ where A_q is uniquely determined by*

$$\mathcal{D}(A_q) = \{v \in \mathcal{D}(q) : q(v,\cdot) \text{ is } H \text{ - continuous}\} \text{ and}$$
$$(A_q v, w) = q(v,w) \text{ for } v \in \mathcal{D}(A_q) \text{ and } w \in \mathcal{D}(q).$$

Proof. From Lemma 29.20, $\hat{q}_A \in \mathcal{Q}$ and $\hat{q}_A$ is the closure of q_A. From Theorem 29.21 $A_q \in \mathcal{P}$ and

$$q\,(\cdot,\cdot) = \left(\sqrt{A_q}\cdot, \sqrt{A_q}\cdot\right) = \hat{q}_{A_q}.$$

So to finish the proof it suffices to show $A \in \mathcal{P} \to \hat{q}_A \in \mathcal{Q}$ is injective. However, again by Theorem 29.21, if $q \in \mathcal{Q}$ and $A \in \mathcal{P}$ such that $q = \hat{q}_A$, then $v \in \mathcal{D}(A_q)$ and $A_q v = w$ iff

$$(\sqrt{A}v, \sqrt{A}\cdot) = q(v,\cdot) = (A_q v, \cdot)|_X.$$

But this implies $\sqrt{A}v \in \mathcal{D}\left(\sqrt{A}\right)$ and $A_q v = \sqrt{A}\sqrt{A}v = Av$. But by the spectral theorem, $D\left(\sqrt{A}\sqrt{A}\right) = D(A)$ and so we have proved $A_q = A$. ■

29.4. **Construction of positive self-adjoint operators.** The main theorem concerning closed symmetric semi-bounded quadratic forms q is Friederich's extension theorem.

Corollary 29.24 (The Friederich's extension)**.** *Suppose that $A : H \to H$ is a densely defined positive symmetric operator. Then A has a positive self-adjoint extension $\hat{A}$. Moreover, $\hat{A}$ is the only self-adjoint extension of A such that $\mathcal{D}(\hat{A}) \subset \mathcal{D}(\hat{q}_A)$.*

Proof. By Proposition 29.19, $q := \hat{q}_A$ exists in $\mathcal{Q}$. By Theorem 29.23, there exists a unique positive self-adjoint operator B on H such that $\hat{q}_B = q$. Since for $v \in \mathcal{D}(A)$, $q(v,w) = (Av, w)$ for all $w \in X$, it follows from Eq. (G.66) and (G.67) that $v \in \mathcal{D}(B)$ and $Bv = Av$. Therefore $\hat{A} := B$ is a self-adjoint extension of A.

Suppose that C is another self-adjoint extension of A such that $\mathcal{D}(C) \subset X$. Then $\hat{q}_C$ is a closed extension of q_A. Thus $q = \hat{q}_A \subset \hat{q}_C$, i.e. $\mathcal{D}(\hat{q}_A) \subset \mathcal{D}(\hat{q}_C)$ and $\hat{q}_A = \hat{q}_C$ on $\mathcal{D}(\hat{q}_A) \times \mathcal{D}(\hat{q}_A)$. For $v \in \mathcal{D}(C)$ and $w \in \mathcal{D}(A)$, we have that

$$\hat{q}_C(v,w) = (Cv, w) = (v, Cw) = (v, Aw) = (v, Bw) = q(v,w).$$

By continuity it follows that

$$\hat{q}_C(v,w) = (Cv, w) = (v, Bw) = q(v,w)$$

for all $w \in \mathcal{D}(B)$. Therefore, $v \in \mathcal{D}(B^*) = \mathcal{D}(B)$ and $Bv = B^*v = Cv$. That is $C \subset B$. Taking adjoints of this equation shows that $B = B^* \subset C^* = C$. Thus $C = B$. ■

Corollary 29.25 (von Neumann)**.** *Suppose that $D : H \to K$ is a closed operator, then $A = D^*D$ is a positive self-adjoint operator on H.*

Proof. The operator D^* is densely defined by Lemma 29.6. The quadratic form $q(v,w) := (Dv, Dw)$ for $v, w \in X := \mathcal{D}(D)$ is closed (Example 29.13) and positive. Hence by Theorem 29.23 there exists an $A \in \mathcal{P}$ such that $q = \hat{q}_A$, i.e.

$$(Dv, Dw) = \left(\sqrt{A}v, \sqrt{A}w\right) \text{ for all } v, w \in X = \mathcal{D}(D) = \mathcal{D}(\sqrt{A}). \tag{29.9}$$

Recalling that $v \in D(A) \subset X$ and $Av = g$ happens iff

$$(Dv, Dw) = q(v,w) = (g, w) \text{ for all } w \in X$$

and this happens iff $Dv \in D(D^*)$ and $D^*Dv = g$. Thus we have shown $D^*D = A$ which is self-adjoint and positive. ■

29.5. **Applications to partial differential equations.** Let $U \subset \mathbb{R}^n$ be an open set, $\rho \in C^1(U \to (0,\infty))$ and for $i, j = 1, 2, \ldots, n$ let $a_{ij} \in C^1(U, \mathbb{R})$. Take $H = L^2(U, \rho dx)$ and define

$$q(f,g) := \int_U \sum_{i,j=1}^n a_{ij}(x) \partial_i f(x) \partial_j g(x) \, \rho(x) \, dx$$

for $f, g \in X = C_c^2(U)$.

Proposition 29.26. *Suppose that $a_{ij} = a_{ji}$ and that $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0$ for all $\xi \in \mathbb{R}^n$. Then q is a symmetric closable quadratic form on H. Hence there exists a* *unique self-adjoint operator $\hat{A}$ on H such that $\hat{q} = \hat{q}_{\hat{A}}$. Moreover $\hat{A}$ is an extension of the operator*

$$Af(x) = -\frac{1}{\rho(x)} \sum_{i,j=1}^{n} \partial_j(\rho(x) a_{ij}(x) \partial_i f(x))$$

for $f \in \mathcal{D}(A) = C_c^2(U)$.

Proof. A simple integration by parts argument shows that $q(f, g) = (Af, g)_H = (f, Ag)_H$ for all $f, g \in \mathcal{D}(A) = C_c^2(U)$. Thus by Proposition 29.19, q is closable. The existence of $\hat{A}$ is a result of Theorem 29.23. In fact $\hat{A}$ is the Friederich's extension of A as in Corollary 29.24. ■

Given the above proposition and the spectral theorem, we now know that (at least in some weak sense) we may solve the general heat and wave equations: $u_t = -Au$ for $t \geq 0$ and $u_{tt} = -Au$ for $t \in \mathbb{R}$. Namely, we will take

$$u(t, \cdot) := e^{-t\hat{A}} u(0, \cdot)$$

and

$$u(t, \cdot) = \cos(t\sqrt{\hat{A}}) u(0, \cdot) + \frac{\sin(t\sqrt{\hat{A}})}{\sqrt{\hat{A}}} u_t(0, \cdot)$$

respectively. In order to get classical solutions to the equations we would have to better understand the operator $\hat{A}$ and in particular its domain and the domains of the powers of $\hat{A}$. This will be one of the topics of the next part of the course dealing with Sobolev spaces.

Remark 29.27. By choosing $\mathcal{D}(A) = C_c^2(U)$ we are essentially using Dirichlet boundary conditions for A and $\hat{A}$. If U is a bounded region with C^2–boundary, we could have chosen (for example VERIFY THIS EXAMPLE)

$$\mathcal{D}(A) = \{f \in C^2(U) \cap C^1(\bar{U}) : \text{ with } \partial u/\partial n = 0 \text{ on } \partial U\}.$$

This would correspond to Neumann boundary conditions. Proposition 29.26 would be valid with this domain as well provided we assume that $a_{i,j}$ and ρ are in $C^1(\bar{U})$.

For a second application let $H = L^2(U, \rho dx; \mathbb{R}^N)$ and for $j = 1, 2, \ldots, n$, let $A_j : U \to \mathcal{M}_{N \times N}$ (the $N \times N$ matrices) be a C^1 function. Set $\mathcal{D}(D) := C_c^1(U \to \mathbb{R}^N)$ and for $S \in \mathcal{D}(D)$ let $DS(x) = \sum_{i=1}^n A_i(x) \partial_i S(x)$.

Proposition 29.28 ("Dirac Like Operators")**.** *The operator D on H defined above is closable. Hence $A := D^* \bar{D}$ is a self-adjoint operator on H, where $\bar{D}$ is the closure of D.*

Proof. Again a simple integration by parts argument shows that $\mathcal{D}(D) \subset \mathcal{D}(D^*)$ and that for $S \in \mathcal{D}(D)$,

$$D^* S(x) = \frac{1}{\rho(x)} \sum_{i=1}^{n} -\partial_i(\rho(x) A_i(x) S(x)).$$

In particular D^* is a densely defined operator and hence D is closable. The result now follows from Corollary 29.25. ■