23. Sobolev Spaces

Definition 23.1. For $p \in [1, \infty]$, $k \in \mathbb{N}$ and Ω an open subset of $\mathbb{R}^d$, let

$$W^{k,p}_{loc}(\Omega) := \{f \in L^p(\Omega) : \partial^\alpha f \in L^p_{loc}(\Omega) \text{ (weakly) for all } |\alpha| \le k\},$$

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega) \text{ (weakly) for all } |\alpha| \le k\},$$

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \le k} \|\partial^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p} \text{ if } p < \infty \tag{23.1}$$

and

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^\infty(\Omega)} \text{ if } p = \infty. \tag{23.2}$$

In the special case of $p = 2$, we write $W^{k,2}_{loc}(\Omega) =: H^k_{loc}(\Omega)$ and $W^{k,2}(\Omega) =: H^k(\Omega)$ in which case $\|\cdot\|_{W^{k,2}(\Omega)} = \|\cdot\|_{H^k(\Omega)}$ is a Hilbertian norm associated to the inner product

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha f \cdot \overline{\partial^\alpha g}\, dm. \tag{23.3}$$

Theorem 23.2. *The function, $\|\cdot\|_{W^{k,p}(\Omega)}$, is a norm which makes $W^{k,p}(\Omega)$ into a Banach space.*

Proof. Let $f, g \in W^{k,p}(\Omega)$, then the triangle inequality for the p – norms on $L^p(\Omega)$ and $l^p(\{\alpha : |\alpha| \le k\})$ implies

$$\begin{aligned}
\|f + g\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha| \le k} \|\partial^\alpha f + \partial^\alpha g\|^p_{L^p(\Omega)}\right)^{1/p} \\
&\le \left(\sum_{|\alpha| \le k} \left[\|\partial^\alpha f\|_{L^p(\Omega)} + \|\partial^\alpha g\|_{L^p(\Omega)}\right]^p\right)^{1/p} \\
&\le \left(\sum_{|\alpha| \le k} \|\partial^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p} + \left(\sum_{|\alpha| \le k} \|\partial^\alpha g\|^p_{L^p(\Omega)}\right)^{1/p} \\
&= \|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k,p}(\Omega)}.
\end{aligned}$$

This shows $\|\cdot\|_{W^{k,p}(\Omega)}$ defined in Eq. (23.1) is a norm. We now show completeness.

If $\{f_n\}_{n=1}^\infty \subset W^{k,p}(\Omega)$ is a Cauchy sequence, then $\{\partial^\alpha f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(\Omega)$ for all $|\alpha| \le k$. By the completeness of $L^p(\Omega)$, there exists $g_\alpha \in L^p(\Omega)$ such that $g_\alpha = L^p\text{–}\lim_{n\to\infty} \partial^\alpha f_n$ for all $|\alpha| \le k$. Therefore, for all $\phi \in C_c^\infty(\Omega)$,

$$\langle f, \partial^\alpha \phi\rangle = \lim_{n\to\infty} \langle f_n, \partial^\alpha \phi\rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle \partial^\alpha f_n, \phi\rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle g_\alpha, \phi\rangle.$$

This shows $\partial^\alpha f$ exists weakly and $g_\alpha = \partial^\alpha f$ a.e. This shows $f \in W^{k,p}(\Omega)$ and that $f_n \to f \in W^{k,p}(\Omega)$ as $n \to \infty$. ∎

Example 23.3. Let $u(x) := |x|^{-\alpha}$ for $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$. Then

$$\int_{B(0,R)} |u(x)|^p \, dx = \sigma\left(S^{d-1}\right) \int_0^R \frac{1}{r^{\alpha p}} r^{d-1} dr = \sigma\left(S^{d-1}\right) \int_0^R r^{d-\alpha p-1} dr$$

$$= \sigma\left(S^{d-1}\right) \cdot \begin{cases} \frac{R^{d-\alpha p}}{d-\alpha p} & \text{if} \quad d-\alpha p > 0 \\ \infty & \text{otherwise} \end{cases} \tag{23.4}$$

and hence $u \in L^p_{loc}\left(\mathbb{R}^d\right)$ iff $\alpha < d/p$. Now $\nabla u(x) = -\alpha |x|^{-\alpha-1} \hat{x}$ where $\hat{x} := x/|x|$. Hence if $\nabla u(x)$ is to exist in $L^p_{loc}\left(\mathbb{R}^d\right)$ it is given by $-\alpha |x|^{-\alpha-1} \hat{x}$ which is in $L^p_{loc}\left(\mathbb{R}^d\right)$ iff $\alpha + 1 < d/p$, i.e. if $\alpha < d/p - 1 = \frac{d-p}{p}$. Let us not check that $u \in W^{1,p}_{loc}\left(\mathbb{R}^d\right)$ provided $\alpha < d/p - 1$. To do this suppose $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\epsilon > 0$, then

$$-\langle u, \partial_i \phi \rangle = -\lim_{\epsilon \downarrow 0} \int_{|x|>\epsilon} u(x) \partial_i \phi(x) dx$$
$$= \lim_{\epsilon \downarrow 0} \left\{ \int_{|x|>\epsilon} \partial_i u(x) \phi(x) dx + \int_{|x|=\epsilon} u(x)\phi(x) \frac{x_i}{\epsilon} d\sigma(x) \right\}.$$

Since

$$\left| \int_{|x|=\epsilon} u(x)\phi(x)\frac{x_i}{\epsilon} d\sigma(x) \right| \le \|\phi\|_\infty \sigma\left(S^{d-1}\right) \epsilon^{d-1-\alpha} \to 0 \text{ as } \epsilon \downarrow 0$$

and $\partial_i u(x) = -\alpha |x|^{-\alpha-1} \hat{x} \cdot e_i$ is locally integrable we conclude that

$$-\langle u, \partial_i \phi \rangle = \int_{\mathbb{R}^d} \partial_i u(x) \phi(x) dx$$

showing that the weak derivative $\partial_i u$ exists and is given by the usual pointwise derivative.

23.1. **Mollifications.**

Proposition 23.4 (Mollification). *Let Ω be an open subset of $\mathbb{R}^d$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $p \in [1, \infty)$ and $u \in W^{k,p}_{loc}(\Omega)$. Then there exists $u_n \in C_c^\infty(\Omega)$ such that $u_n \to u$ in $W^{k,p}_{loc}(\Omega)$.*

Proof. Apply Proposition 19.12 with polynomials, $p_\alpha(\xi) = \xi^\alpha$, for $|\alpha| \le k$. ■

Proposition 23.5. *$C_c^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$ for all $1 \le p < \infty$.*

Proof. The proof is similar to the proof of Proposition 23.4 using Exercise 19.2 in place of Proposition 19.12. ■

Proposition 23.6. *Let Ω be an open subset of $\mathbb{R}^d$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $p \ge 1$, then*

1. *for any α with $|\alpha| \le k$, $\partial^\alpha : W^{k,p}(\Omega) \to W^{k-|\alpha|,p}(\Omega)$ is a contraction.*
2. *For any open subset $V \subset \Omega$, the restriction map $u \to u|_V$ is bounded from $W^{k,p}(\Omega) \to W^{k,p}(V)$.*
3. *For any $f \in C^k(\Omega)$ and $u \in W^{k,p}_{loc}(\Omega)$, the $fu \in W^{k,p}_{loc}(\Omega)$ and for $|\alpha| \le k$,*

$$\partial^\alpha (fu) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} u \tag{23.5}$$

where $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

(4) *For any* $f \in BC^k(\Omega)$ *and* $u \in W^{k,p}_{loc}(\Omega)$, *the* $fu \in W^{k,p}_{loc}(\Omega)$ *and for* $|\alpha| \le k$ *Eq. (23.5) still holds. Moreover, the linear map* $u \in W^{k,p}(\Omega) \to fu \in W^{k,p}(\Omega)$ *is a bounded operator.*

Proof. 1. Let $\phi \in C_c^\infty(\Omega)$ and $u \in W^{k,p}(\Omega)$, then for β with $|\beta| \le k - |\alpha|$,

$$\langle \partial^\alpha u, \partial^\beta \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \partial^\beta \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^{\alpha+\beta} \phi \rangle = (-1)^{|\beta|} \langle \partial^{\alpha+\beta} u, \phi \rangle$$

from which it follows that $\partial^\beta(\partial^\alpha u)$ exists weakly and $\partial^\beta(\partial^\alpha u) = \partial^{\alpha+\beta} u$. This shows that $\partial^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and it should be clear that $\|\partial^\alpha u\|_{W^{k-|\alpha|,p}(\Omega)} \le \|u\|_{W^{k,p}(\Omega)}$.

Item 2. is trivial.

3 - 4. Given $u \in W^{k,p}_{loc}(\Omega)$, by Proposition 23.4 there exists $u_n \in C_c^\infty(\Omega)$ such that $u_n \to u$ in $W^{k,p}_{loc}(\Omega)$. From the results in Appendix A.1, $fu_n \in C_c^k(\Omega) \subset W^{k,p}(\Omega)$ and

$$\partial^\alpha (fu_n) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} u_n \tag{23.6}$$

holds. Given $V \subset_o \Omega$ such that $\bar{V}$ is compactly contained in Ω, we may use the above equation to find the estimate

$$\begin{aligned} \|\partial^\alpha (fu_n)\|_{L^p(V)} &\le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left\|\partial^\beta f\right\|_{L^\infty(V)} \left\|\partial^{\alpha-\beta} u_n\right\|_{L^p(V)} \\ &\le C_\alpha(f,V) \sum_{\beta \le \alpha} \left\|\partial^{\alpha-\beta} u_n\right\|_{L^p(V)} \le C_\alpha(f,V) \|u_n\|_{W^{k,p}(V)} \end{aligned}$$

wherein the last equality we have used Exercise 23.1 below. Summing this equation on $|\alpha| \le k$ shows

$$\|fu_n\|_{W^{k,p}(V)} \le C(f,V) \|u_n\|_{W^{k,p}(V)} \text{ for all } n \tag{23.7}$$

where $C(f,V) := \sum_{|\alpha| \le k} C_\alpha(f,V)$. By replacing u_n by $u_n - u_m$ in the above inequality it follows that $\{fu_n\}_{n=1}^\infty$ is convergent in $W^{k,p}(V)$ and since V was arbitrary $fu_n \to fu$ in $W^{k,p}_{loc}(\Omega)$. Moreover, we may pass to the limit in Eq. (23.6) and in Eq. (23.7) to see that Eq. (23.5) holds and that

$$\|fu\|_{W^{k,p}(V)} \le C(f,V) \|u\|_{W^{k,p}(V)} \le C(f,V) \|u\|_{W^{k,p}(\Omega)}$$

Moreover if $f \in BC(\Omega)$ then constant $C(f,V)$ may be chosen to be independent of V and therefore, if $u \in W^{k,p}(\Omega)$ then $fu \in W^{k,p}(\Omega)$.

Alternative direct proof of 4. We will prove this by induction on $|\alpha|$. If $\alpha = e_i$ then, using Lemma 19.9,

$$\begin{aligned} -\langle fu, \partial_i \phi \rangle &= -\langle u, f\partial_i \phi \rangle = -\langle u, \partial_i [f\phi] - \partial_i f \cdot \phi \rangle \\ &= \langle \partial_i u, f\phi \rangle + \langle u, \partial_i f \cdot \phi \rangle = \langle f\partial_i u + \partial_i f \cdot u, \phi \rangle \end{aligned}$$

showing $\partial_i(fu)$ exists weakly and is equal to $\partial_i(fu) = f\partial_i u + \partial_i f \cdot u \in L^p(\Omega)$. Supposing the result has been proved for all α such that $|\alpha| \le m$ with $m \in [1,k)$. Let $\gamma = \alpha + e_i$ with $|\alpha| = m$, then by what we have just proved each summand in Eq. (23.5) satisfies $\partial_i \left[\partial^\beta f \cdot \partial^{\alpha-\beta} u\right]$ exists weakly and

$$\partial_i \left[\partial^\beta f \cdot \partial^{\alpha-\beta} u\right] = \partial^{\beta+e_i} f \cdot \partial^{\alpha-\beta} u + \partial^{\beta_i} f \cdot \partial^{\alpha-\beta+e} u \in L^p(\Omega).$$

Therefore $\partial^\gamma (fu) = \partial_i \partial^\alpha (fu)$ exists weakly in $L^p(\Omega)$ and

$$\partial^\gamma (fu) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left[\partial^{\beta+e_i} f \cdot \partial^{\alpha-\beta} u + \partial^\beta f \cdot \partial^{\alpha-\beta+e_i} u\right] = \sum_{\beta\le\gamma} \binom{\gamma}{\beta} \left[\partial^\beta f \cdot \partial^{\gamma-\beta} u\right].$$

For the last equality see the combinatorics in Appendix A.1. ■

Theorem 23.7. *Let Ω be an open subset of $\mathbb{R}^d$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $p \in [1, \infty)$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $\Omega_n := \{x \in \Omega : \operatorname{dist}(x, \Omega) > 1/n\} \cap B(0, n)$, then

$$\bar{\Omega}_n \subset \{x \in \Omega : \operatorname{dist}(x, \Omega) \ge 1/n\} \cap \overline{B(0, n)} \subset \Omega_{n+1},$$

$\bar{\Omega}_n$ is compact for every n and $\Omega_n \uparrow \Omega$ as $n \to \infty$. Let $V_0 = \Omega_3$, $V_j := \Omega_{j+3} \setminus \bar{\Omega}_j$ for $j \ge 1$, $K_0 := \bar{\Omega}_2$ and $K_j := \bar{\Omega}_{j+2} \setminus \Omega_{j+1}$ for $j \ge 1$ as in figure 41. Then $K_n \sqsubset\sqsubset V_n$

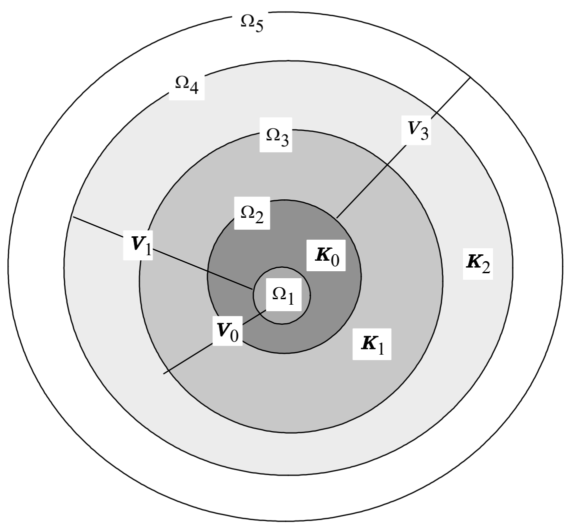

FIGURE 41. Decomposing Ω into compact pieces. The compact sets K_0, K_1 and K_2 are the shaded annular regions while V_0, V_1 and V_2 are the indicated open annular regions.

for all n and $\cup K_n = \Omega$. Choose $\phi_n \in C_c^\infty(V_n, [0, 1])$ such that $\phi_n = 1$ on K_n and set $\psi_0 = \phi_0$ and

$$\psi_j = (1 - \psi_1 - \cdots - \psi_{j-1})\, \phi_j = \phi_j \prod_{k=1}^{j-1} (1 - \phi_k)$$

for $j \ge 1$. Then $\psi_j \in C_c^\infty(V_n, [0, 1])$,

$$1 - \sum_{k=0}^{n} \psi_k = \prod_{k=1}^{n} (1 - \phi_k) \to 0 \text{ as } n \to \infty$$

so that $\sum_{k=0}^\infty \psi_k = 1$ on Ω with the sum being locally finite.

Let $\epsilon > 0$ be given. By Proposition 23.6, $u_n := \psi_n u \in W^{k,p}(\Omega)$ with $\operatorname{supp}(u_n) \sqsubset\sqsubset V_n$. By Proposition 23.4, we may find $v_n \in C_c^\infty(V_n)$ such that

$\|u_n - v_n\|_{W^{k,p}(\Omega)} \le \epsilon/2^{n+1}$ for all n. Let $v := \sum_{n=1}^\infty v_n$, then $v \in C^\infty(\Omega)$ because the sum is locally finite. Since

$$\sum_{n=0}^\infty \|u_n - v_n\|_{W^{k,p}(\Omega)} \le \sum_{n=0}^\infty \epsilon/2^{n+1} = \epsilon < \infty,$$

the sum $\sum_{n=0}^\infty (u_n - v_n)$ converges in $W^{k,p}(\Omega)$. The sum, $\sum_{n=0}^\infty (u_n - v_n)$, also converges pointwise to $u - v$ and hence $u - v = \sum_{n=0}^\infty (u_n - v_n)$ is in $W^{k,p}(\Omega)$. Therefore $v \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ and

$$\|u - v\| \le \sum_{n=0}^\infty \|u_n - v_n\|_{W^{k,p}(\Omega)} \le \epsilon.$$

■

Notation 23.8. Given a closed subset $F \subset \mathbb{R}^d$, let $C^\infty(F)$ denote those $u \in C(F)$ that extend to a C^∞ – function on an open neighborhood of F.

Remark 23.9. It is easy to prove that $u \in C^\infty(F)$ iff there exists $U \in C^\infty(\mathbb{R}^d)$ such that $u = U|_F$. Indeed, suppose Ω is an open neighborhood of F, $f \in C^\infty(\Omega)$ and $u = f|_F \in C^\infty(F)$. Using a partition of unity argument (making use of the open sets V_i constructed in the proof of Theorem 23.7), one may show there exists $\phi \in C^\infty(\Omega, [0,1])$ such that $\operatorname{supp}(\phi) \sqsubset \Omega$ and $\phi = 1$ on a neighborhood of F. Then $U := \phi f$ is the desired function.

Theorem 23.10 (Density of $W^{k,p}(\Omega) \cap C^\infty(\bar\Omega)$ in $W^{k,p}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^d$ be a manifold with C^0 – boundary, then for $k \in \mathbb{N}_0$ and $p \in [1,\infty)$, $W^{k,p}(\Omega^o) \cap C^\infty(\bar\Omega)$ is dense in $W^{k,p}(\Omega^0)$. This may alternatively be stated by assuming $\Omega \subset \mathbb{R}^d$ is an open set such that $\bar\Omega^o = \Omega$ and $\bar\Omega$ is a manifold with C^0 – boundary, then $W^{k,p}(\Omega) \cap C^\infty(\bar\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Before going into the proof, let us point out that some restriction on the boundary of Ω is needed for assertion in Theorem 23.10 to be valid. For example, suppose

$$\Omega_0 := \left\{x \in \mathbb{R}^2 : 1 < |x| < 2\right\} \text{ and } \Omega := \Omega_0 \setminus \{(1,2) \times \{0\}\}$$

and $\theta : \Omega \to (0, 2\pi)$ is defined so that $x_1 = |x| \cos\theta(x)$ and $x_2 = |x| \sin\theta(x)$, see Figure 42. Then $\theta \in BC^\infty(\Omega) \subset W^{k,\infty}(\Omega)$ for all $k \in \mathbb{N}_0$ yet θ can not be

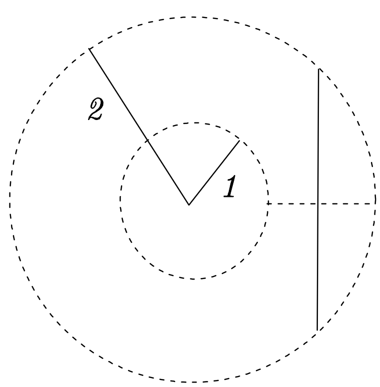

FIGURE 42. The region Ω_0 along with a vertical in Ω.

approximated by functions from $C^\infty(\bar{\Omega}) \subset BC^\infty(\Omega_0)$ in $W^{1,p}(\Omega)$. Indeed, if this were possible, it would follows that $\theta \in W^{1,p}(\Omega_0)$. However, θ is not continuous (and hence not absolutely continuous) on the lines $\{x_1 = \rho\} \cap \Omega$ for all $\rho \in (1, 2)$ and so by Theorem 19.30, $\theta \notin W^{1,p}(\Omega_0)$.

The following is a warm-up to the proof of Theorem 23.10.

Proposition 23.11 (Warm-up). *Let $f : \mathbb{R}^{d-1} \to \mathbb{R}$ be a continuous function and $\Omega := \{x \in \mathbb{R}^d : x_d > f(x_1, \dots, x_{d-1})\}$ and $C^\infty(\bar{\Omega})$ denote those $u \in C(\bar{\Omega})$ which are restrictions of C^∞ – functions defined on an open neighborhood of $\bar{\Omega}$. Then for $p \in [1, \infty)$, $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. By Theorem 23.7, it suffices to show than any $u \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ may be approximated by elements of $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$. For $s > 0$ let $u_s(x) := u(x + se_d)$. Then it is easily seen that $\partial^\alpha u_s = (\partial^\alpha u)_s$ for all α and hence

$$u_s \in W^{k,p}(\Omega - se_d) \cap C^\infty(\Omega - se_d) \subset C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega).$$

These observations along with the strong continuity of translations in L^p (see Proposition 11.13), implies $\lim_{s \downarrow 0} \|u - u_s\|_{W^{k,p}(\Omega)} = 0$. ∎

23.1.1. *Proof of Theorem 23.10.* **Proof.** By Theorem 23.7, it suffices to show than any $u \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ may be approximated by elements of $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$. To understand the main ideas of the proof, suppose that Ω is the triangular region in Figure 43 and suppose that we have used a partition of unity relative to the cover shown so that $u = u_1 + u_2 + u_3$ with $\mathrm{supp}(u_i) \subset B_i$. Now concentrating on

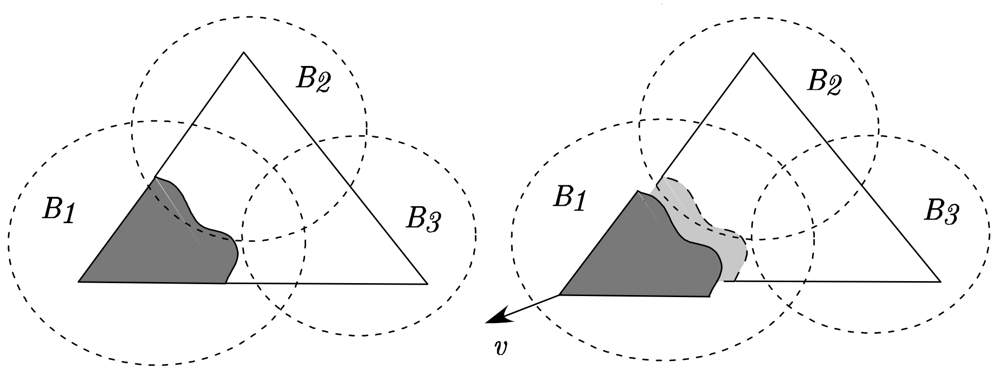

FIGURE 43. Splitting and moving a function in $C^\infty(\Omega)$ so that the result is in $C^\infty(\bar{\Omega})$.

u_1 whose support is depicted as the grey shaded area in Figure 43. We now simply translate u_1 in the direction v shown in Figure 43. That is for any small $s > 0$, let $w_s(x) := u_1(x + sv)$, then v_s lives on the translated grey area as seen in Figure 43. The function w_s extended to be zero off its domain of definition is an element of $C^\infty(\bar{\Omega})$ moreover it is easily seen, using the same methods as in the proof of Proposition 23.11, that $w_s \to u_1$ in $W^{k,p}(\Omega)$.

The formal proof follows along these same lines. To do this choose an at most countable locally finite cover $\{V_i\}_{i=0}^\infty$ of $\bar{\Omega}$ such that $\bar{V}_0 \subset \Omega$ and for each $i \geq 1$, after making an affine change of coordinates, $V_i = (-\epsilon, \epsilon)^d$ for some $\epsilon > 0$ and

$$V_i \cap \bar{\Omega} = \{(y, z) \in V_i : \epsilon > z > f_i(y)\}$$

where $f_i : (-\epsilon, \epsilon)^{d-1} \to (-\epsilon, \epsilon)$, see Figure 44 below. Let $\{\eta_i\}_{i=0}^{\infty}$ be a partition of

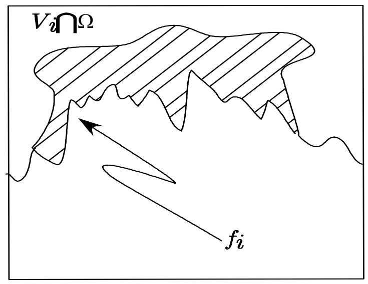

FIGURE 44. The shaded area depicts the support of $u_i = u\eta_i$.

unity subordinated to $\{V_i\}$ and let $u_i := u\eta_i \in C^{\infty}(V_i \cap \Omega)$. Given $\delta > 0$, we choose s so small that $w_i(x) := u_i(x + se_d)$ (extended to be zero off its domain of definition) may be viewed as an element of $C^{\infty}(\bar{\Omega})$ and such that $\|u_i - w_i\|_{W^{k,p}(\Omega)} < \delta/2^i$. For $i = 0$ we set $w_0 := u_0 = u\eta_0$. Then, since $\{V_i\}_{i=1}^{\infty}$ is a locally finite cover of $\bar{\Omega}$, it follows that $w := \sum_{i=0}^{\infty} w_i \in C^{\infty}\left(\bar{\Omega}\right)$ and further we have

$$\sum_{i=0}^{\infty} \|u_i - w_i\|_{W^{k,p}(\Omega)} \le \sum_{i=1}^{\infty} \delta/2^i = \delta.$$

This shows

$$u - w = \sum_{i=0}^{\infty} (u_i - w_i) \in W^{k,p}(\Omega)$$

and $\|u - w\|_{W^{k,p}(\Omega)} < \delta$. Hence $w \in C^{\infty}\left(\bar{\Omega}\right) \cap W^{k,p}(\Omega)$ is a $\delta-$approximation of u and since $\delta > 0$ arbitrary the proof is complete. ■

23.2. **Difference quotients.** Recall from Notation 19.14 that for $h \neq 0$

$$\partial_i^h u(x) := \frac{u(x + he^i) - u(x)}{h}.$$

Remark 23.12 (Adjoints of Finite Differences)*.* For $u \in L^p$ and $g \in L^q$,

$$\int_{\mathbb{R}^d} \partial_i^h u(x)\; g(x)\; dx = \int_{\mathbb{R}^d} \frac{u(x + he_i) - u(x)}{h}\; g(x)\; dx = -\int_{\mathbb{R}^d} u(x)\; \frac{g(x - he_i) - g(x)}{-h}\; dx$$
$$= -\int_{\mathbb{R}^d} u(x) \partial_i^{-h} g(x)\; dx.$$

We summarize this identity by $(\partial_i^h)^* = -\partial_i^{-h}$.

Theorem 23.13. *Suppose $k \in \mathbb{N}_0$, Ω is an open subset of $\mathbb{R}^d$ and V is an open precompact subset of Ω.*

(1) *If $1 \le p < \infty$, $u \in W^{k,p}(\Omega)$ and $\partial_i u \in W^{k,p}(\Omega)$, then*

$$\|\partial_i^h u\|_{W^{k,p}(V)} \le \|\partial_i u\|_{W^{k,p}(\Omega)} \tag{23.8}$$

for all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \Omega^c)$.

(2) *Suppose that* $1 < p \le \infty$, $u \in W^{k,p}(\Omega)$ *and assume there exists a constant* $C(V) < \infty$ *such that*

$$\|\partial_i^h u\|_{W^{k,p}(V)} \le C(V) \text{ for all } 0 < |h| < \frac{1}{2}\mathrm{dist}(V, \Omega^c).$$

Then $\partial_i u \in W^{k,p}(V)$ *and* $\|\partial_i u\|_{W^{k,p}(V)} \le C(V)$. *Moreover if* $C := \sup_{V \subset\subset \Omega} C(V) < \infty$ *then in fact* $\partial_i u \in W^{k,p}(\Omega)$ *and there is a constant* $c < \infty$ *such that*

$$\|\partial_i u\|_{W^{k,p}(\Omega)} \le c \left(C + \|u\|_{L^p(\Omega)} \right).$$

Proof. 1. Let $|\alpha| \le k$, then

$$\|\partial^\alpha \partial_i^h u\|_{L^p(V)} = \|\partial_i^h \partial^\alpha u\|_{L^p(V)} \le \|\partial_i \partial^\alpha u\|_{L^p(\Omega)}$$

wherein we have used Theorem 19.22 for the last inequality. Eq. (23.8) now easily follows.

2. If $\|\partial_i^h u\|_{W^{k,p}(V)} \le C(V)$ then for all $|\alpha| \le k$,

$$\|\partial_i^h \partial^\alpha u\|_{L^p(V)} = \|\partial^\alpha \partial_i^h u\|_{L^p(V)} \le C(V).$$

So by Theorem 19.22, $\partial_i \partial^\alpha u \in L^p(V)$ and $\|\partial_i \partial^\alpha u\|_{L^p(V)} \le C(V)$. From this we conclude that $\|\partial^\beta u\|_{L^p(V)} \le C(V)$ for all $0 < |\beta| \le k+1$ and hence $\|u\|_{W^{k+1,p}(V)} \le c\left[C(V) + \|u\|_{L^p(V)}\right]$ for some constant c. ■

Notation 23.14. Given a multi-index α and $h \ne 0$, let

$$\partial_h^\alpha := \prod_{i=1}^d \left(\partial_i^h\right)^{\alpha_i}.$$

The following theorem is a generalization of Theorem 23.13.

Theorem 23.15. *Suppose* $k \in \mathbb{N}_0$, Ω *is an open subset of* $\mathbb{R}^d$, V *is an open precompact subset of* Ω *and* $u \in W^{k,p}(\Omega)$.

(1) *If* $1 \le p < \infty$ *and* $|\alpha| \le k$, *then* $\|\partial_h^\alpha u\|_{W^{k-|\alpha|}(V)} \le \|u\|_{W^{k,p}(\Omega)}$ *for* h *small.*
(2) *If* $1 < p \le \infty$ *and* $\|\partial_h^\alpha u\|_{W^{k,p}(V)} \le C$ *for all* $|\alpha| \le j$ *and* h *near* 0, *then* $u \in W^{k+j,p}(V)$ *and* $\|\partial^\alpha u\|_{W^{k,p}(V)} \le C$ *for all* $|\alpha| \le j$.

Proof. Since $\partial_h^\alpha = \prod_i \partial_h^{\alpha_i}$, item 1. follows from Item 1. of Theorem 23.13 and induction on $|\alpha|$.

For Item 2., suppose first that $k = 0$ so that $u \in L^p(\Omega)$ and $\|\partial_h^\alpha u\|_{L^p(V)} \le C$ for $|\alpha| \le j$. Then by Proposition 19.16, there exists $\{h_l\}_{l=1}^\infty \subset \mathbb{R} \setminus \{0\}$ and $v \in L^p(V)$ such that $h_l \to 0$ and $\lim_{l\to\infty} \langle \partial_{h_\ell}^\alpha u, \phi\rangle = \langle v, \phi\rangle$ for all $\phi \in C_c^\infty(V)$. Using Remark 23.12,

$$\langle v, \phi\rangle = \lim_{l\to\infty} \langle \partial_{h_\ell}^\alpha u, \phi\rangle = (-1)^{|\alpha|} \lim_{l\to\infty} \langle u, \partial_{-h_\ell}^\alpha \phi\rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi\rangle$$

which shows $\partial^\alpha u = v \in L^p(V)$. Moreover, since weak convergence decreases norms,

$$\|\partial^\alpha u\|_{L^p(V)} = \|v\|_{L^p(V)} \le C.$$

For the general case if $k \in \mathbb{N}$, $u \in W^{k,p}(\Omega)$ such that $\|\partial_h^\alpha u\|_{W^{k,p}(V)} \le C$, then (for $p \in (1,\infty)$, the case $p = \infty$ is similar and left to the reader)

$$\sum_{|\beta|\le k} \|\partial_h^\alpha \partial^\beta u\|_{L^p(V)}^p = \sum_{|\beta|\le k} \|\partial^\beta \partial_h^\alpha u\|_{L^p(V)}^p = \|\partial_h^\alpha u\|_{W^{k,p}(V)}^p \le C^p.$$

As above this implies $\partial^\alpha\partial^\beta u \in L^p(V)$ for all $|\alpha| \le j$ and $|\beta| \le k$ and that

$$\|\partial^\alpha u\|^p_{W^{k,p}(V)} = \sum_{|\beta|\le k} \|\partial^\alpha\partial^\beta u\|^p_{L^p(V)} \le C^p.$$

■

23.3. **Sobolev Spaces on Compact Manifolds.**

Theorem 23.16 (Change of Variables)**.** *Suppose that U and V are open subsets of $\mathbb{R}^d$, $T \in C^k(U,V)$ be a C^k – diffeomorphism such that $\|\partial^\alpha T\|_{BC(U)} < \infty$ for all $1 \le |\alpha| \le k$ and $\epsilon := \inf_U |\det T'| > 0$. Then the map $T^* : W^{k,p}(V) \to W^{k,p}(U)$ defined by $u \in W^{k,p}(V) \to T^*u := u\circ T \in W^{k,p}(U)$ is well defined and is bounded.*

Proof. For $u \in W^{k,p}(V) \cap C^\infty(V)$, repeated use of the chain and product rule implies,

$$\begin{aligned}
(u\circ T)' &= (u'\circ T)\,T' \\
(u\circ T)'' &= (u'\circ T)'\,T' + (u'\circ T)\,T'' = (u''\circ T)\,T'\otimes T' + (u'\circ T)\,T'' \\
(u\circ T)^{(3)} &= \left(u^{(3)}\circ T\right)T'\otimes T'\otimes T' + (u''\circ T)\,(T'\otimes T')' \\
&\quad + (u''\circ T)\,T'\otimes T'' + (u'\circ T)\,T^{(3)} \\
&\ \ \vdots
\end{aligned}$$

(23.9)

$$(u\circ T)^{(l)} = \left(u^{(l)}\circ T\right)\overbrace{T\otimes\cdots\otimes T}^{l\text{ times}} + \sum_{j=1}^{l-1}\left(u^{(j)}\circ T\right)p_j\left(T', T'', \dots, T^{(l+1-j)}\right).$$

This equation and the boundedness assumptions on $T^{(j)}$ for $1 \le j \le k$ implies there is a finite constant K such that

$$\left|(u\circ T)^{(l)}\right| \le K\sum_{j=1}^{l}\left|u^{(j)}\circ T\right| \text{ for all } 1 \le l \le k.$$

By Hölder's inequality for sums we conclude there is a constant K_p such that

$$\sum_{|\alpha|\le k}|\partial^\alpha(u\circ T)|^p \le K_p\sum_{|\alpha|\le k}|\partial^\alpha u|^p\circ T$$

and therefore

$$\|u\circ T\|^p_{W^{k,p}(U)} \le K_p\sum_{|\alpha|\le k}\int_U |\partial^\alpha u|^p\,(T(x))\,dx.$$

Making the change of variables, $y = T(x)$ and using

$$dy = |\det T'(x)|\,dx \ge \epsilon dx,$$

we find

$$\begin{aligned}
\|u\circ T\|^p_{W^{k,p}(U)} &\le K_p\sum_{|\alpha|\le k}\int_U |\partial^\alpha u|^p\,(T(x))\,dx \\
&\le \frac{K_p}{\epsilon}\sum_{|\alpha|\le k}\int_V |\partial^\alpha u|^p\,(y)\,dy = \frac{K_p}{\epsilon}\,\|u\|^p_{W^{k,p}(V)}. \qquad (23.10)
\end{aligned}$$

This shows that $T^* : W^{k,p}(V) \cap C^\infty(V) \to W^{k,p}(U) \cap C^\infty(U)$ is a bounded operator. For general $u \in W^{k,p}(V)$, we may choose $u_n \in W^{k,p}(V) \cap C^\infty(V)$ such that $u_n \to u$ in $W^{k,p}(V)$. Since T^* is bounded, it follows that $T^* u_n$ is Cauchy in $W^{k,p}(U)$ and hence convergent. Finally, using the change of variables theorem again we know,

$$\|T^* u - T^* u_n\|_{L^p(V)}^p \le \epsilon^{-1} \|u - u_n\|_{L^p(U)}^p \to 0 \text{ as } n \to \infty$$

and therefore $T^* u = \lim_{n\to\infty} T^* u_n$ and by continuity Eq. (23.10) still holds for $u \in W^{k,p}(V)$. ■

Let M be a compact C^k – manifolds without boundary, i.e. M is a compact Hausdorff space with a collection of charts x in an "atlas" $\mathcal{A}$ such that $x : D(x) \subset_o M \to R(x) \subset_o \mathbb{R}^d$ is a homeomorphism such that

$$x \circ y^{-1} \in C^k(y(D(x) \cap D(y))), x(D(x) \cap D(y))) \text{ for all } x, y \in \mathcal{A}.$$

Definition 23.17. Let $\{x_i\}_{i=1}^N \subset \mathcal{A}$ such that $M = \cup_{i=1}^N D(x_i)$ and let $\{\phi_i\}_{i=1}^N$ be a partition of unity subordinate do the cover $\{D(x_i)\}_{i=1}^N$. We now define $u \in W^{k,p}(M)$ if $u : M \to \mathbb{C}$ is a function such that

$$\|u\|_{W^{k,p}(M)} := \sum_{i=1}^N \left\|(\phi_i u) \circ x_i^{-1}\right\|_{W^{k,p}(R(x_i))} < \infty. \tag{23.11}$$

Since $\|\cdot\|_{W^{k,p}(R(x_i))}$ is a norm for all i, it easily verified that $\|\cdot\|_{W^{k,p}(M)}$ is a norm on $W^{k,p}(M)$.

Proposition 23.18. *If $f \in C^k(M)$ and $u \in W^{k,p}(M)$ then $fu \in W^{k,p}(M)$ and*

$$\|fu\|_{W^{k,p}(M)} \le C \|u\|_{W^{k,p}(M)} \tag{23.12}$$

where C is a finite constant not depending on u. Recall that $f : M \to \mathbb{R}$ is said to be C^j with $j \le k$ if $f \circ x^{-1} \in C^j(R(x), \mathbb{R})$ for all $x \in \mathcal{A}$.

Proof. Since $\left[f \circ x_i^{-1}\right]$ has bounded derivatives on $\operatorname{supp}(\phi_i \circ x_i^{-1})$, it follows from Proposition 23.6 that there is a constant $C_i < \infty$ such that

$$\left\|(\phi_i f u) \circ x_i^{-1}\right\|_{W^{k,p}(R(x_i))} = \left\|\left[f \circ x_i^{-1}\right](\phi_i u) \circ x_i^{-1}\right\|_{W^{k,p}(R(x_i))} \le C_i \left\|(\phi_i u) \circ x_i^{-1}\right\|_{W^{k,p}(R(x_i))}$$

and summing this equation on i shows Eq. (23.12) holds with $C := \max_i C_i$. ■

Theorem 23.19. *If $\{y_j\}_{j=1}^K \subset \mathcal{A}$ such that $M = \cup_{j=1}^K D(y_j)$ and $\{\psi_j\}_{j=1}^K$ is a partition of unity subordinate to the cover $\{D(y_j)\}_{j=1}^K$, then the norm*

$$|u|_{W^{k,p}(M)} := \sum_{j=1}^K \left\|(\psi_j u) \circ y_j^{-1}\right\|_{W^{k,p}(R(y_j))} \tag{23.13}$$

is equivalent to the norm in Eq. (23.11). That is to say the space $W^{k,p}(M)$ along with its topology is well defined independent of the choice of charts and partitions of unity used in defining the norm on $W^{k,p}(M)$.

Proof. Since $|\cdot|_{W^{k,p}(M)}$ is a norm,

$$
\begin{aligned}
|u|_{W^{k,p}(M)} &= \left|\sum_{i=1}^{N}\phi_i u\right|_{W^{k,p}(M)} \le \sum_{i=1}^{N}|\phi_i u|_{W^{k,p}(M)} \\
&= \sum_{j=1}^{K}\left\|\sum_{i=1}^{N}(\psi_j\phi_i u)\circ y_j^{-1}\right\|_{W^{k,p}(R(y_j))} \\
&\le \sum_{j=1}^{K}\sum_{i=1}^{N}\left\|(\psi_j\phi_i u)\circ y_j^{-1}\right\|_{W^{k,p}(R(y_j))} \qquad (23.14)
\end{aligned}
$$

and since $x_i\circ y_j^{-1}$ and $y_j\circ x_i^{-1}$ are C^k diffeomorphism and the sets $y_j(\mathrm{supp}(\phi_i)\cap\mathrm{supp}(\psi_j))$ and $x_i(\mathrm{supp}(\phi_i)\cap\mathrm{supp}(\psi_j))$ are compact, an application of Theorem 23.16 and Proposition 23.6 shows there are finite constants C_{ij} such that

$$\left\|(\psi_j\phi_i u)\circ y_j^{-1}\right\|_{W^{k,p}(R(y_j))} \le C_{ij}\left\|(\psi_j\phi_i u)\circ x_i^{-1}\right\|_{W^{k,p}(R(x_i))} \le C_{ij}\left\|\phi_i u\circ x_i^{-1}\right\|_{W^{k,p}(R(x_i))}$$

which combined with Eq. (23.14) implies

$$|u|_{W^{k,p}(M)} \le \sum_{j=1}^{K}\sum_{i=1}^{N}C_{ij}\left\|\phi_i u\circ x_i^{-1}\right\|_{W^{k,p}(R(x_i))} \le C\|u\|_{W^{k,p}(M)}$$

where $C := \max_i\sum_{j=1}^{K}C_{ij} < \infty$. Analogously, one shows there is a constant $K<\infty$ such that $\|u\|_{W^{k,p}(M)} \le K|u|_{W^{k,p}(M)}$. ■

Lemma 23.20. *Suppose $x\in\mathcal{A}(M)$ and $U\subset_o M$ such that $U\subset\bar{U}\subset D(x)$, then there is a constant $C<\infty$ such that*

$$\left\|u\circ x^{-1}\right\|_{W^{k,p}(x(U))} \le C\|u\|_{W^{k,p}(M)} \text{ for all } u\in W^{k,p}(M). \qquad (23.15)$$

Conversely a function $u: M\to\mathbb{C}$ with $\mathrm{supp}(u)\subset U$ is in $W^{k,p}(M)$ iff $\left\|u\circ x^{-1}\right\|_{W^{k,p}(x(U))}<\infty$ and in any case there is a finite constant such that

$$\|u\|_{W^{k,p}(M)} \le C\left\|u\circ x^{-1}\right\|_{W^{k,p}(x(U))}. \qquad (23.16)$$

Proof. Choose charts $y_1 := x,\ y_2,\dots,y_K\in\mathcal{A}$ such that $\{D(y_i)\}_{j=1}^{K}$ is an open cover of M and choose a partition of unity $\{\psi_j\}_{j=1}^{K}$ subordinate to the cover $\{D(y_j)\}_{j=1}^{K}$ such that $\psi_1=1$ on a neighborhood of $\bar{U}$. To construct such a partition of unity choose $U_j\subset_o M$ such that $U_j\subset\bar{U}_j\subset D(y_j)$, $\bar{U}\subset U_1$ and $\cup_{j=1}^{K}U_j=M$ and for each j let $\eta_j\in C_c^k(D(y_j),[0,1])$ such that $\eta_j=1$ on a neighborhood of $\bar{U}_j$. Then define $\psi_j := \eta_j(1-\eta_0)\cdots(1-\eta_{j-1})$ where by convention $\eta_0\equiv 0$. Then $\{\psi_j\}_{j=1}^{K}$ is the desired partition, indeed by induction one shows

$$1-\sum_{j=1}^{l}\psi_j = (1-\eta_1)\cdots(1-\eta_l)$$

and in particular

$$1-\sum_{j=1}^{K}\psi_j = (1-\eta_1)\cdots(1-\eta_K) = 0.$$

Using Theorem 23.19, it follows that

$$\begin{aligned}\left\|u\circ x^{-1}\right\|_{W^{k,p}(x(U))} &= \left\|(\psi_1 u)\circ x^{-1}\right\|_{W^{k,p}(x(U))}\\ &\le \left\|(\psi_1 u)\circ x^{-1}\right\|_{W^{k,p}(R(y_1))} \le \sum_{j=1}^{K}\left\|(\psi_j u)\circ y_j^{-1}\right\|_{W^{k,p}(R(y_j))}\\ &= |u|_{W^{k,p}(M)} \le C\,\|u\|_{W^{k,p}(M)}\end{aligned}$$

which proves Eq. (23.15).

Using Theorems 23.19 and 23.16 there are constants C_j for $j = 0, 1, 2\dots, N$ such that

$$\begin{aligned}\|u\|_{W^{k,p}(M)} &\le C_0\sum_{j=1}^{K}\left\|(\psi_j u)\circ y_j^{-1}\right\|_{W^{k,p}(R(y_j))} = C_0\sum_{j=1}^{K}\left\|(\psi_j u)\circ y_1^{-1}\circ y_1\circ y_j^{-1}\right\|_{W^{k,p}(R(y_j))}\\ &\le C_0\sum_{j=1}^{K}C_j\left\|(\psi_j u)\circ x^{-1}\right\|_{W^{k,p}(R(y_1))} = C_0\sum_{j=1}^{K}C_j\left\|\psi_j\circ x^{-1}\cdot u\circ x^{-1}\right\|_{W^{k,p}(R(y_1))}.\end{aligned}$$

This inequality along with $K-$applications of Proposition 23.6 proves Eq. (23.16). ∎

Theorem 23.21. *The space* $(W^{k,p}(M), \|\cdot\|_{W^{k,p}(M)})$ *is a Banach space.*

Proof. Let $\{x_i\}_{i=1}^N \subset \mathcal{A}$ and $\{\phi_i\}_{i=1}^N$ be as in Definition 23.17 and choose $U_i \subset_o M$ such that $\mathrm{supp}(\phi_i) \subset U_i \subset \bar{U}_i \subset D(x_i)$. If $\{u_n\}_{n=1}^\infty \subset W^{k,p}(M)$ is a Cauchy sequence, then by Lemma 23.20, $\left\{u_n\circ x_i^{-1}\right\}_{n=1}^\infty \subset W^{k,p}(x_i(U_i))$ is a Cauchy sequence for all i. Since $W^{k,p}(x_i(U_i))$ is complete, there exists $v_i \in W^{k,p}(x_i(U_i))$ such that $u_n\circ x_i^{-1} \to \tilde{v}_i$ in $W^{k,p}(x_i(U_i))$. For each i let $v_i := \phi_i\left(\tilde{v}_i\circ x_i\right)$ and notice by Lemma 23.20 that

$$\|v_i\|_{W^{k,p}(M)} \le C\left\|v_i\circ x_i^{-1}\right\|_{W^{k,p}(x_i(U_i))} = C\,\|\tilde{v}_i\|_{W^{k,p}(x_i(U_i))} < \infty$$

so that $u := \sum_{i=1}^N v_i \in W^{k,p}(M)$. Since $\mathrm{supp}(v_i - \phi_i u_n) \subset U_i$, it follows that

$$\begin{aligned}\|u-u_n\|_{W^{k,p}(M)} &= \left\|\sum_{i=1}^N v_i - \sum_{i=1}^N \phi_i u_n\right\|_{W^{k,p}(M)}\\ &\le \sum_{i=1}^N\|v_i-\phi_i u_n\|_{W^{k,p}(M)} \le C\sum_{i=1}^N\left\|\left[\phi_i\left(\tilde{v}_i\circ x_i - u_n\right)\right]\circ x_i^{-1}\right\|_{W^{k,p}(x_i(U_i))}\\ &= C\sum_{i=1}^N\left\|\left[\phi_i\circ x_i^{-1}\left(\tilde{v}_i - u_n\circ x_i^{-1}\right)\right]\right\|_{W^{k,p}(x_i(U_i))}\\ &\le C\sum_{i=1}^N C_i\left\|\tilde{v}_i - u_n\circ x_i^{-1}\right\|_{W^{k,p}(x_i(U_i))} \to 0 \text{ as } n\to\infty\end{aligned}$$

wherein the last inequality we have used Proposition 23.6 again. ∎

23.4. **Trace Theorems.** For many more general results on this subject matter, see E. Stein [7, Chapter VI].

Lemma 23.22. *Suppose* $k \geq 1$, $\mathbb{H}^d := \left\{x \in \mathbb{R}^d : x_d > 0\right\} \subset_o \mathbb{R}^d$, $u \in C_c^k\left(\overline{\mathbb{H}^d}\right)$ *and* D *is the smallest constant so that* $\operatorname{supp}(u) \subset \mathbb{R}^{d-1} \times [0, D]$. *Then there is a constant* $C = C(p, k, D, d)$ *such that*

$$\|u\|_{W^{k-1,p}(\partial\mathbb{H}^d)} \leq C(p, D, k, d) \|u\|_{W^{k,p}(\mathbb{H}^d)}. \tag{23.17}$$

Proof. Write $x \in \overline{\mathbb{H}^d}$ as $x = (y, z) \in \mathbb{R}^{d-1} \times [0, \infty)$, then by the fundamental theorem of calculus we have for any $\alpha \in \mathbb{N}_0^{d-1}$ with $|\alpha| \leq k - 1$ that

$$\partial_y^\alpha u(y, 0) = \partial_y^\alpha u(y, z) - \int_0^z \partial_y^\alpha u_t(y, t) dt. \tag{23.18}$$

Therefore, for $p \in [1, \infty)$

$$\begin{aligned}
\left|\partial_y^\alpha u(y, 0)\right|^p &\leq 2^{p/q} \cdot \left[\left|\partial_y^\alpha u(y, z)\right|^p + \left|\int_0^z \partial_y^\alpha u_t(y, t) dt\right|^p\right] \\
&\leq 2^{p/q} \cdot \left[\left|\partial_y^\alpha u(y, z)\right|^p + \int_0^z \left|\partial_y^\alpha u_t(y, t)\right|^p dt \cdot |z|^{q/p}\right] \\
&\leq 2^{p-1} \cdot \left[\left|\partial_y^\alpha u(y, z)\right|^p + \int_0^D \left|\partial_y^\alpha u_t(y, t)\right|^p dt \cdot z^{p-1}\right]
\end{aligned}$$

where $q := \frac{p}{p-1}$ is the conjugate exponent to p. Integrating this inequality over $\mathbb{R}^{d-1} \times [0, D]$ implies

$$D \|\partial^\alpha u\|_{L^p(\partial\mathbb{H}^d)}^p \leq 2^{p-1} \left[\|\partial^\alpha u\|_{L^p(\mathbb{H}^d)}^p + \left\|\partial^{\alpha+e_d} u\right\|_{L^p(\mathbb{H}^d)}^p \frac{D^p}{p}\right]$$

or equivalently that

$$\|\partial^\alpha u\|_{L^p(\partial\mathbb{H}^d)}^p \leq 2^{p-1} D^{-1} \|\partial^\alpha u\|_{L^p(\mathbb{H}^d)}^p + 2^{p-1} \frac{D^{p-1}}{p} \left\|\partial^{\alpha+e_d} u\right\|_{L^p(\mathbb{H}^d)}^p$$

from which implies Eq. (23.17).

Similarly, if $p = \infty$, then from Eq. (23.18) we find

$$\|\partial^\alpha u\|_{L^\infty(\partial\mathbb{H}^d)} = \|\partial^\alpha u\|_{L^\infty(\mathbb{H}^d)} + D \left\|\partial^{\alpha+e_d} u\right\|_{L^\infty(\mathbb{H}^d)}$$

and again the result follows. ■

Theorem 23.23 (Trace Theorem)**.** *Suppose* $k \geq 1$ *and* $\Omega \subset_o \mathbb{R}^d$ *such that* $\bar{\Omega}$ *is a compact manifold with* C^k *– boundary. Then there exists a unique linear map* $T : W^{k,p}(\Omega) \to W^{k-1,p}(\partial\Omega)$ *such that* $Tu = u|_{\partial\Omega}$ *for all* $u \in C^k\left(\bar{\Omega}\right)$.

Proof. Choose a covering $\{V_i\}_{i=0}^N$ of $\bar{\Omega}$ such that $\bar{V}_0 \subset \Omega$ and for each $i \geq 1$, there is C^k – diffeomorphism $x_i : V_i \to R(x_i) \subset_o \mathbb{R}^d$ such that

$$\begin{aligned}
x_i(\partial\Omega \cap V_i) &= R(x_i) \cap \operatorname{bd}(\mathbb{H}^d) \text{ and} \\
x_i(\Omega \cap V_i) &= R(x_i) \cap \mathbb{H}^d
\end{aligned}$$

as in Figure 45. Further choose $\phi_i \in C_c^\infty(V_i, [0, 1])$ such that $\sum_{i=0}^N \phi_i = 1$ on a

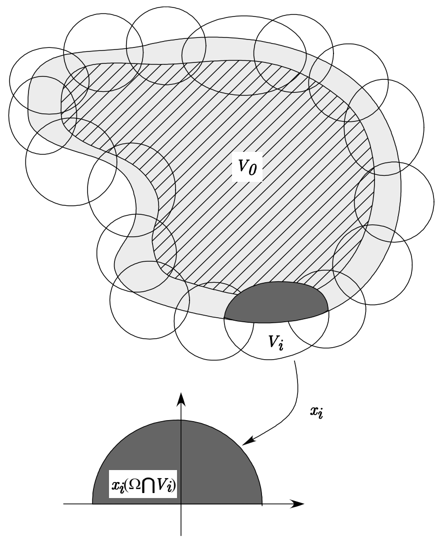

FIGURE 45. Covering Ω (the shaded region) as described in the text.

neighborhood of $\bar{\Omega}$ and set $y_i := x_i|_{\partial\Omega\cap V_i}$ for $i \geq 1$. Given $u \in C^k\left(\bar{\Omega}\right)$, we compute

$$\begin{aligned}
\|u|_{\partial\bar{\Omega}}\|_{W^{k-1,p}\left(\partial\bar{\Omega}\right)} &= \sum_{i=1}^N \left\|(\phi_i u)|_{\partial\bar{\Omega}} \circ y_i^{-1}\right\|_{W^{k-1,p}(R(x_i)\cap \mathrm{bd}(\mathbb{H}^d))} \\
&= \sum_{i=1}^N \left\|\left[(\phi_i u)\circ x_i^{-1}\right]|_{\mathrm{bd}(\mathbb{H}^d)}\right\|_{W^{k-1,p}(R(x_i)\cap \mathrm{bd}(\mathbb{H}^d))} \\
&\leq \sum_{i=1}^N C_i \left\|\left[(\phi_i u)\circ x_i^{-1}\right]\right\|_{W^{k,p}(R(x_i))} \\
&\leq \max C_i \cdot \sum_{i=1}^N \left\|\left[(\phi_i u)\circ x_i^{-1}\right]\right\|_{W^{k,p}(R(x_i)\cap\mathbb{H}^d)} + \left\|\left[(\phi_0 u)\circ x_0^{-1}\right]\right\|_{W^{k,p}(R(x_0))} \\
&\leq C\,\|u\|_{W^{k,p}(\Omega)}
\end{aligned}$$

where $C = \max\{1, C_1, \ldots, C_N\}$. The result now follows by the B.L.T. Theorem 4.1 and the fact that $C^k\left(\bar{\Omega}\right)$ is dense inside $W^{k,p}(\Omega)$. ■

Notation 23.24. In the sequel will often abuse notation and simply write $u|_{\partial\bar{\Omega}}$ for the "function" $Tu \in W^{k-1,p}(\partial\bar{\Omega})$.

Proposition 23.25 (Integration by parts)**.** *Suppose $\Omega \subset_o \mathbb{R}^d$ such that $\bar{\Omega}$ is a compact manifold with C^1 – boundary, $p \in [1,\infty]$ and $q = \frac{p}{p-1}$ is the conjugate exponent. Then for $u \in W^{k,p}(\Omega)$ and $v \in W^{k,q}(\Omega)$,*

$$\int_\Omega \partial_i u \cdot v dm = -\int_\Omega u \cdot \partial_i v dm + \int_{\partial\bar{\Omega}} u|_{\partial\bar{\Omega}} \cdot v|_{\partial\bar{\Omega}} n_i d\sigma \tag{23.19}$$

where $n : \partial\bar{\Omega} \to \mathbb{R}^d$ is unit outward pointing norm to $\partial\bar{\Omega}$.

Proof. Equation 23.19 holds for $u, v \in C^2(\bar{\Omega})$ and therefore for $(u, v) \in W^{k,p}(\Omega) \times W^{k,q}(\Omega)$ since both sides of the equality are continuous in $(u, v) \in W^{k,p}(\Omega) \times W^{k,q}(\Omega)$ as the reader should verify. ■

Definition 23.26. Let $W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{k,p}(\Omega)}$ be the closure of $C_c^\infty(\Omega)$ inside $W^{k,p}(\Omega)$.

Remark 23.27. Notice that if $T : W^{k,p}(\Omega) \to W^{k-1,p}(\partial\bar{\Omega})$ is the trace operator in Theorem 23.23, then $T\left(W_0^{k,p}(\Omega)\right) = \{0\} \subset W^{k-1,p}(\partial\bar{\Omega})$ since $Tu = u|_{\partial\bar{\Omega}} = 0$ for all $u \in C_c^\infty(\Omega)$.

Corollary 23.28. *Suppose $\Omega \subset_o \mathbb{R}^d$ such that $\bar{\Omega}$ is a compact manifold with C^1 – boundary, $p \in [1, \infty]$ and $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator of Theorem 23.23. Then $W_0^{1,p}(\Omega) = \mathrm{Nul}(T)$.*

Proof. It has already been observed in Remark 23.27 that $W_0^{1,p}(\Omega) \subset \mathrm{Nul}(T)$. Suppose $u \in \mathrm{Nul}(T)$ and $\mathrm{supp}(u)$ is compactly contained in Ω. The mollification $u_\epsilon(x)$ defined in Proposition 23.4 will be in $C_c^\infty(\Omega)$ for $\epsilon > 0$ sufficiently small and by Proposition 23.4, $u_\epsilon \to u$ in $W^{1,p}(\Omega)$. Thus $u \in W_0^{1,p}(\Omega)$. We will now give two proofs for $\mathrm{Nul}(T) \subset W_0^{1,p}(\Omega)$.

Proof 1. For $u \in \mathrm{Nul}(T) \subset W^{1,p}(\Omega)$ define

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for} \quad x \in \bar{\Omega} \\ 0 & \text{for} \quad x \notin \bar{\Omega}. \end{cases}$$

Then clearly $\tilde{u} \in L^p(\mathbb{R}^d)$ and moreover by Proposition 23.25, for $v \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \tilde{u} \cdot \partial_i v dm = \int_\Omega u \cdot \partial_i v dm = -\int_\Omega \partial_i u \cdot v dm$$

from which it follows that $\partial_i \tilde{u}$ exists weakly in $L^p(\mathbb{R}^d)$ and $\partial_i \tilde{u} = 1_\Omega \partial_i u$ a.e.. Thus $\tilde{u} \in W^{1,p}(\mathbb{R}^d)$ with $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^d)} = \|u\|_{W^{1,p}(\Omega)}$ and $\mathrm{supp}(\tilde{u}) \subset \Omega$.

Choose $V \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $V(x) \cdot n(x) > 0$ for all $x \in \partial\bar{\Omega}$ and define

$$\tilde{u}_\epsilon(x) = T_\epsilon \tilde{u}(x) := \tilde{u} \circ e^{\epsilon V}(x).$$

Notice that $\mathrm{supp}(\tilde{u}_\epsilon) \subset e^{-\epsilon V}(\bar{\Omega}) \sqsubset\sqsubset \Omega$ for all ϵ sufficiently small. By the change of variables Theorem 23.16, we know that $\tilde{u}_\epsilon \in W^{1,p}(\Omega)$ and since $\mathrm{supp}(\tilde{u}_\epsilon)$ is a compact subset of Ω, it follows from the first paragraph that $\tilde{u}_\epsilon \in W_0^{1,p}(\Omega)$.

To so finish this proof, it only remains to show $\tilde{u}_\epsilon \to u$ in $W^{1,p}(\Omega)$ as $\epsilon \downarrow 0$. Looking at the proof of Theorem 23.16, the reader may show there are constants $\delta > 0$ and $C < \infty$ such that

$$\|T_\epsilon v\|_{W^{1,p}(\mathbb{R}^d)} \le C \|v\|_{W^{1,p}(\mathbb{R}^d)} \text{ for all } v \in W^{1,p}(\mathbb{R}^d). \tag{23.20}$$

By direct computation along with the dominated convergence it may be shown that

$$T_\epsilon v \to v \text{ in } W^{1,p}(\mathbb{R}^d) \text{ for all } v \in C_c^\infty(\mathbb{R}^d). \tag{23.21}$$

As is now standard, Eqs. (23.20) and (23.21) along with the density of $C_c^\infty(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$ allows us to conclude $T_\epsilon v \to v$ in $W^{1,p}(\mathbb{R}^d)$ for all $v \in W^{1,p}(\mathbb{R}^d)$ which completes the proof that $\tilde{u}_\epsilon \to u$ in $W^{1,p}(\Omega)$ as $\epsilon \to 0$.

Proof 2. As in the first proof it suffices to show that any $u \in W_0^{1,p}(\Omega)$ may be approximated by $v \in W^{1,p}(\Omega)$ with $\mathrm{supp}(v) \sqsubset \Omega$. As above extend u to Ω^c

by 0 so that $\tilde{u} \in W^{1,p}(\mathbb{R}^d)$. Using the notation in the proof of 23.23, it suffices to show $u_i := \phi_i \tilde{u} \in W^{1,p}(\mathbb{R}^d)$ may be approximated by $u_i \in W^{1,p}(\Omega)$ with $\operatorname{supp}(u_i) \sqsubset \Omega$. Using the change of variables Theorem 23.16, the problem may be reduced to working with $w_i = u_i \circ x_i^{-1}$ on $B = R(x_i)$. But in this case we need only define $w_i^\epsilon(y) := w_i^\epsilon(y - \epsilon e_d)$ for $\epsilon > 0$ sufficiently small. Then $\operatorname{supp}(w_i^\epsilon) \subset \mathbb{H}^d \cap B$ and as we have already seen $w_i^\epsilon \to w_i$ in $W^{1,p}(\mathbb{H}^d)$. Thus $u_i^\epsilon := w_i^\epsilon \circ x_i \in W^{1,p}(\Omega)$, $u_i^\epsilon \to u_i$ as $\epsilon \downarrow 0$ with $\operatorname{supp}(u_i) \sqsubset \Omega$. ■

23.5. **Extension Theorems.**

Lemma 23.29. *Let $R > 0$, $B := B(0,R) \subset \mathbb{R}^d$, $B^\pm := \{x \in B : \pm x_d > 0\}$ and $\Gamma := \{x \in B : x_d = 0\}$. Suppose that $u \in C^k(B \setminus \Gamma) \cap C(B)$ and for each $|\alpha| \le k$, $\partial^\alpha u$ extends to a continuous function v_α on B. Then $u \in C^k(B)$ and $\partial^\alpha u = v_\alpha$ for all $|\alpha| \le k$.*

Proof. For $x \in \Gamma$ and $i < d$, then by continuity, the fundamental theorem of calculus and the dominated convergence theorem,

$$\begin{aligned} u(x+\Delta e_i) - u(x) &= \lim_{\substack{y \to x \\ y \in B\setminus\Gamma}} [u(y+\Delta e_i) - u(y)] = \lim_{\substack{y \to x \\ y \in B\setminus\Gamma}} \int_0^\Delta \partial_i u(y+se_i)ds \\ &= \lim_{\substack{y \to x \\ y \in B\setminus\Gamma}} \int_0^\Delta v_{e_i}(y+se_i)ds = \int_0^\Delta v_{e_i}(x+se_i)ds \end{aligned}$$

and similarly, for $i = d$,

$$\begin{aligned} u(x+\Delta e_d) - u(x) &= \lim_{\substack{y \to x \\ y \in B^{\operatorname{sgn}(\Delta)}\setminus\Gamma}} [u(y+\Delta e_d) - u(y)] = \lim_{\substack{y \to x \\ y \in B^{\operatorname{sgn}(\Delta)}\setminus\Gamma}} \int_0^\Delta \partial_d u(y+se_d)ds \\ &= \lim_{\substack{y \to x \\ y \in B^{\operatorname{sgn}(\Delta)}\setminus\Gamma}} \int_0^\Delta v_{e_d}(y+se_d)ds = \int_0^\Delta v_{e_d}(x+se_d)ds. \end{aligned}$$

These two equations show, for each i, $\partial_i u(x)$ exits and $\partial_i u(x) = v_{e_i}(x)$. Hence we have shown $u \in C^1(B)$.

Suppose it has been proven for some $l \ge 1$ that $\partial^\alpha u(x)$ exists and is given by $v_\alpha(x)$ for all $|\alpha| \le l < k$. Then applying the results of the previous paragraph to $\partial^\alpha u(x)$ with $|\alpha| = l$ shows that $\partial_i \partial^\alpha u(x)$ exits and is given by $v_{\alpha+e_i}(x)$ for all i and $x \in B$ and from this we conclude that $\partial^\alpha u(x)$ exists and is given by $v_\alpha(x)$ for all $|\alpha| \le l+1$. So by induction we conclude $\partial^\alpha u(x)$ exists and is given by $v_\alpha(x)$ for all $|\alpha| \le k$, i.e. $u \in C^k(B)$. ■

Lemma 23.30. *Given any $k+1$ distinct points, $\{c_i\}_{i=0}^k$, in $\mathbb{R}\setminus\{0\}$, the $(k+1) \times (k+1)$ matrix C with entries $C_{ij} := (c_i)^j$ is invertible.*

Proof. Let $a \in \mathbb{R}^{k+1}$ and define $p(x) := \sum_{j=0}^k a_j x^j$. If $a \in \operatorname{Nul}(C)$, then

$$0 = \sum_{j=0}^k (c_i)^j a_j = p(c_i) \text{ for } i = 0, 1, \ldots, k.$$

Since $\deg(p) \le k$ and the above equation says that p has $k+1$ distinct roots, we conclude that $a \in \operatorname{Nul}(C)$ implies $p \equiv 0$ which implies $a = 0$. Therefore $\operatorname{Nul}(C) = \{0\}$ and C is invertible. ■

Lemma 23.31. *Let B, $B^{\pm}$ and Γ be as in Lemma 23.29 and $\{c_i\}_{i=0}^k$, be $k+1$ distinct points in $(\infty,-1]$ for example $c_i = -(i+1)$ will work. Also let $a \in \mathbb{R}^{k+1}$ be the unique solution (see Lemma 23.30 to $C^{tr}a = \mathbf{1}$ where $\mathbf{1}$ denotes the vector of all ones in $\mathbb{R}^{k+1}$, i.e. a satisfies*

$$1 = \sum_{j=0}^{k} (c_i)^j a_i \text{ for } j = 0, 1, 2 \ldots, k. \tag{23.22}$$

For $u \in C^k(\mathbb{H}^d) \cap C_c(\overline{\mathbb{H}^d})$ with $\operatorname{supp}(u) \subset B$ and $x = (y,z) \in \mathbb{R}^d$ define

$$\tilde{u}(x) = \tilde{u}(y,z) = \begin{cases} u(y,z) & \text{if} \quad z \geq 0 \\ \sum_{i=0}^{k} a_i u(y, c_i z) & \text{if} \quad z \leq 0. \end{cases} \tag{23.23}$$

Then $\tilde{u} \in C_c^k(\mathbb{R}^d)$ with $\operatorname{supp}(\tilde{u}) \subset B$ and moreover there exists a constant M independent of u such that

$$\|\tilde{u}\|_{W^{k,p}(B)} \leq M \|u\|_{W^{k,p}(B^+)}. \tag{23.24}$$

Proof. By Eq. (23.22) with $j=0$,

$$\sum_{i=0}^{k} a_i u(y, c_i 0) = u(y,0) \sum_{i=0}^{k} a_i = u(y,0).$$

This shows that $\tilde{u}$ in Eq. (23.23) is well defined and that $\tilde{u} \in C\left(\mathbb{H}^d\right)$. Let $K^- := \{(y,z) : (y,-z) \in \operatorname{supp}(u)\}$. Since $c_i \in (\infty,-1]$, if $x = (y,z) \notin K^-$ and $z < 0$ then $(y, c_i z) \notin \operatorname{supp}(u)$ and therefore $\tilde{u}(x) = 0$ and therefore $\operatorname{supp}(\tilde{u})$ is compactly contained inside of B. Similarly if $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, Eq. (23.22) with $j = \alpha_d$ implies

$$v_\alpha(x) := \begin{cases} (\partial^\alpha u)(y,z) & \text{if} \quad z \geq 0 \\ \sum_{i=0}^{k} a_i c_i^{\alpha_d} (\partial^\alpha u)(y, c_i z) & \text{if} \quad z \leq 0. \end{cases}$$

is well defined and $v_\alpha \in C\left(\mathbb{R}^d\right)$. Differentiating Eq. (23.23) shows $\partial^\alpha \tilde{u}(x) = v_\alpha(x)$ for $x \in B \setminus \Gamma$ and therefore we may conclude from Lemma 23.29 that $u \in C_c^k(B) \subset C^k\left(\mathbb{R}^d\right)$ and $\partial^\alpha u = v_\alpha$ for all $|\alpha| \leq k$.

We now verify Eq. (23.24) as follows. For $|\alpha| \leq k$,

$$\begin{aligned} \|\partial^\alpha \tilde{u}\|_{L^p(B^-)}^p &= \int_{\mathbb{R}^d} 1_{z<0} \left| \sum_{i=0}^{k} a_i c_i^{\alpha_d} (\partial^\alpha u)(y, c_i z) \right|^p dydz \\ &\leq C \int_{\mathbb{R}^d} 1_{z<0} \sum_{i=0}^{k} |(\partial^\alpha u)(y, c_i z)|^p dydz \\ &= C \int_{\mathbb{R}^d} 1_{z>0} \sum_{i=0}^{k} \frac{1}{|c_i|} |(\partial^\alpha u)(y,z)|^p dydz \\ &= C \left(\sum_{i=0}^{k} \frac{1}{|c_i|} \right) \|\partial^\alpha u\|_{L^p(B^+)}^p \end{aligned}$$

where $C := \left(\sum_{i=0}^{k} |a_i c_i^{\alpha_d}|^q \right)^{p/q}$. Summing this equation on $|\alpha| \leq k$ shows there exists a constant M' such that $\|\tilde{u}\|_{W^{k,p}(B^-)} \leq M' \|u\|_{W^{k,p}(B^+)}$ and hence Eq. (23.24) holds with $M = M' + 1$. ■

Theorem 23.32 (Extension Theorem)**.** *Suppose $k \geq 1$ and $\Omega \subset_o \mathbb{R}^d$ such that $\bar{\Omega}$ is a compact manifold with C^k – boundary. Given $U \subset_o \mathbb{R}^d$ such that $\bar{\Omega} \subset U$, there exists a bounded linear (extension) operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ such that*

1. *$Eu = u$ a.e. in Ω and*
2. $\mathrm{supp}(Eu) \subset U$.

Proof. As in the proof of Theorem 23.23, choose a covering $\{V_i\}_{i=0}^N$ of $\bar{\Omega}$ such that $\bar{V}_0 \subset \Omega$, $\cup_{i=0}^N \bar{V}_i \subset U$ and for each $i \geq 1$, there is C^k – diffeomorphism $x_i : V_i \to R(x_i) \subset_o \mathbb{R}^d$ such that

$$x_i(\partial\Omega \cap V_i) = R(x_i) \cap \mathrm{bd}(\mathbb{H}^d) \text{ and } x_i(\Omega \cap V_i) = R(x_i) \cap \mathbb{H}^d = B^+$$

where B^+ is as in Lemma 23.31, refer to Figure 45. Further choose $\phi_i \in C_c^\infty(V_i, [0,1])$ such that $\sum_{i=0}^N \phi_i = 1$ on a neighborhood of $\bar{\Omega}$ and set $y_i := x_i|_{\partial\Omega\cap V_i}$ for $i \geq 1$. Given $u \in C^k(\bar{\Omega})$ and $i \geq 1$, the function $v_i := (\phi_i u) \circ x_i^{-1}$ may be viewed as a function in $C^k(\mathbb{H}^d) \cap C_c(\overline{\mathbb{H}^d})$ with $\mathrm{supp}(u) \subset B$. Let $\tilde{v}_i \in C_c^k(B)$ be defined as in Eq. (23.23) above and define $\tilde{u} := \phi_0 u + \sum_{i=1}^N \tilde{v}_i \circ x_i \in C_c^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(u) \subset U$. Notice that $\tilde{u} = u$ on $\bar{\Omega}$ and making use of Lemma 23.20 we learn

$$\begin{aligned}
\|\tilde{u}\|_{W^{k,p}(\mathbb{R}^d)} &\leq \|\phi_0 u\|_{W^{k,p}(\mathbb{R}^d)} + \sum_{i=1}^N \|\tilde{v}_i \circ x_i\|_{W^{k,p}(\mathbb{R}^d)} \\
&\leq \|\phi_0 u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N \|\tilde{v}_i\|_{W^{k,p}(R(x_i))} \\
&\leq C(\phi_0)\|u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N \|v_i\|_{W^{k,p}(B^+)} \\
&= C(\phi_0)\|u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N \left\|(\phi_i u) \circ x_i^{-1}\right\|_{W^{k,p}(B^+)} \\
&\leq C(\phi_0)\|u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N C_i \|u\|_{W^{k,p}(\Omega)}.
\end{aligned}$$

This shows the map $u \in C^k(\bar{\Omega}) \to Eu := \tilde{u} \in C_c^k(U)$ is bounded as map from $W^{k,p}(\Omega)$ to $W^{k,p}(U)$. As usual, we now extend E using the B.L.T. Theorem 4.1 to a bounded linear map from $W^{k,p}(\Omega)$ to $W^{k,p}(U)$. So for general $u \in W^{k,p}(\Omega)$, $Eu = W^{k,p}(U) - \lim_{n\to\infty} \tilde{u}_n$ where $u_n \in C^k(\bar{\Omega})$ and $u = W^{k,p}(\Omega) - \lim_{n\to\infty} u_n$. By passing to a subsequence if necessary, we may assume that $\tilde{u}_n$ converges a.e. to Eu from which it follows that $Eu = u$ a.e. on $\bar{\Omega}$ and $\mathrm{supp}(Eu) \subset U$. ■

23.6. **Exercises.**

Exercise 23.1. Show the norm in Eq. (23.1) is equivalent to the norm

$$|f|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

Solution. 23.1This is a consequence of the fact that all norms on $l^p(\{\alpha : |\alpha| \leq k\})$ are equivalent. To be more explicit, let $a_\alpha = \|\partial^\alpha f\|_{L^p(\Omega)}$, then

$$\sum_{|\alpha|\leq k} |a_\alpha| \leq \left(\sum_{|\alpha|\leq k} |a_\alpha|^p\right)^{1/p} \left(\sum_{|\alpha|\leq k} 1^q\right)^{1/q}$$

while

$$\left(\sum_{|\alpha|\leq k} |a_\alpha|^p\right)^{1/p} \leq \left(\sum_{|\alpha|\leq k}^{p} \left[\sum_{|\beta|\leq k} |a_\beta|\right]^p\right)^{1/p} \leq [\#\{\alpha : |\alpha| \leq k\}]^{1/p} \sum_{|\beta|\leq k} |a_\beta| .$$

■