14. Wave Equation on $\mathbb{R}^n$

(**Ref** Courant & Hilbert Vol II, Chap VI §12.)

We now consider the wave equation

$$u_{tt} - \triangle u = 0 \text{ with } u(0,x) = f(x) \text{ and } u_t(0,x) = g(x) \text{ for } x \in \mathbb{R}^n. \tag{14.1}$$

According to Section 13, the solution (in the L^2 – sense) is given by

$$u(t,\cdot) = (\cos(t\sqrt{-\triangle})f + \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}} g. \tag{14.2}$$

To work out the results in Eq. (14.2) we must diagonalize Δ. This is of course done using the Fourier transform. Let $\mathcal{F}$ denote the Fourier transform in the x – variables only. Then

$$\ddot{\hat{u}}(t,k) + |k|^2 \hat{u}(t,k) = 0 \text{ with}$$
$$\hat{u}(0,k) = \hat{f}(k) \text{ and } \dot{\hat{u}}(t,k) = \hat{g}(k).$$

Therefore

$$\hat{u}(t,k) = \cos(t|k|)\hat{f}(k) + \frac{\sin(t|k|)}{|k|}\hat{g}(k).$$

and so

$$u(t,x) = \mathcal{F}^{-1}\left[\cos(t|k|)\hat{f}(k) + \frac{\sin(t|k|)}{|k|}\ \hat{g}(k)\right](x),$$

i.e.

$$\frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}} g = \mathcal{F}^{-1}\left[\frac{\sin(t|k|)}{|k|}\ \hat{g}(k)\right] \text{ and} \tag{14.3}$$

$$\cos(t\sqrt{-\triangle})f = \mathcal{F}^{-1}\left[\cos(t|k|)\hat{f}(k)\right] = \frac{d}{dt}\mathcal{F}^{-1}\left[\frac{\sin(t|k|)}{|k|}\ \hat{g}(k)\right]. \tag{14.4}$$

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Our next goal is to work out these expressions in x – space alone.

14.1. $n = 1$ Case.

As we see from Eq. (14.4) it suffices to compute:

$$\frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}\ g = \mathcal{F}^{-1}\left(\frac{\sin(t|\xi|)}{|\xi|}\ \hat{g}(\xi)\right) = \lim_{M\to\infty} \mathcal{F}^{-1}\left(1_{|\xi|\le M}\ \frac{\sin(t|\xi|)}{|\xi|}\ \hat{g}(\xi)\right)$$
$$= \lim_{M\to\infty} \mathcal{F}^{-1}\left(1_{|\xi|\le M}\ \frac{\sin(t|\xi|)}{|\xi|}\right) \star g. \tag{14.5}$$

This inverse Fourier transform will be computed in Proposition 14.2 below using the following lemma.

Lemma 14.1. *Let C_M denote the contour shown in Figure 38, then for $\lambda \neq 0$ we have*

$$\lim_{M\to\infty} \int_{C_M} \frac{e^{i\lambda\xi}}{\xi}\ d\xi = 2\pi i 1_{\lambda>0}.$$

Proof. First assume that $\lambda > 0$ and let Γ_M denote the contour shown in Figure 38. Then

$$\left|\int_{\Gamma_M} \frac{e^{i\lambda\xi}}{\xi} d\xi\right| \le \int_0^\pi \left|e^{i\lambda M e^{i\theta}}\right| d\theta = 2\pi\int_0^\pi d\theta e^{-\lambda M \sin\theta} \to 0 \text{ as } M \to \infty.$$

Therefore

$$\lim_{M\to\infty}\int_{C_M}\frac{e^{i\lambda\xi}}{\xi}\,d\xi=\lim_{M\to\infty}\int_{C_M+\Gamma_M}\frac{e^{i\lambda\xi}}{\xi}d\xi=2\pi i\operatorname{res}_{\xi=0}\left(\frac{e^{i\lambda\xi}}{\xi}\right)=2\pi i.$$

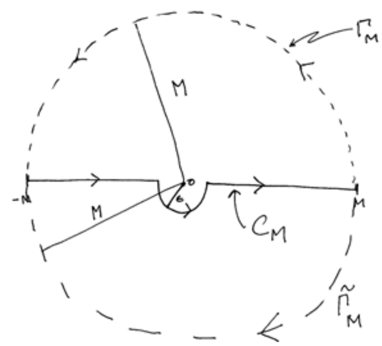

FIGURE 38. A couple of contours in $\mathbb{C}$.

If $\lambda<0$, the same argument shows

$$\lim_{M\to\infty}\int_{C_M}\frac{e^{i\lambda\xi}}{\xi}\,d\xi=\lim_{M\to\infty}\int_{C_M+\tilde{\Gamma}_M}\frac{e^{i\lambda\xi}}{\xi}d\xi$$

and the later integral is 0 since the integrand is holomorphic inside the contour $C_M+\tilde{\Gamma}_M$. ■

Proposition 14.2. $\lim_{M\to\infty}\mathcal{F}^{-1}\left(1_{|\xi|\le M}\ \frac{\sin(t|\xi|)}{|\xi|}\right)(x)=\operatorname{sgn}(t)\frac{\sqrt{\pi}}{\sqrt{2}}\ 1_{|x|<|t|}.$

Proof. Let

$$I_M=\sqrt{2\pi}\mathcal{F}^{-1}\left(1_{|\xi|\le M}\ \frac{\sin(t|\xi|)}{|\xi|}\right)(x)=\int_{|\xi|\le M}\frac{\sin(t\xi)}{\xi}\ e^{i\xi\cdot x}d\xi.$$

Then by deforming the contour we may write

$$\begin{aligned}I_M&=\int_{C_M}\frac{\sin t\xi}{\xi}\ e^{i\xi\cdot x}d\xi=\frac{1}{2i}\int_{C_M}\frac{e^{it\xi}-e^{-it\xi}}{\xi}\ e^{i\xi\cdot x}d\xi\\&=\frac{1}{2i}\int_{C_M}\frac{e^{i(x+t)\xi}-e^{i(x-t)\xi}}{\xi}d\xi\end{aligned}$$

By Lemma 14.1 we conclude that

$$\lim_{M\to\infty}I_M=\frac{1}{2i}2\pi i(1_{(x+t)>0}-1_{(x-t)>0})=\pi\operatorname{sgn}(t)\ 1_{|x|<|t|}.$$

(For the last equality, suppose $t>0$. Then $x-t>0$ implies $x+t>0$ so we get 0 and if $x<-t$, i.e. $x+t<0$ then $x-t<0$ and we get 0 again. If $|x|<t$ the first term is 1 while the second is zero. Similar arguments work when $t<0$ as well.) ■

Theorem 14.3. *For $n = 1$,*

$$\frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}\ g(x) = \frac{1}{2}\int_{x-t}^{x+t} g(y)\,d\lambda(y) \ \text{and} \tag{14.6}$$

$$\cos(t\sqrt{-\triangle})g(x) = \frac{1}{2}\ [g(x+t) + g(x-t)]. \tag{14.7}$$

In particular

$$u(t,x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\ dy \tag{14.8}$$

is the solution to the wave equation (14.2).

Proof. From Eq. (14.5) and Proposition 14.2 we find

$$\begin{aligned}\frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}\ g(x) &= \operatorname{sgn}(t)\frac{1}{2}\int_{\mathbb{R}} 1_{|x-y|>|t|}\ g(y)\,dy \\ &= \operatorname{sgn}(t)\frac{1}{2}\int_{x-|t|}^{x+|t|} g(y)\,dy = \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy.\end{aligned}$$

Differentiating this equation in t gives Eq. (14.7). ■

If we have a forcing term, so $\ddot{u} = u_{x\ x} + h$, with $u(0,\cdot) = 0$ and $u_t(0,\cdot) = 0$, then

$$\begin{aligned}u(t,x) &= \int_0^t \frac{\sin((t-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} h(\tau,x)d\tau = \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} dy h(\tau,y) \\ &= \frac{1}{2}\int_0^t d\tau \int_{-(t+\tau)}^{t-\tau} dr h(\tau, x+r).\end{aligned}$$

14.1.1. *Factorization method for $n = 1$.* Writing the wave equation as

$$0 = \left(\partial_t^2 - \partial_x^2\right) u = (\partial_t + \partial_x)(\partial_t - \partial_x)u = (\partial_t + \partial_x)v$$

with $v := (\partial_t - \partial_x)u$ implies $v(t,x) = v(0, x-t)$ with

$$v(0,x) = u_t(0,x) - u_x(0,x) = g(x) - f'(x).$$

Now u solves $(\partial_t - \partial_x)u = v$, i.e. $\partial_t u = \partial_x u + v$. Therefore

$$\begin{aligned}
u(t,x) &= e^{t\partial_x}u(0,x) + \int_0^t e^{(t-\tau)\partial_x}\ v(\tau,x)d\tau \\
&= u(0,x+t) + \int_0^t v(\tau, x+t-\tau)d\tau \\
&= u(0,x+t) + \int_0^t v(0, x+\underbrace{t-2\tau}_{s})d\tau \\
&= u(0,x+t) + \frac{1}{2}\int_{-t}^t v(0,x+s)ds \\
&= f(x+t) + \frac{1}{2}\int_{-t}^t (g(x+s) - f'(x+s))ds \\
&= f(x+t) - \frac{1}{2}\ f(x+s)\Big|_{s=-t}^{s=t} + \frac{1}{2}\int_{-t}^t g(x+s)ds \\
&= \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{-t}^t g(x+s)ds
\end{aligned}$$

which is equivalent to Eq. (14.8).

14.2. **Solution for $n = 3$.** Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and $t \in \mathbb{R}$ let

$$\bar{f}(x;t) := \fint_{S^2} f(x+t\omega)d\sigma(\omega) = \fint\int_{|y|=|t|} f(x+y)d\sigma(y).$$

Theorem 14.4. *For $f \in L^2(\mathbb{R}^3)$,*

$$\frac{\sin\left(\sqrt{-\Delta}t\right)}{\sqrt{-\Delta}}f = \mathcal{F}^{-1}\left[\frac{\sin|\xi|t}{|\xi|}\hat{f}(\xi)\right](x) = t\bar{f}(x;t)$$

and

$$\cos\left(\sqrt{-\Delta}t\right)g = \frac{d}{dt}\left[t\bar{f}(x;t)\right].$$

In particular the solution to the wave equation (14.1) for $n = 3$ is given by

$$\begin{aligned}
u(t,x) &= \frac{\partial}{\partial t}(t\ \bar{f}(x;t)) + t\ \bar{g}(x;t) \\
&= \frac{1}{4\pi}\int_{|\omega|=1} (tg(x+t\omega) + f(x+t\omega) + t\nabla f(x+t\omega)\cdot\omega)d\sigma(\omega).
\end{aligned}$$

Proof. Let $g_M := \mathcal{F}^{-1}\left[\frac{\sin|\xi|t}{|\xi|}1_{|\xi|\le M}\right]$, then by symmetry and passing to spherical coordinates,

$$\begin{aligned}(2\pi)^{3/2} g_M(x) &= \int_{|\xi|\le M} \frac{\sin|\xi|t}{|\xi|}e^{i\xi\cdot x}d\xi = \int_{|\xi|\le M} \frac{\sin|\xi|t}{|\xi|}\ e^{i|x|\xi_3}d\xi \\ &= \int_0^M d\rho\rho^2\int_0^{2\pi}d\theta\int_0^\pi d\phi\frac{\sin\rho t}{\rho}e^{i\rho|x|\cos\phi}\sin\phi \\ &= 2\pi\int_0^M d\rho\sin\rho t\ \frac{e^{i\rho|x|\cos\phi}}{-i|x|}\ \Big|_0^\pi \\ &= 2\pi\int_0^M d\rho\sin\rho t\frac{e^{i\rho|x|}-e^{-i\rho|x|}}{i|x|} = \frac{4\pi}{|x|}\int_0^M \sin\rho t\sin\rho\,|x|\ \ d\rho.\end{aligned}$$

Using

$$\sin A\sin B = \frac{1}{2}\ [\cos(A-B)-\cos(A+B)]$$

in this last equality, shows

$$\begin{aligned}g_M(x) &= (2\pi)^{-3/2}\frac{2\pi}{|x|}\int_0^M[\cos((t-|x|)\rho)-\cos((t+|x|)\rho)]d\rho \\ &= (2\pi)^{-3/2}\frac{\pi}{|x|}h_M(|x|)\end{aligned}$$

where

$$h_M(r) := \int_{-M}^M[\cos((t-r)\alpha)-\cos((t+r)\alpha)]d\alpha,$$

an odd function in r. Since

$$\mathcal{F}^{-1}\left[\frac{\sin|\xi|t}{|\xi|}\ \hat{f}(\xi)\right] = \lim_{M\to\infty}\ \mathcal{F}^{-1}(\hat{g}_M(\xi)\hat{f}(\xi)) = \lim_{M\to\infty}(g_M\star f)(x)$$

we need to compute $g_M\star f$. To this end

$$\begin{aligned}g_M\star f(x) &= \left(\frac{1}{2\pi}\right)^3\pi\int_{\mathbb{R}^3}\frac{1}{|y|}h_M(|y|)f(x-y)dy \\ &= \left(\frac{1}{2\pi}\right)^3\pi\int_0^\infty d\rho\frac{h_M(\rho)}{\rho}\int_{|y|=\rho}f(x-y)d\sigma(y) \\ &= \left(\frac{1}{2\pi}\right)^3\pi\int_0^\infty d\rho\ \frac{h_M(\rho)}{\rho}4\pi\rho^2\ \fint_{|y|=\rho}f(x-y)d\sigma(y) \\ &= \frac{1}{2\pi}\int_0^\infty d\rho\ h_M(\rho)\rho\bar{f}(x;\rho) = \frac{1}{4\pi}\int_{-\infty}^\infty d\rho\ h_M(\rho)\rho\bar{f}(x;\rho)\end{aligned}$$

where the last equality is a consequence of the fact that $h_M(\rho)\rho\bar{f}(x;\rho)$ is an even function of ρ. Continuing to work on this expression suing $\rho\to\rho\bar{f}(x;\rho)$ is odd implies

$$\begin{aligned} g_M \star f(x) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} d\rho \int_{-M}^{M} [\cos((t-\rho)\alpha) - \cos((t+\rho)\alpha)]d\alpha\, \rho \bar{f}(x;\rho) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho \int_{-M}^{M} \cos((t-\rho)\alpha)\rho \bar{f}(x;\rho)d\alpha \\ &= \frac{1}{2\pi}\mathrm{Re}\int_{-M}^{M} d\rho \int_{-\infty}^{\infty} d\alpha e^{i(t-\rho)\alpha}\rho\bar{f}(x;\rho)d\alpha \to t\bar{f}(x;t) \text{ as } M\to\infty \end{aligned}$$

using the 1 – dimensional Fourier inversion formula. ■

14.2.1. *Alternate Proof of Theorem 14.4.*

Lemma 14.5. $\lim_{M\to\infty} \int_{-M}^{M} \cos(\rho\lambda)d\rho = 2\pi\delta(\lambda)$.

Proof.

$$\int_{-M}^{M} \cos(\rho\lambda)d\rho = \int_{-M}^{M} e^{i\rho\lambda}d\rho$$

so that

$$\int_{\mathbb{R}} \phi(\lambda)\left[\int_{-M}^{M} e^{i\rho\lambda}d\rho\right]d\lambda \to \int_{\mathbb{R}} d\rho \int_{\mathbb{R}} d\lambda\phi(\lambda)e^{i\lambda\rho} = 2\pi\varphi(0)$$

by the Fourier inversion formula. ■

Proof. of Theorem 14.4 again.

$$\begin{aligned} \int \frac{\sin t|\xi|}{|\xi|}\, e^{i\xi\cdot x}d\xi &= \int \frac{\sin t\rho}{\rho}\, e^{i\rho|x|\cos\theta}\sin\theta d\theta\, d\varphi\, \rho^2\, d\rho \\ &= 2\pi\int \frac{\sin t\rho}{\rho}\, \frac{e^{i\rho|x|\lambda}}{i\rho|x|}\, \Big|_{\lambda=-1}^{1}\, d\rho \\ &= \frac{4\pi}{|x|}\int_0^{\infty} \sin t\rho\, \sin\rho|x|\, d\rho \\ &= \frac{2\pi}{|x|}\int_0^{\infty} [\cos(\rho(t-|x|)) - \cos(\rho(t+|x|))]\, d\rho \\ &= \frac{4\pi}{|x|}\int_{-\infty}^{\infty} [\cos(\rho(t-|x|)) - \cos(\rho(t+|x|))]\, d\rho \\ &= \frac{8\pi^2}{|x|}\, (\delta(t-|x|) - \delta(t+|x|)) \end{aligned}$$

Therefore

$$\begin{aligned} &\mathcal{F}^{-1}\left(\frac{\sin t|\xi|}{|\xi|}\right) * g(x) \\ &= \left(\frac{1}{2\pi}\right)^3 2\pi^2 \int_{\mathbb{R}^3} \frac{(\delta(t-|y|) - \delta(t+|y|))}{|y|}\, g(x-y)\, d\lambda(y) \\ &= \frac{1}{4\pi}\int_0^{\infty} (\delta(t-\rho) - \delta(t+\rho))g(x+\rho\omega)\frac{\rho^2}{\rho}\, d\rho\, d\sigma(\omega) \\ &= 1_{t>0}\, t\, \overline{g}(x;t) - 1_{t<0}\, (-t)\, \overline{g}(x;-t) \\ &= t\overline{g}(x;t) \end{aligned}$$

■

14.3. **Du Hamel's Principle.** The solution to

$$u_{tt} = \triangle u + f \text{ with } u(0,x) = 0 \text{ and } u_t(0,x) = 0$$

is given by

$$u(t,x) = \frac{1}{4\pi} \int_{B(x,t)} \frac{f(t-|y-x|, y)}{|y-x|}\, dy = \frac{1}{4\pi} \int_{|z|<t} \frac{f(t-|z|, x+z)}{|z|} dz. \tag{14.9}$$

Indeed, by Du Hamel's principle,

$$\begin{aligned} u(t,x) &= \int_0^t \frac{\sin((t-\tau)\sqrt{-\triangle})}{\sqrt{-\triangle}} f(\tau,x)d\tau = \int_0^t \frac{\sin(\tau\sqrt{-\triangle})}{\sqrt{-\triangle}} f(t-\tau,x)d\tau \\ &= \int_0^t \tau\overline{f}(t-\tau,x;\tau)d\tau = \frac{1}{4\pi}\int_0^t d\tau\ t^2 \int_{|\omega|=1} \frac{f(t-\tau, x+\tau\omega)}{\tau}\, d\sigma(\omega) \\ &= \frac{1}{4\pi} \int_{B(t,x)} \frac{f(t-|y-x|, y)}{|y-x|} dy \text{ (let } y = x+z) \\ &= \frac{1}{4\pi} \int_{|z|<t} \frac{f(t-|z|, x+z)}{|z|} dz. \end{aligned}$$

Thinking of $u(t,x)$ as pressure (14.9) says that the pressure at x at time t is the "average" of the disturbance at time $t-|y-x|$ at location y.

14.4. **Spherical Means.** Let $n \geq 2$ and suppose u solves $u_{tt} = \triangle u$. Since Δ is invariant under rotations, i.e. for $R \in O(n)$ we have $\triangle(u \circ R) = (\triangle u) \circ R$, it follows that $u \circ R$ is also a solution to the wave equation. Indeed,

$$(u(t,\cdot) \circ R)_{tt} = u_{tt}(t,\cdot) \circ R = \Delta u(t,\cdot) \circ R = \triangle(u(t,\cdot) \circ R).$$

By the linearity of the wave equation, this also implies, with dR denoting normalized Haar measure on $O(n)$, that

$$U(t,|x|) := \int_{O(n)} (u(t,Rx) \circ R) dR$$

must be a radial solution of the Wave equation. This implies

$$U_{tt} = \triangle_x U(t,|x|) = \frac{1}{r^{n-1}} \partial_r (r^{n-1} \partial_r U(t,r))_{r=|x|} = \left[\partial_r^2 U(t,r) + \frac{n-1}{r}\ \partial_r U(t,r)\right]_{r=|x|}.$$

Now

$$U(t,|x|) = \int_{0(n)} u(t,Rx) dR = \fint_{B(0,|x|)} u(t,y) d\sigma(y).$$

Using the translation invariance of Δ the same argument as above gives the following theorem**.**

Theorem 14.6. *Suppose $u_{tt} = \triangle u$ and $x \in \mathbb{R}^n$ and let*

$$U(t,r) := \overline{u}(t,x;r) := \fint_{\partial B(x,r)} u(t,y) d\sigma(y) = \fint_{\partial B(0,1)} u(t, x+r\omega) d\sigma(\omega).$$

Then U *solves*

$$U_{tt} = \frac{1}{r^{n-1}}\ \partial_r(r^{n-1}U_r)$$

with

$$U(0,r) = \fint_{\partial B(0,1)} u(0,x+r\omega)d\sigma(\omega) = \bar{f}(x;r)$$
$$U_t(0,r) = \overline{g}(x;r).$$

Proof. This has already been proved, nevertheless, let us give another proof which does not rely on using integration over $O(n)$. To this hence we compute

$$\begin{aligned}
\partial_r U(t,r) &= \partial_r \fint_{\partial B(0,1)} u(t,x+r\omega)d\sigma(\omega) \\
&= \fint_{\partial B(0,1)} \nabla u(t,x+r\omega)\cdot\omega d\sigma(\omega) \\
&= \frac{1}{\sigma\left(S^{n-1}\right)r^{n-1}}\int_{|y|=r}\nabla u(t,x+y)\cdot\hat{y}d\sigma(y) \\
&= \frac{1}{\sigma\left(S^{n-1}\right)r^{n-1}}\int_{|y|\le r}\Delta u(t,x+y)dy \\
&= \frac{1}{\sigma\left(S^{n-1}\right)r^{n-1}}\int_0^r d\rho\int_{|y|=\rho}\Delta u(t,x+y)d\sigma(y)
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{r^{n-1}}\partial_r(r^{n-1}U_r) &= \frac{1}{r^{n-1}}\partial_r\left[\frac{1}{\sigma\left(S^{n-1}\right)}\int_0^r d\rho\int_{|y|=\rho}\Delta u(t,x+y)d\sigma(y)\right] \\
&= \frac{1}{\sigma\left(S^{n-1}\right)r^{n-1}}\int_{|y|=r}\Delta u(t,x+y)d\sigma(y) \\
&= \fint_{|y|=r}\Delta u(t,x+y)d\sigma(y) \\
&= \fint_{|y|=r}u_{tt}(t,x+y)d\sigma(y) = U_{tt}.
\end{aligned}$$

■

We can now use the above result to solve the wave equation. For simplicity, assume $n=3$ and let $V(t,r) = r\,\overline{u}(t,x;r) = r\,U(t,r)$. Then for $r>0$ we have

$$\begin{aligned}
V_{rr} &= 2U_r + r\,U_{rr} = r(U_{rr} + \frac{2}{r}\ U_r) \\
&= r\,U_{tt} = V_{tt}.
\end{aligned}$$

This is also valid for $r<0$ because $V(t,r)$ is odd in r. Indeed for $r<0$, let $v(t,r) = V(t,-r)$, then $V_{rr}(t,r) = V_{rr}(t,-r) = V_{tt}(t,-r) = V_{tt}(t,r)$. By our solution to the one dimensional wave equation we find

$$V(t,r) = \frac{1}{2}(V(0,t+r)+V(0,r-t)) + \frac{1}{2}\int_{r-t}^{r+t}V_t(0,y)dy.$$

Now suppose that $u(0,x)=0$ and $u_t(0,x)=g(x)$, in which case

$$V(0,r)=0 \text{ and } V_t(0,r)=r\overline{g}(x,r)$$

and the previous equation becomes

Then

$$V(t,r)=\frac{1}{2}\int_{r-t}^{r+t} y\overline{g}(x,y)dy$$

and noting that

$$\frac{\partial}{\partial r}\Big|_0 V(t,r)=\overline{u}(t,x;0)=u(t,x)$$

we learn

$$u(t,x)=\frac{1}{2}\left[t\overline{g}(x;t)-(-t)\,\overline{g}(x;-t)\right]=t\overline{g}(x;t)$$

as before.

14.5. **Energy methods.**

Theorem 14.7 (Uniqueness on Bounded Domains)**.** *Let Ω be a bounded domain such that $\bar{\Omega}$ is a submanifold with C^2 – boundary and consider the boundary value problem*

$$\begin{aligned} &u_{tt}-\triangle u=h && \text{on } \Omega_T \\ &u=f && \text{on } (\partial\Omega\times[0,T])\cup(\Omega\times\{t=0\}) \\ &u_t=g && \text{on } \Omega\times\{t=0\} \end{aligned}$$

If $u\in C^2(\overline{\Omega}_T)$ then u is unique.

Proof. As usual, using the linearity of the equation, it suffices to consider the special case where $f=0$, $g=0$ and $h=0$ and to show this implies $u\equiv 0$. Let

$$E_\Omega(t)=\frac{1}{2}\int_\Omega\left[\dot{u}(t,x)^2+|\nabla u(t,x)|^2\right]dx.$$

Clearly by assumption, $E_\Omega(0)=0$ while the usual computation shows

$$\begin{aligned} \dot{E}_\Omega(t)&=(\dot{u},\ddot{u})_{L^2(\Omega)}+(\nabla u(t),\nabla\dot{u}(t))_{L^2(\Omega)} \\ &=(\dot{u},\Delta u)_{L^2(\Omega)}+(\nabla u(t),\nabla\dot{u}(t))_{L^2(\Omega)} \\ &=-(\nabla\dot{u}(t),\nabla u(t))_{L^2(\Omega)}+\int_{\partial\Omega}\dot{u}(t,x)\frac{\partial u(t,x)}{\partial n}d\sigma(x) \\ &\quad+(\nabla u(t),\nabla\dot{u}(t))_{L^2(\Omega)} \\ &=0 \end{aligned}$$

wherein we have used $u(t,x)=0$ implies $\dot{u}(t,x)=0$ for $x\in\partial\Omega$.

From this we conclude that $E_\Omega(t)=0$ and therefore $\dot{u}(t,x)=0$ and hence $u\equiv 0$. ■

The following proposition is expected to hold given the finite speed of propagation we have seen exhibited above for solutions to the wave equation.

Proposition 14.8 (Local Energy)**.** *Let $x\in\mathbb{R}^n$, $T>0$, $u_{tt}=\Delta u$ and define*

$$e(t):=E_{B(x,T-t)}(u;t):=\frac{1}{2}\int_{B(x,T-t)}\left[|\dot{u}(t,y)|^2+|\nabla u(t,y)|^2\right]dy.$$

Then $e(t)$ is decreasing for $0\le t\le T$.

Proof. First recall that

$$\frac{d}{dr}\int_{B(x,r)} f\,dx = \frac{d}{dr}\int_0^r d\rho\int_{|y-x|=\rho} f(y)d\sigma(y) = \int_{\partial B(x,r)} f\,d\sigma.$$

Hence

$$\begin{aligned}\dot{e}(t) &= \frac{d}{dt}\int_{B(x,R-t)} \{|\dot{u}(t,y)|^2 + |\nabla u(t,y)|^2\}dy \\ &= -\frac{1}{2}\int_{\partial B(x,R-t)} (|\dot{u}|^2 + |\nabla u|^2)d\sigma + \int_{B(x,R-t)} [\dot{u}\;\ddot{u} + \nabla u\cdot\nabla\dot{u}]\,dm \\ &= -\frac{1}{2}\int_{\partial B(x,R-t)} (|\dot{u}|^2 + |\nabla u|^2)d\sigma + \int_{B(x,R-t)} [\dot{u}\;\Delta u + \nabla u\cdot\nabla\dot{u}]\,dm \\ &= -\frac{1}{2}\int_{\partial B(x,R-t)} (|\dot{u}|^2 + |\nabla u|^2)d\sigma + 2\int_{\partial B(x,R-t)} \dot{u}\;\frac{\partial u}{\partial n}d\sigma \\ &= \frac{1}{2}\int_{\partial B(x,R-t)} \{2\;\dot{u}\,(\nabla u\cdot n) - (|\dot{u}|^2 + |\nabla u|^2)\}d\sigma \le 0\end{aligned}$$

wherein we have used the elementary estimate,

$$2\,(\nabla u\cdot n)\;\dot{u} \le 2|\nabla u|\;|\dot{u}| \le (|\dot{u}|^2 + |\nabla u|^2).$$

Therefore $e(t) \le e(0) = 0$ for all t i.e. $e(t) := 0$. ■

Corollary 14.9 (Uniqueness of Solutions)**.** *Suppose that u is a classical solution to the wave equation with $u(0,\cdot) = 0 = u_t(0,\cdot)$. Then $u \equiv 0$.*

Proof. Proposition 14.8 shows

$$\frac{1}{2}\int_{B(x,T-t)} \left[|\dot{u}(t,y)|^2 + |\nabla u(t,y)|^2\right] dy = E_{B(x,T)}(0) = 0$$

for all $0 \le t < T$ and $x \in \mathbb{R}^n$. This then implies that $\dot{u}(t,y) = 0$ for all $y \in \mathbb{R}^n$ and $0 \le t \le T$ and hence $u \equiv 0$. ■

Remark 14.10. This result also applies to certain class of weak type solutions in x by first convolving u with an approximate (spatial) delta function, say $u_\epsilon(t,x) = u(t,\cdot)*\delta_\epsilon(x)$. Then u_ϵ satisfies the hypothesis of Corollary 14.9 and hence is 0. Now let $\epsilon \downarrow 0$ to find $u \equiv 0$.

Remark 14.11. Proposition 14.8 also exhibits the finite speed of propagation of the wave equation.

14.6. **Wave Equation in Higher Dimensions.**

14.6.1. *Solution derived from the heat kernel.* Let

$$p_t^n(x) := \frac{1}{(2\pi t)^{n/2}}e^{-\frac{1}{2t}|x|^2}$$

and simply write p_t for p_t^1. Then

$$2\int_0^\infty \cos\omega t\;p_\lambda(t)dt = \int_{\mathbb{R}} e^{it\omega}p_\lambda(t)dt = e^{-\lambda\partial_t^2/2}e^{it\omega}|_{t=0} = e^{-\lambda\omega^2/2}.$$

Taking $\omega = \sqrt{-\Delta}$ and writing $u(t,x) := \cos\left(\sqrt{-\Delta}t\right)g(x)$ the previous identity gives

$$\begin{aligned}
2\int_0^\infty u(t,x)\ \frac{1}{\sqrt{2\pi\lambda}}e^{-\frac{1}{2\lambda}t^2}dt &= 2\int_0^\infty u(t,x)\ p_\lambda(t)dt \\
&= e^{\lambda\Delta/2}g(x) = \int_{\mathbb{R}^n} p_\lambda^n(y)g(x-y)dy \\
&= \int_{\mathbb{R}^n} \frac{1}{(2\pi\lambda)^{n/2}}e^{-\frac{1}{2\lambda}|y|^2}g(x-y)dy \\
&= \frac{1}{(2\pi\lambda)^{n/2}}\int_0^\infty d\rho e^{-\frac{1}{2\lambda}\rho^2}\int_{|y|=\rho} g(x-y)d\sigma(y) \\
&= \frac{\sigma(S^{n-1})}{(2\pi\lambda)^{n/2}}\int_0^\infty d\rho e^{-\frac{1}{2\lambda}\rho^2}\rho^{n-1}\bar{g}(x;\rho),
\end{aligned}$$

and so

$$\begin{aligned}
\int_0^\infty u(t,x)e^{-\frac{1}{2\lambda}t^2}dt &= \sqrt{\frac{\pi\lambda}{2}}\frac{\sigma(S^{n-1})}{(2\pi\lambda)^{n/2}}\int_0^\infty d\rho e^{-\frac{1}{2\lambda}\rho^2}\rho^{n-1}\bar{g}(x;\rho) \\
&= \sqrt{\frac{\pi}{2}}\frac{\sigma(S^{n-1})}{(2\pi)^{n/2}}\lambda^{-(n-1)/2}\int_0^\infty e^{-\frac{1}{2\lambda}t^2}t^{n-1}\bar{g}(x;t)dt.
\end{aligned}$$

Suppose $n = 2k+1$ and let $c_n := \sqrt{\frac{\pi}{2}}\frac{\sigma(S^{n-1})}{(2\pi)^{n/2}}$, then the above equation reads

$$\begin{aligned}
\int_0^\infty u(t,x)e^{-\frac{1}{2\lambda}t^2}dt &= c_n\lambda^{-k}\int_0^\infty e^{-\frac{1}{2\lambda}t^2}t^{2k}\bar{g}(x;t)dt \\
&= c_n\int_0^\infty \left(-\frac{1}{t}\partial_t\right)^k e^{-\frac{1}{2\lambda}t^2}t^{2k}\bar{g}(x;t)dt \\
&\overset{\text{I.B.P.}}{=} c_n\int_0^\infty e^{-\frac{1}{2\lambda}t^2}\left(\partial_t M_{t^{-1}}\right)^k\left[t^{2k}\bar{g}(x;t)\right]dt.
\end{aligned}$$

By the injectivity of the Laplace transform (after making the substitution $t \to \sqrt{t}$, this implies

$$\begin{aligned}
\cos\left(\sqrt{-\Delta}t\right)g(x) = u(t,x) &= c_n\left(\partial_t M_{t^{-1}}\right)^k\left[t^{2k}\bar{g}(x;t)\right] \\
&= c_n\left(\partial_t M_{t^{-1}}\partial_t M_{t^{-1}}\dots\partial_t M_{t^{-1}}\right)\left[t^{2k}\bar{g}(x;t)\right] \\
&= c_n\partial_t\left(\overbrace{M_{t^{-1}}\partial_t M_{t^{-1}}\dots M_{t^{-1}}\partial_t}^{k-1 \text{ times}}\right)\left[t^{2k-1}\bar{g}(x;t)\right] \\
&= c_n\partial_t\left(\frac{1}{t}\partial_t\right)^{k-1}\left[t^{2k-1}\bar{g}(x;t)\right].
\end{aligned}$$

Hence we have derived the following theorem.

Theorem 14.12. *Suppose $n = 2k+1$ is odd and let $c_n := \sqrt{\frac{\pi}{2}}\frac{\sigma(S^{n-1})}{(2\pi)^{n/2}}$, then*

$$\cos\left(\sqrt{-\Delta}t\right)g(x) = c_n\partial_t\left(\frac{1}{t}\partial_t\right)^{k-1}\left[t^{2k-1}\bar{g}(x;t)\right]$$

and

$$\frac{\sin\left(\sqrt{-\Delta}t\right)}{\sqrt{-\Delta}}f(x) = \int_0^t \cos\left(\sqrt{-\Delta}\tau\right) f(x)d\tau = c_n\left(\frac{1}{t}\partial_t\right)^{k-1}\left[t^{2k-1}\bar{g}(x;t)\right].$$

Proof. For the last equality we have used

$$\left(\frac{1}{t}\partial_t\right)^{k-1} t^{2k-1} = \text{const.} * t^{2k-1-2(k-1)} = \text{const.} * t$$

so that $\left(\frac{1}{t}\partial_t\right)^{k-1}\left[t^{2k-1}\bar{g}(x;t)\right] = O(t)$ and in particular is 0 at $t=0$. ■

14.6.2. *Solution derived from the Poisson kernel.* Suppose we want to write

$$e^{-|x|} = \int_0^\infty \phi(s)p_s(x)ds.$$

Since

$$\int_{\mathbb{R}} e^{-|x|}e^{i\lambda x}dx = 2\operatorname{Re}\int_0^\infty e^{-x}e^{i\lambda x}dx = 2\operatorname{Re}\left(\frac{1}{1-i\lambda}\right) = \frac{2}{1+\lambda^2}$$

and

$$\int_{\mathbb{R}} p_s(x)e^{i\lambda x}dx = e^{s\partial_x^2/2}e^{i\lambda x}|_{x=0} = e^{-s\lambda^2/2}$$

ϕ must satisfy

$$\int_0^\infty \phi(s)e^{-s\lambda^2/2}ds = \frac{2}{1+\lambda^2} = \int_0^\infty e^{-s\left(1+\lambda^2\right)/2}ds = \int_0^\infty e^{-s/2}e^{-s\lambda^2/2}ds.$$

from which it follows that $\phi(s) = e^{-s/2}$. Thus we have derived the formula

$$e^{-|x|} = \int_0^\infty (2\pi s)^{-1/2}e^{-s/2}e^{-\frac{1}{2s}x^2}ds \tag{14.10}$$

Let : $H \to H$ such that $A = A^*$ and $A \leq 0$. By the spectral theorem, we may "substitute" $x = t\sqrt{-A}$ into Eq. (14.10) to learn

$$e^{-t\sqrt{-A}} = \int_0^\infty (2\pi s)^{-1/2}e^{-s/2}e^{\frac{t^2}{2s}A}ds$$

and in particular taking $A = \Delta$ one finds

$$e^{-t\sqrt{-\Delta}} = \int_0^\infty (2\pi s)^{-1/2}e^{-s/2}e^{\frac{t^2}{2s}\Delta}ds$$

from which we conclude the convolution kernel $Q_t(x)$ for $e^{-t\sqrt{-\Delta}}$ is given by

$$\begin{aligned}
Q_t(x) &= \int_0^\infty (2\pi s)^{-1/2}e^{-s/2}p^n_{t^2s^{-1}}(x)ds = \int_0^\infty (2\pi s)^{-1/2}e^{-s/2}\frac{e^{-\frac{s}{2t^2}|x|^2}}{\left(2\pi t^2s^{-1}\right)^{n/2}}ds \\
&= (2\pi)^{-1/2}\left(2\pi t^2\right)^{-n/2}\int_0^\infty s^{\frac{n-1}{2}}e^{-s\frac{1}{2}\left(1+\frac{|x|^2}{t^2}\right)}ds \\
&= (2\pi)^{-1/2}\left(2\pi t^2\right)^{-n/2}\int_0^\infty s^{\frac{n+1}{2}}e^{-s\frac{1}{2}\left(1+\frac{|x|^2}{t^2}\right)}\frac{ds}{s}.
\end{aligned}$$

Making the substitution, $u = s\frac{1}{2}\left(1+\frac{|x|^2}{t^2}\right)$ in the previous integral shows

$$\begin{aligned}
Q_t(x) &= (2\pi)^{-1/2}\left(2\pi t^2\right)^{-n/2}\left[\frac{1}{2}\left(1+\frac{|x|^2}{t^2}\right)\right]^{-\frac{n+1}{2}}\int_0^\infty s^{\frac{n+1}{2}}e^{-s}\frac{ds}{s} \\
&= (2\pi)^{-1/2}2^{\frac{n+1}{2}}\left(2\pi\right)^{-n/2}t\left(t^2\right)^{-\frac{n+1}{2}}\left(1+\frac{|x|^2}{t^2}\right)^{-\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right) \\
&= 2^{\frac{n+1}{2}}\left(2\pi\right)^{-\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)\frac{t}{\left(t^2+|x|^2\right)^{\frac{n+1}{2}}} \\
&= \Gamma\left(\frac{n+1}{2}\right)\frac{t}{\pi^{\frac{n+1}{2}}\left(t^2+|x|^2\right)^{\frac{n+1}{2}}}.
\end{aligned}$$

Theorem 14.13. *Let*

$$c_n := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$$

$$Q_t(x) = c_n\frac{t}{\left(t^2+|x|^2\right)^{\frac{n+1}{2}}} \tag{14.11}$$

then

$$e^{-t\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^n} Q_t(x-y)f(y)dy. \tag{14.12}$$

Notice that if $u(t,x) := e^{-t\sqrt{-\Delta}}f(x)$, we have $\partial_t^2 u(t,x) = \left(\sqrt{-\Delta}\right)^2 u(t,x) = -\Delta u(t,x)$ with $u(0,x) = f(x)$. This explains why Q_t is the same Poisson kernel which we already saw in Eq. (9.36) of Theorem 9.31 above. To match the two results, observe Theorem 9.31 is for "spatial dimension" $n-1$ not n as in Theorem 14.13.

Integrating Eq. (14.12) from t to ∞ then implies

$$\begin{aligned}
\frac{1}{\sqrt{-\Delta}}e^{-t\sqrt{-\Delta}}f(x) &= \frac{-1}{\sqrt{-\Delta}}e^{-\tau\sqrt{-\Delta}}f(x)|_{\tau=t}^{\infty} \\
&= \int_t^\infty e^{-\tau\sqrt{-\Delta}}f(x)d\tau \\
&= \int_{\mathbb{R}^n}\int_t^\infty d\tau Q_\tau(x-y)f(y)dy.
\end{aligned}$$

Now

$$\begin{aligned}
\int_t^\infty Q_\tau(x-y)d\tau &= c_n\int_t^\infty \frac{\tau}{\left(\tau^2+|x|^2\right)^{\frac{n+1}{2}}}d\tau = \frac{c_n}{1-n}\left(\tau^2+|x|^2\right)^{\frac{1-n}{2}}|_{\tau=t}^{\infty} \\
&= \frac{c_n}{n-1}\left(t^2+|x|^2\right)^{-\frac{n-1}{2}}
\end{aligned}$$

and hence

$$\frac{1}{\sqrt{-\Delta}}e^{-t\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^n}\frac{c_n}{n-1}\left(t^2+|y|^2\right)^{-\frac{n-1}{2}}f(x-y)dy$$

and by analytic continuation,

$$\begin{aligned}\frac{1}{\sqrt{-\Delta}}e^{(it-\epsilon)\sqrt{-\Delta}}f(x) &= \frac{1}{\sqrt{-\Delta}}e^{-(\epsilon-it)\sqrt{-\Delta}}f(x) \\ &= \frac{c_n}{n-1}\int_{\mathbb{R}^n}\left((\epsilon-it)^2+|y|^2\right)^{-\frac{n-1}{2}}f(x-y)dy \\ &= \frac{c_n}{n-1}\int_{\mathbb{R}^n}\left(|y|^2-(t-i\epsilon)^2\right)^{-\frac{n-1}{2}}f(x-y)dy\end{aligned}$$

and hence

$$\frac{1}{\sqrt{-\Delta}}\sin\left(t\sqrt{-\Delta}\right)f(x) = c_n'\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n}\operatorname{Im}\left(|y|^2-(t-i\epsilon)^2\right)^{-\frac{n-1}{2}}f(x-y)dy.$$

Now if $|y|>|t|$ then

$$\lim_{\epsilon\downarrow 0}\left(|y|^2-(t-i\epsilon)^2\right)^{-\frac{n-1}{2}} = \left(|y|^2-t^2\right)^{-\frac{n-1}{2}}$$

is real so

$$\lim_{\epsilon\downarrow 0}\operatorname{Im}\left(|y|^2-(t-i\epsilon)^2\right)^{-\frac{n-1}{2}} = 0 \text{ if } |y|>|t|.$$

Similarly if n is odd $\lim_{\epsilon\downarrow 0}\left(|y|^2-(t-i\epsilon)^2\right)^{-\frac{n-1}{2}} = \left(|y|^2-t^2\right)^{-\frac{n-1}{2}}\in\mathbb{R}$ and so

$$\lim_{\epsilon\downarrow 0}\operatorname{Im}\left(|y|^2-(t-i\epsilon)^2\right)^{-\frac{n-1}{2}}$$

is a distribution concentrated on the sphere $|y|=|t|$ which is the sharp propagation again. See Taylor Vol. 1., p. 221– 225 for more on this approach. Let us examine here the special case $n=3$,

$$\operatorname{Im}\left(\frac{1}{|y|^2-(t-i\epsilon)^2}\right) = \operatorname{Im}\left(\frac{1}{|y|^2-t^2+\epsilon^2+2i\epsilon t}\right) = \frac{-2\epsilon t}{\left(|y|^2-t^2+\epsilon^2\right)^2+4\epsilon^2t^2}$$

so

$$\begin{aligned}I &:= \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n}\operatorname{Im}\left(\frac{1}{|y|^2-(t-i\epsilon)^2}\right)f(x-y)dy \\ &= \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n}\frac{-2\epsilon t}{\left(|y|^2-t^2+\epsilon^2\right)^2+4\epsilon^2t^2}f(x-y)dy \\ &= 4\pi\lim_{\epsilon\downarrow 0}\int_0^\infty\rho^2\frac{-2\epsilon t}{\left(\rho^2-t^2+\epsilon^2\right)^2+4\epsilon^2t^2}\bar{f}(x;\rho)d\rho \\ &= ct\lim_{\epsilon\downarrow 0}\int_0^\infty\rho^2\frac{\epsilon}{\left(\rho^2-t^2+\epsilon^2\right)^2+4\epsilon^2t^2}\bar{f}(x;\rho)d\rho.\end{aligned}$$

Make the change of variables $\rho = t + \epsilon s$ above to find

$$\begin{aligned} I &= ct \lim_{\epsilon \downarrow 0} \int_{-t/\epsilon}^{\infty} \frac{(t+\epsilon s)^2 \epsilon^2}{(2\epsilon s t + \epsilon^2 s^2 + \epsilon^2)^2 + 4\epsilon^2 t^2} \bar{f}(x; t+\epsilon s) ds \\ &= ct \lim_{\epsilon \downarrow 0} \int_{-t/\epsilon}^{\infty} \frac{(t+\epsilon s)^2}{(2st + \epsilon s^2 + \epsilon)^2 + 4t^2} \bar{f}(x; t+\epsilon s) ds \\ &= ct\bar{f}(x;t) \int_{-\infty}^{\infty} \frac{t^2}{4t^2 s^2 + 4t^2} ds = \frac{c}{4} t \bar{f}(x;t) \int_{-\infty}^{\infty} \frac{1}{s^2+1} ds \\ &= \frac{c}{4} \pi t \bar{f}(x;t) \end{aligned}$$

which up to an overall constant is the result that we have seen before.

14.7. **Explain Method of descent** $n = 2$**.**

$$u(t,x) = \frac{1}{2} \fint_{B(x,t)} \frac{t\, g(y) + t^2\, h(y) + t \nabla g(y) \cdot (y - x)}{(t^2 - |y-x|^2)^{1/2}}\, dy.$$

See constant coefficient PDE notes for more details on this.