1. 2^{ND} ORDER DIFFERENTIAL OPERATORS

Notations 1.1. Let Ω be a precompact open subset of $\mathbb{R}^d$, $A_{ij} = A_{ji}, A_i, A_0 \in BC^\infty(\overline{\Omega})$ for $i, j = 1, \ldots, d$,

$$p(x,\xi) := -\sum_{i,j=1}^{d} A_{ij}\xi_i\xi_j + \sum_{i=1}^{d} A_i\xi_i + A_0$$

and

$$L = p(x,\partial) = -\sum_{i,j=1}^{d} A_{ij}\partial_i\partial_j + \sum_{i=1}^{d} A_i\partial_i + A_0.$$

We also let

$$L^\dagger = -\sum_{i,j=1}^{d} \partial_i\partial_j M_{A_{ij}} - \sum_{i=1}^{d} \partial_i M_{A_i} + A_0.$$

Remark 1.2. The operators L and $L^\dagger$ have the following properties.

(1) The operator $L^\dagger$ is the formal adjoint of L, i.e.
$$(Lu, v) = (u, L^\dagger v) \text{ for all } u, v \in \mathcal{D}(\Omega) = C_c^\infty(\Omega).$$
(2) We may view L as an operator on $\mathcal{D}'(\Omega)$ via the formula $u \in \mathcal{D}'(\Omega) \to Lu \in \mathcal{D}'(\Omega)$ where
$$\langle Lu, \phi\rangle := \langle u, L^\dagger\phi\rangle \text{ for all } \phi \in C_c^\infty(\Omega).$$
(3) The restriction of L to $H^{k+2}(\Omega)$ gives a bounded linear transformation
$$L : H^{k+2}(\Omega) \to H^k(\Omega) \text{ for } k \in \mathbb{N}_0.$$
Indeed, L may be written as
$$L = -\sum_{i,j=1}^{d} M_{A_{ij}}\partial_i\partial_j + \sum_{i=1}^{d} M_{A_i}\partial_i + M_{A_0}.$$
Now $\partial_i : H^k(\Omega) \to H^{k+1}(\Omega)$ is bounded and $M_\psi : H^k(\Omega) \to H^k(\Omega)$ is bounded where $\psi \in BC^\infty(\overline{\Omega})$. Therefore, for $k \in \mathbb{N}_0$, $L : H^{k+2}(\Omega) \to H^k(\Omega)$ is bounded.

Definition 1.3. For $u \in \mathcal{D}'(\Omega)$, let

$$\|u\|_{H^{-1}(\Omega)} := \sup_{0\neq\phi\in\mathcal{D}(\Omega)} \frac{|\langle u,\phi\rangle|}{\|\phi\|_{H_0^1(\Omega)}}$$

and

$$H^{-1}(\Omega) := \left\{u \in \mathcal{D}'(\Omega) : \|u\|_{H^{-1}(\Omega)} < \infty\right\}.$$

Example 1.4. Let $\Omega = \mathbb{R}^d$ and $S \subset \Omega$ be the unit sphere in $\mathbb{R}^d$. Then define $\sigma \in \mathcal{D}'(\Omega)$ by

$$\langle\sigma,\phi\rangle := \int_S \phi d\sigma.$$

Let us shows that $\sigma \in H^{-1}(\Omega)$. For this let $T : H^1(\Omega) \to L^2(S, d\sigma)$ denote the trace operator, i.e. the unique bounded linear operator such that $T\phi = \phi|_S$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$. Since T is bounded,

$$|\langle\sigma,\phi\rangle| \le \sigma(S)^{1/2}\|T\phi\|_{L^2(S)} \le \sigma(S)^{1/2}\|T\|_{L(H^1(\Omega),L^2(S))}\|\phi\|_{H^{-1}(\Omega)}.$$

This shows $\sigma \in H^{-1}(\Omega)$ and $\|\sigma\|_{H^{-1}(\Omega)} \le \sigma(S)^{1/2} \|T\|_{L(H^1(\Omega),L^2(S))}$.

Lemma 1.5. *Suppose Ω is an open subset of $\mathbb{R}^d$ such that $\bar{\Omega}$ is a manifold with C^0 – boundary and $\Omega = \bar{\Omega}^o$, then the map $u \in \left[H_0^1(\Omega)\right]^* \to u|_{\mathcal{D}(\Omega)} \in H^{-1}(\Omega)$ is a unitary map of Hilbert spaces.*

Proof. By definition $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, and hence it follows that the map $u \in \left[H_0^1(\Omega)\right]^* \to u|_{\mathcal{D}(\Omega)} \in H^{-1}(\Omega)$ is isometric. If $u \in H^{-1}(\Omega)$, it has a unique extension to $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$ and this provides the inverse map. ∎

If we identify $L^2(\Omega) = H^0(\Omega)$ with elements of $\mathcal{D}'(\Omega)$ via $u \to (u, \cdot)_{L^2(\Omega)}$, then

$$\mathcal{D}'(\Omega) \supset H^{-1}(\Omega) \supset H^0(\Omega) = L^2(\Omega) \supset H^1(\Omega) \supset H^2(\Omega) \supset \dots$$

Proposition 1.6. *The following mapping properties hold:*

1. *If $\chi \in BC^1(\overline{\Omega})$. Then $M_\chi : H^{-1}(\Omega) \to H^{-1}(\Omega)$ is a bounded operator.*
2. *If $V = \sum_{i=1}^d M_{A_i}\partial_i + M_{A_0}$ with $A_i, A_0 \in BC^1(\bar{\Omega})$, then $V : L^2(\Omega) \to H^{-1}(\Omega)$ is a bounded operator.*
3. *The map $L : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ restricts to a bounded linear map from $H^1(\Omega)$ to $H^{-1}(\Omega)$. Also*

Proof. Let us begin by showing $M_\chi : H_0^1(\Omega) \to H_0^1(\Omega)$ is a bounded linear map. In order to do this choose $\chi_n \in C_c^\infty\left(\mathbb{R}^d\right)$ such that $\chi_n \to \chi$ in $BC^1(\overline{\Omega})$. Then for $\phi \in C_c^\infty(\Omega)$, $\chi_n\phi \in C_c^\infty(\Omega) \subset H_0^1(\Omega)$ and there is a constant $K < \infty$ such that

$$\|\chi_n\phi\|_{H_0^1(\Omega)}^2 \le K \|\chi_n\|_{BC^1(\overline{\Omega})} \|\phi\|_{H_0^1(\Omega)}^2 .$$

By density this estimate holds for all $\phi \in H_0^1(\Omega)$ and by replacing χ_n by $\chi_n - \chi_m$ we also learn that

$$\|(\chi_n - \chi_m)\phi\|_{H_0^1(\Omega)}^2 \le K \|\chi_n - \chi_m\|_{BC^1(\overline{\Omega})} \|\phi\|_{H_0^1(\Omega)}^2 \to 0 \text{ as } m, n \to \infty.$$

By completeness of $H_0^1(\Omega)$ it follows that $\chi\phi \in H_0^1(\Omega)$ for all $\phi \in H_0^1(\Omega)$ and

$$\|\chi_n\phi\|_{H_0^1(\Omega)}^2 \le K \|\chi\|_{BC^1(\overline{\Omega})} \|\phi\|_{H_0^1(\Omega)}^2 .$$

1. If $u \in H^{-1}(\Omega)$ and $\phi \in H_0^1(\Omega)$, then by definition, $\langle M_\chi u, \phi\rangle = \langle u, \chi\phi\rangle$ and therefore,

$$|\langle M_\chi u, \phi\rangle| = |\langle u, \chi\phi\rangle| \le \|u\|_{H^{-1}(\Omega)} \|\chi\phi\|_{H_0^1(\Omega)} \le K \|\chi\|_{BC^1(\overline{\Omega})} \|u\|_{H^{-1}(\Omega)} \|\phi\|_{H_0^1(\Omega)}$$

which implies $M_\chi u \in H^{-1}(\Omega)$ and

$$\|M_\chi u\|_{H^{-1}(\Omega)} \le K \|\chi\|_{BC^1(\overline{\Omega})} \|u\|_{H^{-1}(\Omega)}.$$

2. For $u \in L^2(\Omega)$ and $\phi \in C_c^\infty(\Omega)$

$$|\langle \partial_i u, \phi\rangle| = |\langle u, \partial_i\phi\rangle| \le \|u\|_{L^2(\Omega)} \cdot \|\partial_i\phi\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)} \|\phi\|_{H_0^1(\Omega)}$$

and therefore $\|\partial_i u\|_{H^{-1}(\Omega)} \le \|u\|_{L^2(\Omega)}$. For general $V = \sum_{i=1}^d M_{A_i}\partial_i + M_{A_0}$, we have

$$\begin{aligned}\|Au\|_{H^{-1}(\Omega)} &\le \sum_{i=1}^d K \|A_i\|_{BC^1(\overline{\Omega})} \|\partial_i u\|_{H^{-1}(\Omega)} + \|A_0\|_\infty \|u\|_{L^2(\Omega)} \\ &\le \left[\sum_{i=1}^d K \|A_i\|_{BC^1(\overline{\Omega})} + \|A_0\|_\infty\right] \|u\|_{L^2(\Omega)} .\end{aligned}$$

(3) Since $V : H^1(\Omega) \to L^2(\Omega)$ and $i : L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ are both bounded maps, to prove $L = -\sum_{i,j=1}^d M_{A_{ij}} \partial_i \partial_j + V$ is bounded from $H^1(\Omega) \to H^{-1}(\Omega)$ it suffices to show $M_{A_{ij}} \partial_i \partial_j : H^1(\Omega) \to H^{-1}(\Omega)$ is a bounded. But $M_{A_{ij}} \partial_i \partial_j : H^1(\Omega) \to H^{-1}(\Omega)$ is bounded since it is the composition of the following bounded maps:

$$H^1(\Omega) \overset{\partial_j}{\to} L^2(\Omega) \overset{\partial_i}{\to} H^{-1}(\Omega) \overset{M_{A_{ij}}}{\to} H^{-1}(\Omega).$$

■

Lemma 1.7. *Suppose $\chi \in BC^\infty(\overline{\Omega})$ then*

(1) *$[L, M_\chi] = V$ is a first order differential operator acting on $\mathcal{D}'(\Omega)$ which necessarily satisfies $V : H^k(\Omega) \to H^{k-1}(\Omega)$ for $k = 0, 1, 2, \ldots$ etc.*

(2) *If $u \in H^k(\Omega)$, then*

$$[L, M_\chi]u \in H^{k-1}(\Omega) \text{ for } k = 0, 1, 2, \ldots$$

and

$$\|[L, M_\chi]u\|_{H^{k-1}(\Omega)} \le C_k(\chi)\|u\|_{H^k(\Omega)}.$$

Proof. On smooth functions $u \in C^\infty(\Omega)$,

$$L(\chi u) = \chi L u - 2\sum_{i,j=1}^d A_{ij}\partial_i\chi \cdot \partial_j u + \left(\sum_{i=1}^d A_i \partial_i \chi - \sum_{i,j=1}^d A_{ij}\partial_i\partial_j\chi\right) \cdot u$$

and therefore

$$[L, M_\chi]u = -2\sum_{i,j=1}^d A_{ij}\partial_i\chi \cdot \partial_j u + \left(\sum_{i=1}^d A_i \partial_i \chi - \sum_{i,j=1}^d A_{ij}\partial_i\partial_j\chi\right) \cdot u =: Vu.$$

Similarly,

$$L^\dagger(\chi u) = -\sum_{i,j=1}^d \partial_i\partial_j[\chi A_{ij} u] - \sum_{i=1}^d \partial_i(\chi A_i u) + A_0 \chi u$$

$$= \chi L^\dagger u - 2\sum_{i,j=1}^d \partial_i\chi \cdot \partial_j[A_{ij}u] - \sum_{i=1}^d A_i\partial_i\chi \cdot u - \left(\sum_{i,j=1}^d A_{ij}\partial_i\partial_j\chi\right) u. \tag{1.1}$$

Noting that

$$\begin{aligned} V^\dagger u &= 2\sum_{i,j=1}^d \partial_j[\partial_i\chi \cdot A_{ij}u] + \left(\sum_{i=1}^d A_i\partial_i\chi - \sum_{i,j=1}^d A_{ij}\partial_i\partial_j\chi\right)\cdot u \\ &= 2\sum_{i,j=1}^d \partial_i\chi \cdot \partial_j[A_{ij}u] + \left(\sum_{i=1}^d A_i\partial_i\chi + \sum_{i,j=1}^d A_{ij}\partial_i\partial_j\chi\right)\cdot u, \end{aligned}$$

Eq. (1.1) may be written as

$$[L^\dagger, M_\chi] = -V^\dagger.$$

Now suppose $k = 0$, then in this case for $\phi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} |\langle [L, M_\chi]u, \phi\rangle| &= \left|\langle u, [M_\chi, L^\dagger]\phi\rangle\right| = \left|\langle u, V^\dagger\phi\rangle\right| \\ &\le \|u\|_{L^2(\Omega)}\|V^\dagger\phi\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega)}\|\phi\|_{H_0^1(\Omega)}. \end{aligned}$$

This implies $\|[L, M_\chi]u\|_{H^{-1}(\Omega)} \le C\|u\|_{L^2}$ and in particular $[L, M_\chi]u \in H^{-1}(\Omega)$. For $k > 0$, $[L, M_\chi]u = Vu$ with V as above and therefore by Proposition26.6, there exists $C < \infty$ such that $\|Vu\|_{H^{k-1}(\Omega)} \le C\|u\|_{H^k(\Omega)}$. ■

Definition 1.8. The operator L is **uniformly elliptic** on Ω if there exists $\epsilon > 0$ such that $(A_{ij}(x)) \ge \epsilon I$ for all $x \in \Omega$, i.e. $A_{ij}(x)\xi_i\xi_j \ge \epsilon |\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$.

Suppose now that L is uniformly elliptic. Let us outline the results to be proved below.

1.1. Outline of future results.

(1) We consider L with Dirichlet boundary conditions meaning we will view L as a mapping from $H_0^1(\Omega) \to H^{-1}(\Omega) = \left[H_0^1(\Omega)\right]^*$. Proposition 2.13 below states there exists $C = C(L) < \infty$ such that $(L + C) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism of Hilbert spaces. The proof uses the Dirichlet form

$$\mathcal{E}(u, v) := \langle Lu, v\rangle \text{ for } u, v \in H_0^1(\Omega).$$

Notice for $v \in \mathcal{D}(\Omega)$ and $u \in H_0^1(\Omega)$,

$$\begin{aligned}
\mathcal{E}(u, v) &= \langle Lu, v\rangle = \langle u, L^\dagger v\rangle \\
&= \int_\Omega u\left(-\partial_i\partial_j(A_{ij}v) - \partial_i\left(A_i v\right) + A_0 v\right) dm \\
&= \int_\Omega \left[\partial_i u \cdot \partial_j(A_{ij}v) - u\partial_i\left(A_i v\right) + uA_0 v\right] dm \\
&= \int_\Omega \left[A_{ij}\partial_i u \cdot \partial_j v + \left(A_i + \partial_j A_{ij}\right)\partial_i u \cdot v + A_0 uv\right] dm.
\end{aligned}$$

Since the last expression is continuous for $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we have shown

$$\mathcal{E}(u, v) = \int_\Omega \left[A_{ij}\partial_i u \cdot \partial_j v + \left(A_i + \partial_j A_{ij}\right)\partial_i u \cdot v + A_0 uv\right] dm$$

for all $u, v \in H_0^1(\Omega)$.

(2) To implement other boundary conditions, we will need to consider L acting on subspaces of $H^2(\Omega)$ which are determined by the boundary conditions. Rather than describe the general case here, let us consider an example where $L = -\Delta$ and the boundary condition is $\frac{\partial u}{\partial n} = \rho u$ on $\partial\Omega$ where $\partial_n u = \nabla u \cdot n$, n is the outward normal on $\partial\Omega$ and $\rho : \partial\Omega \to \mathbb{R}$ is a smooth function. In this case, let

$$D := \left\{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = \rho u \text{ on } \partial\Omega\right\}.$$

We will eventually see that D is a dense subspace of $H^1(\Omega)$. For $u \in D$ and $v \in H^1(\Omega)$,

$$\begin{aligned}
(-\Delta u, v) &= \int_\Omega \nabla u \cdot \nabla v \; dm - \int_{\partial\Omega} v\partial_n u \; d\sigma \\
&= \int_\Omega \nabla u \cdot \nabla v \; dm - \int_{\partial\Omega} \rho uv \; d\sigma =: \mathcal{E}(u, v).
\end{aligned} \tag{1.2}$$

The latter expression extends by continuity to all $u \in H^1(\Omega)$. Given $\mathcal{E}$ as in Eq. (1.2) let $-\Delta_{\mathcal{E}} : H^1(\Omega) \to \left[H^1(\Omega)\right]^*$ be defined by $-\Delta_{\mathcal{E}} u := \mathcal{E}(u, \cdot)$ so that $-\Delta_{\mathcal{E}} u$ is an extension of $-\Delta u$ as a linear functional on $H_0^1(\Omega)$ to one on $H^1(\Omega) \supset H_0^1(\Omega)$. It will be shown below that there exists $C < \infty$ such that $(-\Delta_{\mathcal{E}} + C) : H^1(\Omega) \to \left[H^1(\Omega)\right]^*$ is an isomorphism of Hilbert spaces.

(3) The Dirichlet form $\mathcal{E}$ in Eq. (1.2) may be rewritten in a way as to avoid the surface integral term. To do this, extend the normal vector field n along $\partial\Omega$ to a smooth vector field on $\bar{\Omega}$. Then by integration by parts,

$$\int_{\partial\Omega} \rho uv \, d\sigma = \int_{\partial\Omega} n_i^2 \rho uv \, d\sigma = \int_\Omega \partial_i \left[n_i \rho uv\right] dm$$
$$= \left[\int_\Omega \nabla \cdot (\rho n) \ uv + \rho n_i \partial_i u \cdot v + \rho n_i u \cdot \partial_i v\right] dm.$$

In this way we see that the Dirichlet form $\mathcal{E}$ in Eq. (1.2) may be written as

$$\mathcal{E}(u,v) = \int_\Omega \left[\nabla u \cdot \nabla v + a_{i0} \partial_i u \cdot v + a_{0i} u \partial_i v + a_{00} uv\right] \ dm \tag{1.3}$$

with $a_{00} = \nabla \cdot (\rho n)$, $a_{i0} = \rho n_i = a_{0i}$. This should motivate the next section where we consider generalizations of the form $\mathcal{E}$ in Eq. (1.3).

2. Dirichlet Forms

In this section Ω will be an open subset of $\mathbb{R}^d$.

2.1. Basics.

Notation 2.1 (Dirichlet Forms)**.** For $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha|, |\beta| \leq 1$, suppose $a_{\alpha,\beta} \in BC^\infty\left(\bar{\Omega}\right)$ and $\rho \in BC^\infty\left(\bar{\Omega}\right)$ with $\rho > 0$, let

$$\mathcal{E}(u,v) = \sum_{|\alpha|,|\beta| \leq 1} \int_\Omega a_{\alpha\beta} \partial^\alpha u \cdot \partial^\beta v \ d\mu \tag{2.1}$$

where $d\mu := \rho dm$. We will also write $(u,v) := \int_\Omega uv \ d\mu$ and L^2 for $L^2(\Omega, \mu)$. In the sequel we will often write $a_{i,\beta}$ for $a_{e_i,\beta}$, $a_{\alpha,j}$ for a_{α,e_j} and a_{ij} for a_{e_i,e_j}.

Proposition 2.2. *Let $\mathcal{E}$ be as in Notation 2.1 then*

$$|\mathcal{E}(u,v)| \leq C \|u\|_{H^1} \|v\|_{H^1} \text{ for all } u, v \in H^1$$

where C is a constant depending on d and upper bounds for $\left\{\|a_{\alpha\beta}\|_{BC(\bar{\Omega})} : |\alpha|, |\beta| \leq 1\right\}$.

Proof. To simplify notation in the proof, let $\|\cdot\|$ denote the $L^2(\Omega, \mu)$ – norm. Then

$$|\mathcal{E}(u,v)| \leq C \sum_{ij} \{\|\partial_i u\| \ \|\partial_j v\| + \|\partial_i u\| \ \|v\| + \|u\| \ \|\partial_i v\| + \|u\| \ \|v\|\}$$
$$\leq C \|u\|_{H^1} \cdot \|v\|_{H^1}.$$

■

Notation 2.3. Let $\mathcal{E}$ be a Dirichlet form as in Proposition 2.2, then we define bounded linear operators $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{L}_{\mathcal{E}}^\dagger$ from $H^1(\Omega) \to \left[H^1(\Omega)\right]^*$ by

$$\mathcal{L}_{\mathcal{E}} u := \mathcal{E}(u, \cdot) \text{ and } \mathcal{L}_{\mathcal{E}}^\dagger u := \mathcal{E}(\cdot, u).$$

It follows directly from the definitions that $\langle \mathcal{L}_{\mathcal{E}} u, v\rangle = \langle u, \mathcal{L}_{\mathcal{E}}^{\dagger} v\rangle$ for all $u, v \in H^1(\Omega)$. The Einstein summation convention will be used below when convenient.

Proposition 2.4. *Suppose Ω is a precompact open subset of $\mathbb{R}^d$ such that $\bar{\Omega}$ is a manifold with C^2 – boundary, Then for all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$,*

$$\langle \mathcal{L}_{\mathcal{E}} u, v\rangle = \mathcal{E}(u,v) = (Lu, v) + \int_{\partial\Omega} Bu \cdot v \;\rho d\sigma \tag{2.2}$$

and for all $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$,

$$\langle u, \mathcal{L}_{\mathcal{E}}^{\dagger} v\rangle = \mathcal{E}(u,v) = (u, L^{\dagger} v) + \int_{\partial\Omega} u \cdot B^{\dagger} v \;\rho d\sigma, \tag{2.3}$$

where

$$B = n_j a_{ij} \partial_i + n_j a_{0j} = n \cdot a\nabla + n \cdot a_{0,\cdot}, \tag{2.4}$$

$$B^{\dagger} = n_i \left[a_{ij}\partial_j + a_{i0}\right] = an \cdot \nabla + n \cdot a_{\cdot,0}, \tag{2.5}$$

$$Lu := \rho^{-1} \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\beta|} \partial^{\beta} \left[\rho a_{\alpha\beta} \partial^{\alpha} u\right] \tag{2.6}$$

and

$$L^{\dagger} v := \rho^{-1} \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\alpha|} \partial^{\alpha} \left[\rho a_{\alpha\beta} \partial^{\beta} v\right] \tag{2.7}$$

We may also write L, $L^{\dagger}$ as

$$L = -a_{ij}\partial_j\partial_i + \left(a_{i0} - a_{0j} - \rho^{-1}\partial_j\left[\rho a_{ij}\right]\right)\partial_i + \left(a_{00} - \rho^{-1}\partial_j\left[\rho a_{0j}\right]\right), \tag{2.8}$$

$$L^{\dagger} = -a_{ij}\partial_i\partial_j + \left(a_{0j} - a_{j0} - \rho^{-1}\partial_i\left[\rho a_{ij}\right]\right)\partial_j + \left(a_{00} - \rho^{-1}\partial_i\left[\rho a_{i0}\right]\right). \tag{2.9}$$

Proof. Suppose $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then by integration by parts,

$$\begin{aligned}
\mathcal{E}(u,v) &= \sum_{|\alpha|,|\beta|\le 1} \int_{\Omega} (-1)^{|\beta|} \rho^{-1} \partial^{\beta}\left[\rho a_{\alpha\beta}\partial^{\alpha} u\right] \cdot v \; d\mu + \sum_{|\alpha|\le 1}\sum_{j=1}^{d} \int_{\partial\Omega} n_j \left[a_{\alpha j}\partial^{\alpha} u\right] \cdot v \;\rho d\sigma \\
&= (Lu, v) + \int_{\partial\Omega} Bu \cdot v \;\rho d\sigma,
\end{aligned}$$

where

$$\begin{aligned}
Lu &= \rho^{-1} \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\beta|} \partial^{\beta}\left[\rho a_{\alpha\beta}\partial^{\alpha} u\right] = -\rho^{-1} \sum_{|\alpha|\le 1}\sum_{j=1}^{d} \partial_j\left(\rho a_{\alpha j}\partial^{\alpha} u\right) + \sum_{|\alpha|\le 1} a_{\alpha 0}\partial^{\alpha} u \\
&= -\rho^{-1}\sum_{i,j=1}^{d} \partial_j\left(\rho a_{ij}\partial_i u\right) - \rho^{-1}\sum_{j=1}^{d}\partial_j\left(\rho a_{0j} u\right) + \sum_{i=1}^{d} a_{i0}\partial_i u + a_{00} u \\
&= -\sum_{i,j=1}^{d} a_{ij}\partial_j\partial_i u - \rho^{-1}\sum_{i,j=1}^{d}\left(\partial_j\left[\rho a_{ij}\right]\right)\partial_i u - \sum_{j=1}^{d} a_{0j}\partial_j u \\
&\quad - \rho^{-1}\sum_{j=1}^{d}\left(\partial_j\left[\rho a_{0j}\right]\right) u + \sum_{i=1}^{d} a_{i0}\partial_i u + a_{00} u
\end{aligned}$$

and

$$Bu = \sum_{|\alpha|\le 1}\sum_{j=1}^{d} n_j\left(a_{\alpha j}\partial^{\alpha}u\right) = \sum_{i,j=1}^{d} n_j a_{ij}\partial_i u + \sum_{j=1}^{d} n_j a_{0j} u.$$

Similarly for $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$,

$$\begin{aligned}\mathcal{E}(u,v) &= \sum_{|\alpha|,|\beta|\le 1}\int_{\Omega} u\cdot(-1)^{|\alpha|}\rho^{-1}\partial^{\alpha}\left[\rho a_{\alpha\beta}\partial^{\beta}v\right]\ d\mu + \sum_{i=1}^{d}\sum_{|\beta|\le 1}\int_{\Omega} u\cdot n_i\left[a_{i\beta}\partial^{\beta}v\right]\ \rho d\sigma\\ &= (u, L^{\dagger}v) + \int_{\partial\Omega} u\cdot B^{\dagger}v\ \rho d\sigma,\end{aligned}$$

where $B^{\dagger}v = n_i\left[a_{ij}\partial_j + a_{i0}\right]$ and

$$\begin{aligned}L^{\dagger}v &= -\rho^{-1}\partial_i\left(\rho a_{ij}\partial_j v\right) + a_{0j}\partial_j v - \rho^{-1}\partial_i\left(\rho a_{i0}v\right) + a_{00}v\\ &= -a_{ij}\partial_i\partial_j v - \rho^{-1}\left(\partial_i\left[\rho a_{ij}\right]\right)\partial_j v + a_{0j}\partial_j v - a_{i0}\partial_i v - \rho^{-1}\left(\partial_i\left[\rho a_{i0}\right]\right)v + a_{00}v\\ &= \left[-a_{ij}\partial_i\partial_j + \left(a_{0j} - a_{j0} - \rho^{-1}\partial_i\left[\rho a_{ij}\right]\right)\partial_j + a_{00} - \rho^{-1}\partial_i\left[\rho a_{i0}\right]\right]v.\end{aligned}$$

■

Proposition 2.4 shows that to the Dirichlet form $\mathcal{E}$ there is an associated second order elliptic operator L along with boundary conditions B as in Eqs. (2.6) and (2.4). The next proposition shows how to reverse this procedure and associate a Dirichlet form $\mathcal{E}$ to a second order elliptic operator L with boundary conditions.

Proposition 2.5 (Following Folland p. 240.). *Let $A_j, A_0\rho \in BC^{\infty}(\Omega)$ and $A_{ij} = A_{ji} \in BC^{\infty}(\Omega)$ with $(A_{ij}) > 0$ and $\rho > 0$ and let*

$$L = -A_{ij}\partial_i\partial_j + A_i\partial_i + A_0 \tag{2.10}$$

and $(u,v) := \int_{\Omega} uv\rho dm$. Also suppose $\alpha : \partial\Omega \to \mathbb{R}$ and $V : \partial\Omega \to \mathbb{R}^d$ are smooth functions such that $V(x)\cdot n(x) > 0$ for all $x \in \partial\Omega$ and let $B_0 u := V\cdot\nabla u + \alpha u$. Then there exists a Dirichlet form $\mathcal{E}$ as in Notation 2.1 and $\beta \in C^{\infty}(\partial\Omega \to (0,\infty))$ such that Eq. (2.2) holds with $Bu = \beta B_0 u$. In particular if $u \in H^2(\Omega)$, then $Bu = 0$ iff $B_0 u = 0$ on $\partial\Omega$.

Proof. Since mixed partial derivatives commute on $H^2(\Omega)$, the term $a_{ij}\partial_j\partial_i$ in Eq. (2.8) may be written as

$$\frac{1}{2}\left(a_{ij} + a_{ji}\right)\partial_j\partial_i.$$

With this in mind we must find coefficients $\{a_{\alpha,\beta} : |\alpha|, |\beta| \le 1\}$ as in Notation 2.1, such that

$$A_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right), \tag{2.11}$$

$$A_i = \left(a_{i0} - a_{0j} - \rho^{-1}\partial_j\left[\rho a_{ij}\right]\right), \tag{2.12}$$

$$A_0 = a_{00} - \rho^{-1}\partial_j\left[\rho a_{0j}\right], \tag{2.13}$$

$$a^{\mathrm{tr}}n = \beta V \text{ and} \tag{2.14}$$

$$n_i a_{0i} = \beta\alpha. \tag{2.15}$$

Eq. (2.11) will be satisfied if

$$a_{ij} = A_{ij} + c_{ij}$$

where $c_{ij} = -c_{ji}$ are any functions in $BC^\infty(\Omega)$. Dotting Eq. (2.14) with n shows that

$$\beta = \frac{a^{\mathrm{tr}} n \cdot n}{V \cdot n} = \frac{n \cdot an}{V \cdot n} = \frac{n \cdot An}{V \cdot n} \tag{2.16}$$

and Eq. (2.14) may now be written as

$$w := An - \frac{n \cdot An}{V \cdot n} V = cn \tag{2.17}$$

which means we have to choose $c = (c_{ij})$ so that Eq. (2.17) holds. This is easily done, since $w \cdot n = 0$ by construction we may define $c\xi := w(n \cdot \xi) - n(w \cdot \xi)$ for $\xi \in \mathbb{R}^d$. Then c is skew symmetric and $cn = w$ as desired. Since c_{ij} are smooth functions on $\partial\Omega$, a partition of unity argument shows that $c_{ij} = -c_{ji}$ may be extended to element of $C^\infty(\bar{\Omega})$. (These extensions are highly non-unique but it does not matter.) With these choices, Eq. (2.11) and Eq. (2.14) now hold with β as in Eq. (2.16). We now choose $a_{0i} \in C^\infty(\bar{\Omega})$ such that $a_{0i} = \beta\alpha n_i$ on $\partial\Omega$. Once these choices are made, it should be clear that Eqs. (2.13) and (2.14) may be solved uniquely for the functions a_{0j} and a_{00}. ■

2.2. **Weak Solutions for Elliptic Operators.** For the rest of this subsection we will assume $\rho = 1$. This can be done here by absorbing ρ into the coefficient $a_{\alpha\beta}$.

Definition 2.6. The Dirichlet for $\mathcal{E}$ is **uniformly elliptic** on Ω if there exists $\epsilon > 0$ such that $(a_{ij}(x)) \geq \epsilon I$ for all $x \in \Omega$, i.e. $a_{ij}(x)\xi_i\xi_j \geq \epsilon\,|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$.

Assumption 1. For the remainder of this chapter, it will be assumed that $\mathcal{E}$ is uniformly elliptic on Ω.

Lemma 2.7. *If* $\xi^2 \leq A\xi + B$ *then* $\xi^2 \leq A^2 + 2B$.

Proof. $\xi^2 \leq \frac{1}{2}A^2 + \frac{1}{2}\xi^2 + B$. Therefore $\frac{1}{2}\xi^2 \leq \frac{1}{2}A^2 + B$ or $\xi^2 \leq A^2 + 2B$. ■

Theorem 2.8. *Keeping the notation and assumptions of Proposition 2.2 along with Assumption 1, then*

$$\mathcal{E}(u,u) + C_\epsilon \|u\|_{L^2(\Omega)} \geq \frac{\epsilon}{2}\|u\|_{H^1(\Omega)}, \tag{2.18}$$

where $C_\epsilon = \frac{2C^2}{\epsilon} + C + \frac{\epsilon}{2}$.

Proof. To simplify notation in the proof, let $\|\cdot\|$ denote the $L^2(\Omega)$ – norm. Since

$$\int_\Omega a_{ij}\partial_i u \cdot \partial_j u \; dm \geq \epsilon \int_\Omega |\nabla u|^2 dx = \epsilon \|\nabla u\|_{L^2}^2,$$

$$\mathcal{E}(u,u) \geq \epsilon\|\nabla u\|_{L^2}^2 - C(\|\nabla u\| \; \|u\| + \|u\|^2)$$

and so

$$\|\nabla u\|^2 \leq \frac{C}{\epsilon}\|u\|\|\nabla u\| + \left(\frac{1}{\epsilon}\mathcal{E}(u,u) + \frac{C}{\epsilon}\|u\|^2\right).$$

Therefore by Lemma 2.7 with $A = \frac{C}{\epsilon}\|u\|$, $B = \left(\frac{1}{\epsilon}\mathcal{E}(u,u) + \frac{C}{\epsilon}\|u\|^2\right)$ and $\xi = \|\nabla u\|$,

$$\|\nabla u\|^2 \leq \frac{C^2}{\epsilon^2}\|u\|^2 + \frac{2}{\epsilon}(\mathcal{E}(u,u) + \frac{C}{\epsilon}\|u\|^2) = \frac{2}{\epsilon}\mathcal{E}(u,u) + \left(\frac{C^2}{\epsilon^2} + \frac{2C}{\epsilon}\right)\|u\|^2.$$

Hence

$$\frac{\epsilon}{2}\|\nabla u\|^2 \le \mathcal{E}(u,u) + \left(\frac{2C^2}{\epsilon} + C\right)\|u\|^2$$

which, after adding $\frac{\epsilon}{2}\|u\|^2$ to both sides of this equation, gives Eq. (2.18). ■

The following theorem is an immediate consequence of Theorem 2.8 and the Lax-Milgram Theorem 29.9.

Corollary 2.9. *The quadratic form*

$$Q(u,v) := \mathcal{E}(u,v) + C_\epsilon(u,v)$$

satisfies the assumptions of the Lax Milgram Theorem 29.9 on $H^1(\Omega)$ or any closed subspace X of $H^1(\Omega)$.

Theorem 2.10 (Weak Solutions)**.** *Let $\mathcal{E}$ be as in Notation 2.1 and C_ϵ be as in Theorem 2.8,*

$$Q(u,v) := \mathcal{E}(u,v) + C_\epsilon(u,v) \text{ for } u,v \in H^1(\Omega)$$

and X be a closed subspace of $H^1(\Omega)$. Then the maps $\mathcal{L} : X \to X^*$ and $\mathcal{L}^\dagger : X \to X^*$ defined by

$$\mathcal{L}v := Q(v,\cdot) = (\mathcal{L}_{\mathcal{E}} + C)\,v \text{ and}$$
$$\mathcal{L}^\dagger v := Q(\cdot,v) = \left(\mathcal{L}_{\mathcal{E}}^\dagger + C\right) v$$

are linear isomorphisms of Hilbert spaces satisfying

$$\left\|\mathcal{L}^{-1}\right\|_{L(X^*,X)} \le \frac{2}{\epsilon} \text{ and } \left\|(\mathcal{L}^\dagger)^{-1}\right\|_{L(X^*,X)} \le \frac{2}{\epsilon}.$$

In particular for $f \in X^*$, there exist a unique solution $u \in X$ to $\mathcal{L}u = f$ and this solution satisfies the estimate

$$\|u\|_{H^1(\Omega)} \le \frac{2}{\epsilon}\|f\|_{X^*}.$$

Remark 2.11. If $X \supset H_0^1(\Omega)$ and $u \in X$ then for $\phi \in C_c^\infty(\Omega) \subset X$,

$$\langle \mathcal{L}u, \phi\rangle = Q(u,\phi) = \left(u, \left(L^\dagger + C\right)\phi\right) = \langle (L+C)u, \phi\rangle.$$

That is to say $\mathcal{L}u|_{C_c^\infty(\Omega)} = (L+C)\,u$. In particular any solution $u \in X$ to $\mathcal{L}u = f \in X^*$ solves

$$(L+C)\,u = f|_{C_c^\infty(\Omega)} \in \mathcal{D}'(\Omega)\,.$$

Remark 2.12. Suppose that $\Gamma \subset \partial\Omega$ is a measurable set such that $\sigma(\Gamma) > 0$ and $X_\Gamma := \left\{u \in H^1(\Omega) : 1_\Gamma u|_{\partial\Omega} = 0\right\}$. If $u \in H^2(\Omega)$ solves $\mathcal{L}u = f$ for some $f \in L^2(\Omega) \subset X^*$, then by Proposition 15.6,

$$(f,u) := \langle \mathcal{L}u, v\rangle = \mathcal{E}(u,v) + C(u,v) = ((L+C)\,u, v) + \int_{\partial\Omega} Bu \cdot v \, d\sigma \tag{2.19}$$

for all $v \in X_\Gamma \subset H^1(\Omega)$. Taking $v \in \mathcal{D}(\Omega) \subset X_\Gamma$ in Eq. (2.19) shows $(L+C)\,u = f$ a.e. and

$$\int_{\partial\Omega} Bu \cdot v \, d\sigma = 0 \text{ for all } v \in X_\Gamma.$$

Therefore we may conclude, u solves

$$\begin{aligned}(L+C)\,u &= f \text{ a.e. with}\\ Bu(x) &= 0 \text{ for } \sigma-\text{a.e. } x \in \partial\Omega\setminus\Gamma \text{ and}\\ u(x) &= 0 \text{ for } \sigma-\text{a.e. } x\in\Gamma.\end{aligned}$$

The following proposition records the important special case of Theorem 2.10 when $X = H_0^1(\Omega)$ and hence $X^* = H^{-1}(\Omega)$. The point to note here is that $\mathcal{L}u = (L+C)\,u$ when $X = H_0^1(\Omega)$, i.e. $\mathcal{L}u$ equals $[(L+C)\,u]$ extended by continuity to a linear functional on $X^* = \left[H_0^1(\Omega\right]^*$.

Proposition 2.13. *Assume L is elliptic as above. Then there exist $C > 0$ sufficiently large such that $(L+C) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is bijective with bounded inverse. Moreover*

$$\|(L+C)^{-1}\|_{L\left(H_0^1(\Omega),H^{-1}(\Omega)\right)} \le 2/\epsilon$$

or equivalently

$$\|u\|_{H_0^1(\Omega)} \le \frac{2}{\epsilon}\|(L+C)u\|_{H^{-1}(\Omega)} \text{ for all } u \in H_0^1(\Omega).$$

Our next goal, see Theorem 3.13, is to prove the elliptic regularity result, namely if $X = H_0^1(\Omega)$ or $X = H^1(\Omega)$ and $u \in X$ satisfies $\mathcal{L}u \in H^k(\Omega)$, then $u \in H^{k+2}(\Omega)\cap X$.

3. Elliptic Regularity

Assume that $\bar{\Omega}$ is a compact manifold with C^2 – boundary and satisfying $\bar{\Omega}^o = \Omega$ and let $\mathcal{E}$ be the Dirichlet form defined in Notation 15.3 and L be as in Eq. (15.12) or Eq. (15.14). We will assume $\mathcal{E}$ or equivalently that L is uniformly elliptic on Ω. This section is devoted to proving the following elliptic regularity theorem.

Theorem 3.1 (Elliptic Regularity Theorem). *Suppose $X = H_0^1(\Omega)$ or $H^1(\Omega)$ and $\mathcal{E}$ is as above. If $u \in X$ such that $\mathcal{L}_{\mathcal{E}} u \in H^k(\Omega)$ for some $k \in \mathbb{N}_0 \cup \{-1\}$, then $u \in H^{k+2}(\Omega)$ and*

$$\|u\|_{H^{k+2}}(\Omega) \le C(\|\mathcal{L}_{\mathcal{E}} u\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{3.1}$$

3.1. Interior Regularity.

Theorem 3.2. *To each $\chi \in C_c^\infty(\Omega)$ there exist a constant $C = C(\chi)$ such that*

$$\|\chi u\|_{H^1(\Omega)} \le C\{\|Lu\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Omega)}\} \text{ for all } u \in H^1(\Omega). \tag{3.2}$$

In particular, if W is a precompact open subset of Ω, then

$$\|u\|_{H^1(W)} \le C\{\|Lu\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Omega)}\}. \tag{3.3}$$

Proof. For $u \in H^1(\Omega)$, $\chi u \in H_0^1(\Omega)$ and hence by Proposition 2.13, Proposition 1.6 and Lemma 1.7,

$$\begin{aligned}
\|\chi u\|_{H^1(\Omega)} &\le \frac{2}{\epsilon}\|(L + C_\epsilon)(\chi u)\|_{H^{-1}(\Omega)} = \frac{2}{\epsilon}\|\chi(L + C_\epsilon)u + [L, M_\chi]u\|_{H^{-1}(\Omega)} \\
&\le \frac{2}{\epsilon}C(\chi)\left\{\|(L + C_\epsilon)u\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Omega)}\right\} \\
&\le \frac{2}{\epsilon}C(\chi)\left\{\|Lu\|_{H^{-1}(\Omega)} + C_\epsilon\|u\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Omega)}\right\}
\end{aligned}$$

from which Eq. (3.2) follows. To prove Eq. (3.3), choose $\chi \in C_c^\infty(\Omega, [0,1])$ such that $\chi = 1$ on a neighborhood of $\bar{W}$ in which case

$$\|u\|_{H^1(W)} = \|\chi u\|_{H^1(W)} \le \|\chi u\|_{H^1(\Omega)} \le C\{\|Lu\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Omega)}\}.$$

■

Exercise 3.1. Let $v \in \mathbb{R}^d$ with $|v| = 1$, $u \in L^2(\Omega)$ and W be an open set such that $\bar{W} \sqsubset\sqsubset \Omega$. Show, for all $h \ne 0$ sufficiently small, that $\partial_v^h u \in H^{-1}(W)$ and

$$\left\|\partial_v^h u\right\|_{H^{-1}(W)} \le \|u\|_{L^2(\Omega)}.$$

3.1. Let W_1 be a precompact open subset of Ω such that $\bar{W} \subset W_1 \subset \bar{W}_1 \subset \Omega$. Then for $\phi \in \mathcal{D}(W)$ and h close to zero,

$$\begin{aligned}
\left|\langle \partial_v^h u, \phi\rangle\right| &= \left|\langle u, \partial_v^{-h}\phi\rangle\right| \le \|u\|_{L^2(W_1)}\left\|\partial_v^{-h}\phi\right\|_{L^2(W_1)} \\
&\le \|u\|_{L^2(W_1)}\|\partial_v\phi\|_{L^2(\Omega)} \text{ (Theorem 26.13)} \\
&\le \|u\|_{L^2(\Omega)}\|\phi\|_{H^1(\Omega)}.
\end{aligned}$$

Hence

$$\left\|\partial_v^h u\right\|_{H^{-1}(W)} = \sup\left\{\left|\langle \partial_v^h u, \phi\rangle\right| : \phi \in \mathcal{D}(W) \text{ with } \|\phi\|_{H^1(\Omega)} = 1\right\} \le \|u\|_{L^2(\Omega)}.$$

■

Theorem 3.3 (Interior Regularity)**.** *Suppose L is 2^{nd} order uniformly elliptic operator on Ω and $u \in H^1(\Omega)$ satisfies $Lu \in H^k(\Omega)$*[1] *for some $k = -1, 0, 1, 2, \ldots,$ then $u \in H^{k+2}_{loc}(\Omega)$. Moreover, if $W \subset\subset \Omega$ then there exists $C = C_k(W) < \infty$ such that*

$$\|u\|_{H^{k+2}(W)} \le C(\|Lu\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{3.4}$$

Proof. The proof is by induction on k with Theorem 3.2 being the case $k = -1$. Suppose that the interior regularity theorem holds for $-1 \le k \le k_0$. We will now complete the induction proof by showing it holds for $k = k_0 + 1$.

So suppose that $u \in H^1(\Omega)$ such that $Lu \in H^{k_0+1}(\Omega)$ and $W = W_0 \subset \Omega$ is fixed. Choose open sets W_1, W_2 and W_3 such that $\bar{W}_0 \subset W_1 \subset \bar{W}_1 \subset W_2 \subset \bar{W}_2 \subset W_3 \subset \bar{W}_3 \subset \Omega$ as in Figure 1. The idea now is to apply the induction hypothesis to the

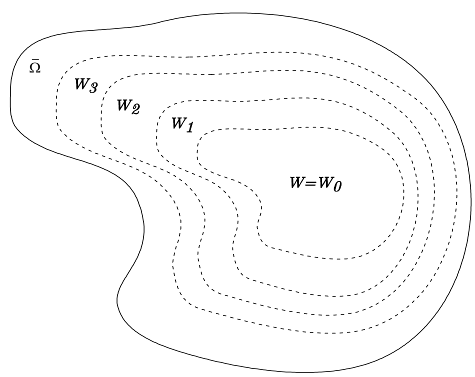

FIGURE 1. The sets W_i for $i = 0, 1, 2$.

function $\partial_v^h u$ where $v \in \mathbb{R}^d$ and ∂_v^h is the finite difference operator in Definition 22.14. For the remainder of the proof $h \neq 0$ will be assumed to be sufficiently small so that the following computations make sense. To simplify notation let $D^h = \partial_v^h$.

For h small, $D^h u \in H^1(W_1)$ and $D^h Lu \in H^{k_0+1}(W_2)$ and by Exercise 3.1 for $k_0 = -1$ and Theorem 26.13 for $k_0 \ge 0$,

$$\|D^h Lu\|_{H^{k_0}(W_1)} \le \|Lu\|_{H^{k_0+1}(W_2)}. \tag{3.5}$$

We now compute $LD^h u$ as

$$LD^h u = D^h Lu + [L, D^h]u, \tag{3.6}$$

where

$$\begin{aligned}
[L, D^h]u &= LD^h u - D^h Lu \\
&= P(x,\partial) D^h u(x) - D^h P(x,\partial) u(x) \\
&= P(x,\partial)\left(\frac{u(x+hv) - u(x)}{h}\right) - \frac{P(x+hv,\partial)u(x+hv) - P(x,\partial)u(x)}{h} \\
&= \frac{P(x,\partial) - P(x+hv, D)}{h}\, u(x+hv) = L^h \tau_v^h u(x), \\
&\qquad \tau_v^h u(x) = u(x+hv)
\end{aligned}$$

[1] A priori, $Lu \in H^{-1}(\Omega) \subset \mathcal{D}'(\Omega)$).

and

$$L^h u := \sum_{|\alpha|\leq 2} \frac{A_\alpha(x) - A_\alpha(x+he)}{h}\ \partial^\alpha u.$$

The meaning of Eq. (3.6) and the above computations require a bit more explanation in the case $k_0 = -1$ in which case $Lu \in L^2(\Omega)$. What is being claimed is that

$$LD^h u = D^h Lu + L^h \tau_v^h u$$

as elements of $H^{-1}(W_3)$. By definition this means that

$$\begin{aligned} -\langle u, D^{-h} L^\dagger \phi\rangle &= \langle LD^h u, \phi\rangle = \langle D^h Lu + L^h \tau_v^h u, \phi\rangle \\ &= -\langle u, L^\dagger D^{-h}\phi\rangle + \langle \tau_v^h u, \left(L^h\right)^\dagger \phi\rangle. \end{aligned}$$

So the real identity which needs to be proved here is that $\left[D^{-h}, L^\dagger\right]\phi = -\tau_v^{-h}\left(L^h\right)^\dagger \phi$ for all $\phi \in \mathcal{D}(W_3)$. This can be done as above or it can be inferred (making use of the properties $L^\dagger$ is the formal adjoint of L and $-D^{-h}$ is the formal adjoint of D^h) from the computations already done in the previous paragraph with u being a smooth function.

Since L^h is a second order differential operator with coefficients which have bounded derivatives to all orders with bounds independent of h small, $[L, D^h]u = L^h \tau_v^h u \in H^{k_0}(W_1)$ and there is a constant $C < \infty$ such that

(3.7)
$$\|[L, D^h]u\|_{H^{k_0}(W_1)} = \|L^h \tau_v^h u\|_{H^{k_0}(W_1)} \leq C\|\tau_v^h u\|_{H^{k_0+2}(W_2)} \leq C\|u\|_{H^{k_0+2}(W_3)}.$$

Combining Eqs. (3.5 – 3.7) implies that $LD^h u \in H^{k_0}(W_2)$ and

$$\|LD^h u\|_{H^{k_0}(W_1)} \lesssim \|Lu\|_{H^{k_0+1}(W_2)} + \|u\|_{H^{k_0+2}(W_3)}.$$

Therefore by the induction hypothesis, $D^h u \in H^{k_0+2}(W_0)$ and

$$\begin{aligned} \left\|D^h u\right\|_{H^{k_0+2}(W_0)} &\lesssim \|LD^h u\|_{H^{k_0}(W_1)} + \|D^h u\|_{L^2(W_1)} \\ &\lesssim \|Lu\|_{H^{k_0+1}(W_2)} + \|u\|_{H^{k_0+2}(W_3)} + \|u\|_{H^1(W_2)} \\ &\lesssim \|Lu\|_{H^{k_0+1}(W_2)} + \|u\|_{H^{k_0+2}(W_3)} \\ &\lesssim \|Lu\|_{H^{k_0+1}(\Omega)} + \|Lu\|_{H^{k_0}(\Omega)} + \|u\|_{L^2(\Omega)} \ \text{(by induction hypothesis)} \\ &\lesssim \|Lu\|_{H^{k_0+1}(\Omega)} + \|u\|_{L^2(\Omega)}. \end{aligned}$$

So by Theorem 26.13, $\partial_v u \in H^{k_0+2}(W_0)$ for all $v = e_i$ with $i = 1, 2, \ldots, d$ and

$$\|\partial_i u\|_{H^{k_0+2}(W)} = \|\partial_i u\|_{H^{k_0+2}(W_0)} \lesssim \|Lu\|_{H^{k_0+1}(\Omega)} + \|u\|_{L^2(\Omega)}.$$

Thus $u \in H^{k_0+3}(W_0)$ and Eq. (3.4) holds. ■

Corollary 3.4. *Suppose L is as above and $u \in H^1(\Omega)$ such that $Lu \in BC^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.*

Proof. Choose $\Omega_0 \subset\subset \Omega$ so $Lu \in BC^\infty(\overline{\Omega}_0)$. Therefore $Lu \in H^k(\Omega_0)$ for all $k = 0, 1, 2, \ldots.$ Hence $u \in H^{k+2}_{loc}(\Omega_0)$ for all $k = 0, 1, 2, \ldots.$ Then by Sobolev embedding Theorem 28.18, $u \in C^\infty(\Omega_0)$. Since Ω_0 is an arbitrary precompact open subset of Ω, $u \in C^\infty(\Omega)$. ■

3.2. **Boundary Regularity Theorem.**

Example 3.5. Let $\Omega = D(0,1)$ and $u(z) = (1+z)\log(1+z)$. Then $\Delta u = 0$ and moreover $\frac{\partial u}{\partial z} = \log(1+z)+1$ so $u_x = 1+\log(1+z)$ and $u_y = i(1+\log(1+z))$ and hence $u \in H^1(\Omega)$ with $\Delta u = 0 \in H^k(\Omega)$ for all k but $u \notin H^2(\Omega)$. Indeed,

$$u_{xx} = \frac{\partial^2}{\partial z^2} u = \frac{1}{1+z}.$$

Now

$$\int_\Omega \left|\frac{1}{1+z}\right|^2 dx\ dy \geq \int_\Omega \frac{1}{(1+z)^2}\ dx\ dy = \int \frac{1}{|z|^2}\ dx\ dy$$
$$\cong 2\int_{\theta=\frac{\pi}{2}}^{\pi} \int_{r=0}^{f(\theta)} \frac{1}{r^2}\ r\ dr\ d\theta = 2\int_{\frac{\pi}{2}}^{T/2} d\theta\ \ln(r)\Big|_{r=0}^{r=f(\theta)} = \infty$$

so $u \notin H^2(\Omega)$.

This example shows that in order to get an elliptic regularity result which is valid all the way up to the boundary, it is necessary to impose some sort of boundary conditions on the solution which will rule out the bad behavior of the example. Since the Dirichlet form contains boundary information, we will do this by working with $\mathcal{E}$ rather than the operator L on $\mathcal{D}'(\Omega)$ associated to $\mathcal{E}$. Having to work with the quadratic form makes life a bit more difficult.

Notations 3.6. Let

(1) $N_r := \{x \in \mathbb{H}^n : |x| < r\}$.
(2) $X = H_0^1(N_r)$ or X be the closed subspace $H^1(N_r)$ given by

$$X = \left\{u \in H^1(N_r) : u|_{\partial\mathbb{H}^n \cap \bar{N}_r} = 0\right\}. \tag{3.8}$$

(3) For $s \leq r$ let $X_s = \{u \in X : u = 0 \text{ on } \mathbb{H}^n \setminus N_\rho \text{ for some } \rho < s\}$.

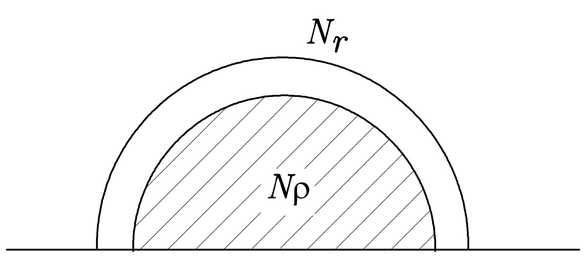

Figure 2. Nested half balls.

Remark 3.7. (1) If $\phi \in C^\infty(\overline{\mathbb{H}}^n)$ and vanishes on $\mathbb{H}^n \setminus N_\rho$ for some $\rho < r$ then $\phi u \in X_r$ for all $u \in X$.
(2) If $u \in X_r$ then $\partial_h^\alpha u \in X_r$ for all α such that $\alpha_d = 0$ and $|h|$ sufficiently small.

Lemma 3.8 (Commutator). *If $\psi \in C^\infty(\overline{N}_r)$ then for $\gamma \in \mathbb{N}_0^{d-1} \times \{0\}$ there exists $C_\gamma(\psi) < \infty$ such that*

$$\|[\psi, \partial_h^\gamma] f\|_{L^2(N_\rho)} \leq C_\gamma(\psi) \sum_{\alpha<\gamma} \|\partial^\alpha f\|_{L^2(N_r)}. \tag{3.9}$$

for all $f \in L^2(N_r)$.

Proof. The proof will be by induction on $|\gamma|$. If $\gamma = e_i$ for some $i < d$, then

$$\begin{aligned}\partial_h^i(\psi f)(x) &:= \frac{\psi(x+he_i)f(x+he_i) - \psi(x)f(x)}{h} \\ &= \frac{[\psi(x+he_i) - \psi(x)]\, f(x+he_i) + \psi(x)\,[f(x+he_i) - f(x)]}{h} \\ &= \partial_h^i\psi(x) f(x+he_i) + \psi(x)\partial_h^i f(x)\end{aligned}$$

which gives

$$[\partial_h^i, \psi]f = (\partial_h^i\psi)\tau_h^i f. \tag{3.10}$$

This then implies that

$$\|[\psi, \partial_h^i]f\|_{L^2(N_\rho)} \le C(\psi)\|f\|_{L^2(N_r)}.$$

Now suppose $|\gamma| > 1$ with $\gamma = e_i + \gamma'$ so that $\partial_h^\gamma = \partial_h^{\gamma'}\partial_h^i$ with$|\gamma'| = |\gamma| - 1$. Then

$$[\psi, \partial_h^\gamma] = [\psi, \partial_h^{\gamma'}]\partial_h^i + \partial_h^{\gamma'}[\psi, \partial_h^i]$$

and therefore by the induction hypothesis and Theorems 26.13 and 26.15,

$$\begin{aligned}\|[\psi, \partial_h^\gamma]f\|_{L^2} &\le C_{\gamma'}(\psi)\sum_{\alpha<\gamma'}\|\partial^\alpha\partial_h^i f\|_{L^2} + \|\partial^{\gamma'}[\psi, \partial_h^i]f\|_{L^2} \\ &\le C_{\gamma'}(\psi)\sum_{\alpha<\gamma'}\|\partial^{\alpha+e_i} f\|_{L^2} + \|\partial^{\gamma'}\left[(\partial_h^i\psi)\tau_h^i f\right]\|_{L^2}. \end{aligned}\tag{3.11}$$

But

$$\partial^{\gamma'}\left[(\partial_h^i\psi)\tau_h^i f\right] = \sum_{\beta_1+\beta_2=\gamma'}\frac{\gamma'!}{\beta_1!\beta_2!}(\partial_h^i\partial^{\beta_1}\psi)\tau_h^i\partial^{\beta_2} f$$

and hence

$$\|\partial^{\gamma'}\left[(\partial_h^i\psi)\tau_h^i f\right]\| \le C\sum_{\beta\le\gamma'}\|\partial^\beta f\|_{L^2}. \tag{3.12}$$

Combining Eqs. (3.11) and (3.12) gives the desired result,

$$\|[\psi, \partial_h^\gamma]f\|_{L^2} \le C_\gamma(\psi)\sum_{\alpha<\gamma}\|\partial^\alpha f\|_{L^2}.$$

■

Lemma 3.9 (Warmup for Proposition 3.10)**.** *Let* $a_{\alpha\beta} \in BC^\infty\left(\mathbb{H}^d\right)$ *with* $(a_{ij}) \ge \epsilon\delta_{ij}$ *for some* $\epsilon > 0$,

$$\langle\mathcal{L}u, v\rangle = \mathcal{E}(u,v) = \int_{\mathbb{H}^d}\sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}\partial^\alpha u\cdot\partial^\beta v\, dx, \tag{3.13}$$

$X = H_0^1(\mathbb{H}^d)$ *or* $H^1(\mathbb{H}^d)$. *There exists* $C < \infty$ *such that if* $u \in X$ *such that* $\mathcal{L}u =: f \in L^2(\mathbb{H}^d)$, *then*

$$\|u\|_{H^2(\mathbb{H}^d)} \le C(\|f\|_{L^2(\mathbb{H}^d)} + \|u\|_{X^*}). \tag{3.14}$$

Proof. If $\mathcal{L}u = f \in X^*$ then $(\mathcal{L} + C)\, u = f + Cu$, so by the Lax-Milgram method,

$$\|u\|_X \lesssim \|f + Cu\|_{X^*} \le \|f\|_{X^*} + C\,\|u\|_{X^*} \lesssim \|\mathcal{L}u\|_{X^*} + \|u\|_{X^*}\,.$$

Suppose first $k = 0$, in which case we wish to prove $\partial_i u \in H^1(\mathbb{H}^d)$ for all $i < d$ and

$$\|\partial_i u\|_{H^1(\mathbb{H}^d)} \lesssim \|\mathcal{L}u\|_{L^2(\mathbb{H}^d)} + \|u\|_{X^*}.$$

To do this consider

$$\begin{aligned}
\langle \mathcal{L}\partial_i^h u, v\rangle &= \int\limits_{\mathbb{H}^d} \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}\partial_i^h\partial^\alpha u \cdot \partial^\beta v \; dx \\
&= \int\limits_{\mathbb{H}^d} \sum_{|\alpha|,|\beta|\le 1} \left\{\partial_i^h\left(a_{\alpha\beta}\partial^\alpha u\right) + \left[a_{\alpha\beta}, \partial_i^h\right]\partial^\alpha u\right\} \cdot \partial^\beta v \; dx \\
&= -\langle \mathcal{L}u, \partial_i^{-h} v\rangle - \int\limits_{\mathbb{H}^d} \sum_{|\alpha|,|\beta|\le 1} (\partial_h^i a_{\alpha\beta})\tau_h^i \partial^\alpha u \cdot \partial^\beta v \; dx \\
&= -\langle \mathcal{L}u, \partial_i^{-h} v\rangle - \mathcal{E}_{\partial_h^i a}(\tau_h^i u, v = -(f, \partial_i^{-h} v) - \mathcal{E}_{\partial_h^i a}(\tau_h^i u, v) \\
&= (\partial_i^h f, v) - \mathcal{E}_{\partial_h^i a}(\tau_h^i u, v)
\end{aligned}$$

wherein we have made use of Eq. (3.10) in the third equality. From this it follows that

$$\mathcal{L}\partial_i^h u = \partial_i^h \mathcal{L}u - \mathcal{E}_{\partial_h^i a}(\tau_h^i u, \cdot) \in X^*$$

and

$$\begin{aligned}
\left\|\mathcal{L}\partial_i^h u\right\|_{X^*} &\le \left\|\partial_i^h \mathcal{L}u\right\|_{X^*} + \left\|\mathcal{E}_{\partial_h^i a}(\tau_h^i u, \cdot)\right\|_{X^*} \lesssim \|\mathcal{L}u\|_{L^2} + \|u\|_X \\
&\lesssim \|\mathcal{L}u\|_{L^2} + \|\mathcal{L}u\|_{X^*} + \|u\|_{X^*} \lesssim \|\mathcal{L}u\|_{L^2} + \|u\|_{X^*}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\partial_i^h u\right\|_X &\lesssim \left\|\mathcal{L}\partial_i^h u\right\|_{X^*} + \left\|\partial_i^h u\right\|_{X^*} \lesssim \|\mathcal{L}u\|_{L^2} + \|u\|_{X^*} + \|u\|_{L^2} \\
&\lesssim \|\mathcal{L}u\|_{L^2} + \|u\|_{X^*}\,.
\end{aligned}$$

Since h is small but arbitrary we conclude that $\partial_i u \in X$ and

$$\|\partial_i u\|_X \lesssim \|\mathcal{L}u\|_{L^2} + \|u\|_{X^*} \text{ for all } i < d.$$

Finally if $i = d$, we have that $f = \mathcal{L}u = \sum_{\alpha \ne 2e_d} A_\alpha \partial^\alpha u + \partial_d^2 u$ which implies (writing $A_{d,d}$ for A_{2e_d})

$$\partial_d^2 u = A_{d,d}^{-1}\left(f - \sum_{\alpha\ne 2e_d} A_\alpha \partial^\alpha u\right) \in L^2$$

because we have shown that $\partial_i\partial_j u \in L^2$ if $\{i, j\} \ne \{d\}$. Moreover we have the estimate that

$$\begin{aligned}
\left\|\partial_d^2 u\right\|_{L^2} &\lesssim \left\|f - \sum_{\alpha\ne 2e_d} A_\alpha\partial^\alpha u\right\|_{L^2} \lesssim \|f\|_{L^2} + \sum_{\alpha\ne 2e_d}\|\partial^\alpha u\|_{L^2} \\
&\lesssim \|f\|_{L^2} + \sum_{j<d}\left\|\partial^j u\right\|_{X^*} \lesssim \|\mathcal{L}u\|_{L^2(\mathbb{H}^d)} + \|u\|_{X^*}.
\end{aligned}$$

Thus we have shown that $u \in X \cap H^2(\mathbb{H}^d)$ and

$$\|u\|_{H^2(\mathbb{H}^d)} \lesssim \|\mathcal{L}u\|_{L^2(\mathbb{H}^d)} + \|u\|_{X^*}.$$

■

If we try to use the above proof inductively to get higher regularity we run into a snag. To see this suppose now that $f \in H^1$.Then as above

$$\mathcal{L}\partial_j^h u = \partial_j^h \mathcal{L}u - \mathcal{E}_{\partial_h^j a}(\tau_h^j u, \cdot) = \partial_j^h f - \mathcal{E}_{\partial_h^j a}(\tau_h^j u, \cdot).$$

Let $\partial_h^j a = b$ and $\tau_h^j u = w$ and consider

$$\mathcal{E}_b(w, v) = \int_{\mathbb{H}^d} b_{\alpha,\beta}\partial^\alpha w \cdot \partial^\beta v dm.$$

Since $w \in H^2$ we may integrate by parts to find

$$\mathcal{E}_b(w, v) = \int_{\mathbb{H}^d} (-1)^{|\beta|} \partial^\beta \left(b_{\alpha,\beta}\partial^\alpha w\right) \cdot v dm - \int_{\partial\mathbb{H}^d} b_{\alpha,d}\partial^\alpha w \cdot v d\sigma.$$

This shows that $\mathcal{E}_b(w, \cdot)$ is representable by $(-1)^{|\beta|} \partial^\beta \left(b_{\alpha,\beta}\partial^\alpha w\right) \in L^2$ plus the boundary term

$$v \to \int_{\partial\mathbb{H}^d} b_{\alpha,d}\partial^\alpha w \cdot v d\sigma.$$

To continue on by this method, we would have to show that the boundary term is representable by an element of L^2. This should be the case since $v|_{\partial\mathbb{H}^d} \in H^{-1/2}\left(\mathbb{H}^d\right)$ while $\partial^\alpha w \in H^{1/2}\left(\mathbb{H}^d\right)$ with bounds. However we have not proven such statements so we will proceed by a different but closely related approach.

Proposition 3.10 (Local Tangential Boundary Regularity)**.** *Let $a_{\alpha,\beta} \in C^\infty\left(\bar{N}_t\right)$ with $a_{ij}\, \xi_i\, \xi_j \geq 2\epsilon|\xi|^2$,*

$$Q(u, v) = \int_{N_t} \sum_{|\alpha|,|\beta|\leq 1} a_{\alpha\beta}\partial^\alpha u \cdot \partial^\beta v \; dx, \tag{3.15}$$

$X = H^1(N_t)$ or X be the closed subspace of $H^1\left(N_t\right)$ defined in Eq. (3.8) of Notation 3.6. Suppose $k \in \mathbb{N}_0$, $u \in X$ and $f \in H^k(N_t)$ satisfy,

$$Q(u, v) = \int_{N_t} f v \; dx \text{ for all } v \in X_t. \tag{3.16}$$

Given $\rho < t$, there exists $C < \infty$ such that for all $\gamma \in \mathbb{N}_0^{d-1} \times \{0\}$ with $|\gamma| \leq k+1$, $\partial^\gamma u \in H^1(N_\rho)$ and

$$\|\partial^\gamma u\|_{H^1(N_\rho)} \leq C(\|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}) \tag{3.17}$$

Proof. Let $\rho < r < s < t$ and consider the half nested balls as in Figure 3 below. The proof will be by induction on $j = |\gamma|$. When $j = 0$ the assertion is trivial. Assume now there exists $j \in [1, k+1] \cap \mathbb{N}$ such that $\partial^\gamma u \in H^1(N_s)$ for all $|\gamma| < j$ with $\gamma_d = 0$ and

$$\|\partial^\gamma u\|_{H^1(N_s)} \leq C(\|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}).$$

Fix $\phi \in C_c^\infty(\overline{N}_t)$ such that $\phi = 0$ on $\bar{N}_t \setminus \bar{N}_r$ and $\phi = 1$ in a neighborhood of $\overline{N}_\rho$. Suppose γ is a multi-index such that $|\gamma| = j$ and $\gamma_d = 0$. Then $\partial_h^\gamma(\phi u) \in X_r$ for h sufficiently small.

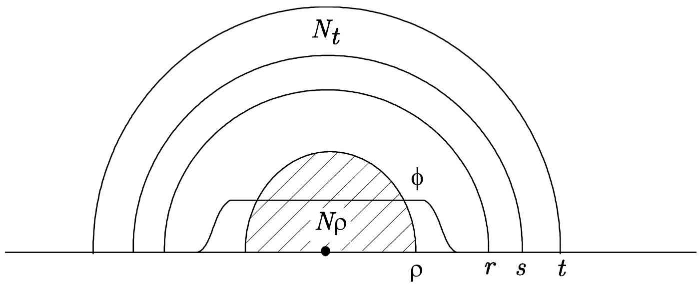

FIGURE 3. A collection of nested half balls along with the cutoff function ϕ.

With out loss of generality we may assume $\gamma_1 > 0$ and write $\gamma = e_1 + \gamma'$ and $\partial_h^\gamma = \partial_h^1 \partial_h^{\gamma'}$. For $v \in X_r$,

$$
\begin{aligned}
Q(\partial_h^\gamma(\phi u), v) &= \int_{N_t} \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}\partial^\alpha \partial_h^\gamma(\phi u)\cdot \partial^\beta v = \int_{N_t} \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}\partial_h^\gamma \partial^\alpha(\phi u)\cdot \partial^\beta v \\
&= \sum_{|\alpha|,|\beta|\le 1} \int_{N_t} \partial_h^\gamma(a_{\alpha\beta}\partial^\alpha(\phi u))\cdot \partial^\beta v + \overbrace{\sum_{|\alpha|,|\beta|\le 1} \int_{N_t} [a_{\alpha\beta}, \partial_h^\gamma]\partial^\alpha(\phi u)\cdot \partial^\beta v}^{E_1} \\
&= \sum_{|\alpha|,|\beta|\le 1} \int_{N_t} \partial_h^\gamma(a_{\alpha\beta}\phi\partial^\alpha u)\cdot \partial^\beta v + \overbrace{\sum_{|\alpha|,|\beta|\le 1} \int_{N_t} \partial_h^\gamma(a_{\alpha\beta}[\partial^\alpha, \phi]u)\cdot \partial^\beta v}^{E_2} + E_1 \\
&= \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\gamma|} \int_{N_t} a_{\alpha\beta}\partial^\alpha u\cdot \phi\partial^\beta \partial_{-h}^\gamma v + E_1 + E_2 \\
&= \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\gamma|} \int_{N_t} a_{\alpha\beta}\partial^\alpha u\cdot \partial^\beta \left[\phi\partial_{-h}^\gamma v\right] + E_1 + E_2 \\
&\qquad + \overbrace{\sum_{|\alpha|,|\beta|\le 1} (-1)^{|\gamma|} \int_{N_t} a_{\alpha\beta}\partial^\alpha u\cdot \left[\phi, \partial^\beta\right]\partial_{-h}^\gamma v}^{E_3} \\
&= (-1)^{|\gamma|} Q(u, \phi\partial_{-h}^\gamma v) + E_1 + E_2 + E_3 \\
&= (-1)^{|\gamma|} \int_{N_t} \phi f \partial_{-h}^\gamma v + E_1 + E_2 + E_3 \\
&= E_1 + E_2 + E_3 \overbrace{- (-1)^{|\gamma|} \int_{N_t} \partial_h^{\gamma'}[\phi f]\cdot \partial_{-h}^i v}^{E_4}. \\
&= E_1 + E_2 + E_3 + E_4.
\end{aligned}
$$

To summarize,

$$Q(\partial_h^\gamma(\phi u), v) = E_1 + E_2 + E_3 + E_4$$

where

$$
\begin{aligned}
E_1 &:= \sum_{|\alpha|,|\beta|\le 1} \int_{N_t} [a_{\alpha\beta}, \partial_h^\gamma]\partial^\alpha(\phi u)\cdot\partial^\beta v \\
E_2 &:= \sum_{|\alpha|,|\beta|\le 1} \int_{N_t} \partial_h^\gamma(a_{\alpha\beta}[\partial^\alpha,\phi]u)\cdot\partial^\beta v \\
E_3 &:= \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\gamma|}\int_{N_t} a_{\alpha\beta}\partial^\alpha u\cdot\left[\phi,\partial^\beta\right]\partial_{-h}^\gamma v \text{ and} \\
E_4 &:= -(-1)^{|\gamma|}\int_{N_t}\partial_h^{\gamma'}[\phi f]\cdot\partial_{-h}^i v.
\end{aligned}
$$

To finish the proof we will estimate each of the terms E_i for $i = 1,\ldots,4$. Using Lemma 3.8,

$$
\begin{aligned}
|E_1| &\le \sum_{|\alpha|,|\beta|\le 1}\int_{N_t}\left|[a_{\alpha\beta},\partial_h^\gamma]\partial^\alpha(\phi u)\cdot\partial^\beta v\right| \le \|v\|_{H^1(N_r)}\sum_{|\alpha|,|\beta|\le 1}\|[a_{\alpha\beta},\partial_h^\gamma]\partial^\alpha(\phi u)\|_{L^2(N_r)} \\
&\le \|v\|_{H^1(N_r)}\sum_{|\alpha|,|\beta|\le 1}\sum_{\delta<\gamma} C_\gamma(a_{\alpha\beta})\|\partial^\delta\partial^\alpha(\phi u)\|_{L^2(N_r)} \\
&\lesssim \|v\|_{H^1(N_r)}\sum_{\delta<\gamma}\|\partial^\delta u\|_{H^1(N_r)} \\
&\lesssim \|v\|_{H^1(N_s)}\left(\|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)}\right) \text{ (by the induction hypothesis).}
\end{aligned}
$$

For E_2,

$$
\begin{aligned}
|E_2| &= \left|\sum_{|\beta|\le 1,|\alpha|=1}\int_{N_t}\partial_h^\gamma(a_{\alpha\beta}\,(\partial^\alpha\phi)\,u)\cdot\partial^\beta v\right| \\
&\le \|v\|_{H^1(N_r)}\sum_{|\beta|\le 1,|\alpha|=1}\|\partial_h^\gamma[a_{\alpha\beta}(\partial^\alpha\phi)u]\|_{L^2(N_r)} \\
&\le C\|v\|_{H^1(N_r)}\sum_{|\beta|\le 1,|\alpha|=1}\|\partial^\gamma[a_{\alpha\beta}(\partial^\alpha\phi)u]\|_{L^2(N_r)} \\
&\le C\|v\|_{H^1(N_r)}\sum_{\delta\le\gamma}\|\partial^\delta u\|_{L^2(N_r)} \\
&\le C\|v\|_{H^1(N_r)}\sum_{|\delta|\le j-1,\delta_{n=0}}\|\partial^\delta u\|_{H^1(N_r)}\ (\,|\delta|\le j-1 \text{ since } L^2(N_r)\to H^1(N_r)\,) \\
&\le C\|v\|_{H^1(N_s)}(\|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)}) \text{ (by the induction hypothesis).}
\end{aligned}
$$

For E_3,

$$\begin{aligned}
|E_3| &\le \sum_{|\alpha|\le 1, |\beta|=1} \left| \int_{N_t} a_{\alpha\beta}\partial^\alpha u \cdot \left(\partial^\beta \phi\right) \partial_{-h}^\gamma v \right| \\
&= \sum_{|\alpha|\le 1, |\beta|=1} \left| \int_{N_t} \partial_h^{\gamma'} \left[a_{\alpha\beta}\partial^\alpha u \cdot \left(\partial^\beta \phi\right)\right] \cdot \partial_{-h}^i v \right| \\
&\le \sum_{|\alpha|\le 1, |\beta|=1} \|v\|_{H^1(N_r)} \|\partial^{\gamma'} \left[a_{\alpha\beta}\partial^\alpha u \cdot \left(\partial^\beta \phi\right)\right] \|_{L^2(N_r)} \\
&\le C\|v\|_{H^1(N_r)} \sum_{|\alpha|\le 1} \sum_{\delta\le\gamma'} \|\partial^{\delta+\alpha} u\|_{L^2(N_r)} \\
&\le C\|v\|_{H^1(N_r)} \sum_{\delta\le\gamma'} \|\partial^{\delta} u\|_{H^1(N_r)} \\
&\le C\|v\|_{H^1(N_s)} (\|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)}) \text{ (by the induction hypothesis).}
\end{aligned}$$

Finally for E_4,

$$\begin{aligned}
|E_4| = \left| \int_{N_t} \partial_h^{\gamma'} [\phi f] \cdot \partial_{-h}^i v \right| &\le \|\partial_{-h}^i v\|_{L^2(N_r)} \ \|\partial_h^{\gamma'}(\phi f)\|_{L^2(N_r)} \\
&\le \|v\|_{H^1(N_r)} \ \|\partial^{\gamma'}(\phi f)\|_{L^2(N_r)} \\
&\le \|v\|_{H^1(N_s)} \ \|\phi f\|_{H^{j-1}(N_r)} \le C\|v\|_{H^1(N_s)} \|f\|_{H^k(N_s)}.
\end{aligned}$$

Putting all of these estimates together proves, whenever $|\gamma| = j$,

$$|Q(\partial_h^\gamma(\phi u), v)| \le C\|v\|_{H^1(N_s)} (\|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)}) \tag{3.18}$$

for all $v \in X_s$. In particular we may take $v = \partial_h^\gamma(\phi u) \in X_s$ in the above inequality to learn

$$Q\left(\partial_h^\gamma(\phi u), \partial_h^\gamma(\phi u)\right) \le C\|\partial_h^\gamma(\phi u)\|_{H^1(N_s)} (\|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)}). \tag{3.19}$$

But by coercivity of Q,

$$\begin{aligned}
\|\partial_h^\gamma(\phi u)\|_{H^1(N_s)}^2 &\le C \left[Q(\partial_h^\gamma(\phi u), \partial_h^\gamma(\phi u)) + \|\partial_h^\gamma(\phi u)\|_{L^2(N_s)}^2 \right] \\
&\lesssim \|\partial_h^\gamma(\phi u)\|_{H^1(N_s)} (\|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)}) \\
&\qquad + \|\partial_h^\gamma(\phi u)\|_{H^1(N_s)} \|\partial_h^\gamma(\phi u)\|_{L^2(N_s)} \\
&\lesssim \|\partial_h^\gamma(\phi u)\|_{H^1(N_s)} (\|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)} + \|\partial_h^\gamma(\phi u)\|_{L^2(N_s)}) \qquad (3.20)
\end{aligned}$$

and hence

$$\|\partial_h^\gamma(\phi u)\|_{H^1(N_s)} \lesssim \|f\|_{H^k(N_s)} + \|u\|_{H^1(N_s)} + \|\partial_h^\gamma(\phi u)\|_{L^2(N_s)}. \tag{3.21}$$

Now

$$\begin{aligned}
\|\partial_h^\gamma(\phi u)\|_{L^2(N_s)} = \|\partial_h^i \partial_h^{\gamma'}(\phi u)\|_{L^2(N_s)} &\le \|\partial_h^{\gamma'}(\phi u)\|_{H^1(N_s)} \le \|\partial^{\gamma'}(\phi u)\|_{H^1(N_s)} \\
&\le C \sum_{\alpha\le\gamma'} \|\partial^\alpha u\|_{H^1(N_s)} \text{ by the chain rule} \\
&\le C(\|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}) \text{ by the induction hypothesis.}
\end{aligned}$$

This last estimated combined with Eq. (3.21) shows

$$\|\partial_h^\gamma(\phi u)\|_{H^1(N_s)} \lesssim \|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}$$

and therefore $\partial^\gamma(\phi u) \in H^1(N_s)$ and

$$\|\partial^\gamma(\phi u)\|_{H^1(N_s)} \lesssim \|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}.$$

This proves the proposition since $\phi \equiv 1$ on N_ρ so

$$\begin{aligned}\|\partial^\gamma u\|_{H^1(N_\rho)} &= \|\partial^\gamma(\phi u)\|_{H^1(N_\rho)} \leq \|\partial^\gamma(\phi u)\|_{H^1(N_s)} \\ &\lesssim \|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}.\end{aligned}$$

■

Theorem 3.11 (Local Boundary Regularly)**.** *As in Proposition 3.10, let* $a_{\alpha,\beta} \in C^\infty(\bar{N}_t)$ *with* $a_{ij}\,\xi_i\,\xi_j \geq 2\epsilon|\xi|^2$,

$$Q(u,v) = \sum_{|\alpha|,|\beta|\leq 1} \int_{N_t} a_{\alpha\beta}\partial^\alpha u \cdot \partial^\beta v\ dx$$

and $X = H_0^1(N_t)$ *or* $X \subset H^1(N_t)$ *as in Eq. (3.8) . If* $f \in H^k(N_t)$ *for some* $k \geq 0$ *and* $u \in X$ *solves* Q

$$Q(u,v) = (f,v) \text{ for all } v \in X_t$$

then for all $\rho < t$, $u \in H^{k+2}(N_\rho)$ *and there exists* $C < \infty$ *such that*

$$\|u\|_{H^{k+2}(N_\rho)} \leq C(\|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}).$$

Proof. The theorem will be proved by showing $\partial^\gamma u \in L^2(N_\rho)$ for all $|\gamma| \leq k+2$ and

$$\|\partial^\gamma u\|_{L^2(N_\rho)} \lesssim \|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}. \tag{3.22}$$

The proof of Eq. (3.22) will be by induction on $j = \gamma_d$. The case $j = 0, 1$ follows from Proposition 3.10. Suppose $j = \gamma_d \geq 2$ and $\gamma' = \gamma - 2e_d$ so $\partial^\gamma = \partial^{\gamma'}\partial_d^2$. Now letting

$$L = \sum_{|\alpha|,|\beta|\leq 1} (-1)^{|\beta|}\partial^\beta a_{\alpha\beta}\partial_\alpha = \sum_{|\alpha|\leq 2} A_\alpha \partial^\alpha,$$

then $Lu = f$ in the distributional sense. Writing $\tilde{A}$ for $A_{(0,0,\ldots,0,2)}$,

$$f = \tilde{A}\partial_d^2 u + \sum_{|\alpha|\leq 2,\alpha_d<2} A_\alpha \partial^\alpha u$$

so that

$$\partial_d^2 u = \frac{1}{\tilde{A}}(f - \sum_{|\alpha|\leq 2,\alpha_d<2} A_\alpha \partial^\alpha u)$$

and

$$\partial^\gamma u = \partial^{\gamma'}\partial_d^2 u = \partial^{\gamma'}\left(\frac{1}{\tilde{A}}f - \sum_{|\alpha|\leq 2,\alpha_d<2} \frac{A_\alpha}{\tilde{A}}\partial^\alpha u\right). \tag{3.23}$$

Now by the product rule

$$\sum_{|\alpha|\leq 2,\alpha_d<2} \partial^{\gamma'}\left(\frac{A_\alpha}{\tilde{A}}\partial^\alpha u\right) := \sum_{|\alpha|\leq 2,\alpha_d<2,\delta\leq\gamma'} \binom{\gamma'}{\delta}\partial^{(\gamma'-\delta+\alpha)}\left(\frac{A_\alpha}{\tilde{A}}\right)\cdot\partial^{(\delta+\alpha)}u. \tag{3.24}$$

Since $(\gamma' + \alpha)_d < j$, the induction hypothesis (i.e. Eq. (3.22) is valid for $|\gamma| < j$) shows the right member of Eq. (3.24) is in $L^2(N_\rho)$ and gives the estimate

$$\left\| \sum_{|\alpha|\le 2, \alpha_d<2} \partial^{\gamma'} \left(\frac{A_\alpha}{\tilde{A}} \partial^\alpha u \right) \right\|_{L^2(N_\rho)} \lesssim \sum_{|\alpha|\le 2, \alpha_d<2, \delta\le\gamma'} \left\| \partial^{(\delta+\alpha)} u \right\|_{L^2(N_\rho)}$$
$$\lesssim \|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}.$$

Combining this with Eq. (3.23) gives $\partial^\gamma u \in L^2(N_\rho)$ and

$$\|\partial^\gamma u\|_{L^2(N_\rho)} \lesssim \|f\|_{H^{|\gamma'|}(N_t)} + \left\| \sum_{|\alpha|\le 2, \alpha_d<2} \partial^{\gamma'} \left(\frac{A_\alpha}{\tilde{A}} \partial^\alpha u \right) \right\|_{L^2(N_\rho)}$$
$$\lesssim \|f\|_{H^k(N_t)} + \|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)} \lesssim \|f\|_{H^k(N_t)} + \|u\|_{H^1(N_t)}. \tag{3.25}$$

■

The following assumptions an notation will be in force for the remainder of this chapter.

Assumption 2. Let Ω be a bounded open subset such that $\bar{\Omega}^o = \Omega$ and $\bar{\Omega}$ is a C^∞ – manifold with boundary, X be either $H_0^1(\Omega)$ or $H^1(\Omega)$ and $\mathcal{E}$ be a Dirichlet form as in Notation 15.3 which is assumed to be elliptic. Also if W is an open subset of $\mathbb{R}^d$ let

$$X_W := \{v \in X : \mathrm{supp}(v) \sqsubset\sqsubset W \cap \bar{\Omega}\}.$$

Lemma 3.12. *For each $p \in \partial\Omega$ there exists precompact open neighborhoods V and W in $\mathbb{R}^d$ such that $\bar{V} \subset W$, for each $k \in \mathbb{N}$ there is a constant $C_k < \infty$ such that if $u \in X$ and $f \in H^k(\Omega)$ satisfies*

$$\mathcal{E}(u, v) = \int_\Omega f v \; dx \text{ for all } v \in X_W \tag{3.26}$$

then $u \in H^{k+2}(V \cap \Omega)$ and

$$\|u\|_{H^{k+2}(V\cap\Omega)} \le C(\|f\|_{H^k(\Omega)} + \|u\|_{H^1(\Omega)}) \tag{3.27}$$

Proof. Let W be an open neighborhood of p such that there exists a chart $\psi : W \to B(0, r)$ with inverse $\phi := \psi^{-1} : B(0, r) \to W$ satisfying:

(1) The maps ψ and ϕ has bounded derivatives to all orders.
(2) $\psi(W \cap \Omega) = B(0, r) \cap \mathbb{H}^d = N_r$ and $\psi\,(W \cap \mathrm{bd}(\Omega)) = B(0, r) \cap \mathrm{bd}(\mathbb{H}^d))$.

Now let $\rho < r$ and define $V := \phi(B(0, \rho))$, see Figure 4.

Suppose that $u \in X$ satisfies Eq. (3.26) and $v \in X_W$. Then making the change of variables $x = \phi(y)$,

$$\int_\Omega f v dm = \int_{N_r} f\,(\phi(y))\, v\,(\phi(y))\, J(y) dy = \int_{N_r} \tilde{f}(y) \tilde{v}(y) dy$$

where $J(y) := |\det \phi'(y)|$, $\tilde{f}(y) := J(y) f\,(\phi(y))$ and $\tilde{v}(y) = v\,(\phi(y))$. By the change of variables theorem, $\phi^* v := v \circ \phi$ is the generic element of $X_r(N_r)$ and $\tilde{f} \in H^k(N_r)$. We also define a quadratic form on $X(N_r)$ by

$$Q(\tilde{u}, \tilde{v}) := \sum_{|\alpha|,|\beta|\le 1} \int_W a_{\alpha\beta} \partial^\alpha\,(\tilde{u} \circ \psi) \cdot \partial^\beta\,(\tilde{v} \circ \psi)\, dm.$$

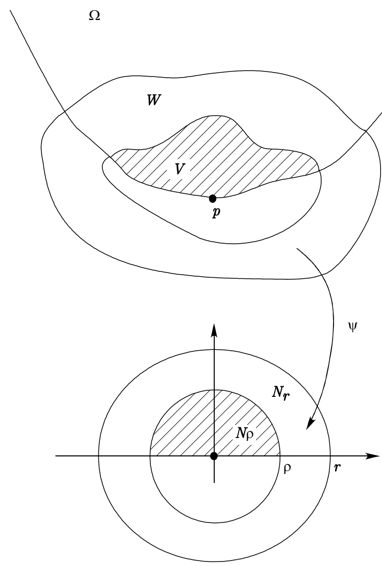

FIGURE 4. Flattening out the boundary of Ω in a neighborhood of p.

Again by making change of variables (using Theorem 26.16 along with the change of variables theorem for integrals) this quadratic form may be written in the standard form,

$$Q(\tilde{u}, \tilde{v}) = \sum_{|\alpha|, |\beta| \le 1} \int_{N_r} \tilde{a}_{\alpha,\beta} \partial^\alpha \tilde{u} \cdot \partial^\beta \tilde{v} \; dm.$$

This new form is still elliptic. To see this let Γ denote the matrix (a_{ij}), then

$$\begin{aligned} \sum_{i,j=1}^{d} a_{ij} \partial_i (\tilde{u} \circ \psi) \cdot \partial_j (\tilde{v} \circ \psi) &= \Gamma \nabla (\tilde{u} \circ \psi) \cdot \nabla (\tilde{u} \circ \psi) \\ &= \Gamma \left[\psi'\right]^{tr} \nabla \tilde{u} \circ \psi \cdot \left[\psi'\right]^{tr} \nabla \tilde{v} \circ \psi \end{aligned}$$

which shows

$$\tilde{a}_{ij} = \Gamma \left[\psi'\right]^{tr} e_i \cdot \left[\psi'\right]^{tr} e_i$$

and

$$\sum_{i,j=1}^{d} \tilde{a}_{ij} \xi_i \xi_j = \Gamma \left[\psi'\right]^{tr} \xi \cdot \left[\psi'\right]^{tr} \xi \ge \epsilon \left| \left[\psi'\right]^{tr} \xi \right|^2 \ge \epsilon \delta \left| \xi \right|^2$$

where

$$\delta = \inf \left\{ \left| \left[\psi'(x)\right]^{tr} \xi \right|^2 : |\xi| = 1 \;\&\; x \in W \right\} > 0.$$

Then Eq. (3.26) implies

$$Q(\tilde{u}, \tilde{v}) = \int_{N_r} \tilde{f}(y) \tilde{v}(y) dy \text{ for all } \tilde{v} \in X_r.$$

Therefore by local boundary regularity Theorem 3.11, $\tilde{u} \in H^{k+2}(N_\rho)$ and there exists $C < \infty$ such that

$$\|\tilde{u}\|_{H^{k+2}(N_\rho)} \le C(\|\tilde{f}\|_{H^k(N_t)} + \|\tilde{u}\|_{H^1(N_t)}). \tag{3.28}$$

Invoking the change of variables Theorem 26.16 again shows $u \in H^k(V)$ and the estimate in Eq. (3.28) implies the estimated in Eq. (3.27). ■

Theorem 3.13 (Elliptic Regularity)**.** *Let Ω be a bounded open subset such that $\bar{\Omega}^o = \Omega$ and $\bar{\Omega}$ is a C^∞ – manifold with boundary, X be either $H_0^1(\Omega)$ or $H^1(\Omega)$ and $\mathcal{E}$ be a Dirichlet form as in Notation 15.3. If $k \in \mathbb{N}$ and $u \in X$ such that $\mathcal{L}_\mathcal{E} u \in H^k(\Omega)$ then $u \in H^{k+2}(\Omega)$ and*

$$\|u\|_{H^{k+2}(\Omega)} \leq C(\|f\|_{H^k(\Omega)} + \|u\|_{X^*}) \leq C(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{3.29}$$

Proof. Cover $\partial\Omega$ with $\{V_i\}_{i=1}^N$ and $\{W_i\}_{i=1}^N$ as in the above Lemma 3.12 such that $\bar{V}_i \sqsubset\sqsubset W_i$. Also choose a precompact open subset V_0 contained in Ω such that $\{V_i\}_{i=0}^N$ covers $\overline{\Omega}$. Choose W_0 such that $\overline{V}_0 \subset W_0$ and $\bar{W}_0 \subset \Omega$. If $\mathcal{L}_\mathcal{E} u =: f \in H^k(\Omega)$, then by Lemma 3.12 for $i \geq 1$ and Theorem 3.3 for $i = 0$, $u \in H^{k+2}(V_i)$ and there exist $C_i < \infty$ such that

$$\|u\|_{H^{k+2}(V_i\cap\Omega)} \leq C_i(\|f\|_{H^k(W_i\cap\Omega)} + \|u\|_{H^1(W_i\cap\Omega)}). \tag{3.30}$$

Summing Eq. (3.30) on i implies $u \in H^{k+2}(\Omega)$ and

$$\|u\|_{H^{k+2}(\Omega)} \leq C(\|f\|_{H^k(\Omega)} + \|u\|_X). \tag{3.31}$$

Finally

$$\begin{aligned}
\|u\|_X^2 &\leq C(\mathcal{E}(u,u) + \|u\|_{H^{-1}(\Omega)}^2) \\
&= C((f,u)_{L^2(\Omega)} + \|u\|_{H^{-1}(\Omega)}^2) \\
&\leq C(\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|_{H^{-1}(\Omega)}^2) \\
&\leq C(\frac{1}{2\delta}\|f\|_{L^2(\Omega)} + \frac{\delta}{2}\|u\|_{L^2(\Omega)} + \|u\|_{H^{-1}(\Omega)}^2) \\
&\leq C(\frac{1}{2\delta}\|f\|_{L^2(\Omega)}^2 + \frac{\delta}{2}\|u\|_X^2 + \|u\|_{H^{-1}(\Omega)}^2)
\end{aligned}$$

for any $\delta > 0$. Choosing δ so that $C\delta = 1$, we find

$$\frac{1}{2}\|u\|_X^2 \leq C(\frac{1}{2\delta}\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^{-1}(\Omega)}^2)$$

which implies with a new constant C that

$$\|u\|_X \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{H^{-1}(\Omega)}\right). \tag{3.32}$$

Combining Eqs. (3.31) and (3.32) implies Eq. (3.29). ■