20. Fourier Transform

The underlying space in this section is $\mathbb{R}^n$ with Lebesgue measure. The Fourier inversion formula is going to state that

$$f(x)=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n}d\xi e^{i\xi x}\int_{\mathbb{R}^n}dyf(y)e^{-iy\xi}. \tag{20.1}$$

If we let $\xi=2\pi\eta$, this may be written as

$$f(x)=\int_{\mathbb{R}^n}d\eta e^{i2\pi\eta x}\int_{\mathbb{R}^n}dyf(y)e^{-iy2\pi\eta}$$

and we have removed the multiplicative factor of $\left(\frac{1}{2\pi}\right)^n$ in Eq. (20.1) at the expense of placing factors of 2π in the arguments of the exponential. Another way to avoid writing the 2π's altogether is to redefine dx and $d\xi$ and this is what we will do here.

Notation 20.1. Let m be Lebesgue measure on $\mathbb{R}^n$ and define:

$$\mathbf{d}x=\left(\frac{1}{\sqrt{2\pi}}\right)^n dm(x)\text{ and }\mathbf{d}\xi\equiv\left(\frac{1}{\sqrt{2\pi}}\right)^n dm(\xi).$$

To be consistent with this new normalization of Lebesgue measure we will redefine $\|f\|_p$ and $\langle f,g\rangle$ as

$$\|f\|_p=\left(\int_{\mathbb{R}^n}|f(x)|^p\,\mathbf{d}x\right)^{1/p}=\left(\left(\frac{1}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n}|f(x)|^p\,dm(x)\right)^{1/p}$$

and

$$\langle f,g\rangle:=\int_{\mathbb{R}^n}f(x)g(x)\mathbf{d}x\text{ when }fg\in L^1.$$

Similarly we will define the convolution relative to these normalizations by $f\bigstar\mathbf{g}:=\left(\frac{1}{2\pi}\right)^{n/2}f*g$, i.e.

$$f\bigstar g(x)=\int_{\mathbb{R}^n}f(x-y)g(y)\mathbf{d}y=\int_{\mathbb{R}^n}f(x-y)g(y)\left(\frac{1}{2\pi}\right)^{n/2}dm(y).$$

The following notation will also be convenient; given a multi-index $\alpha\in\mathbb{Z}_+^n$, let $|\alpha|=\alpha_1+\cdots+\alpha_n$,

$$x^\alpha:=\prod_{j=1}^n x_j^{\alpha_j},\ \partial_x^\alpha=\left(\frac{\partial}{\partial x}\right)^\alpha:=\prod_{j=1}^n\left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}\text{ and}$$

$$D_x^\alpha=\left(\frac{1}{i}\right)^{|\alpha|}\left(\frac{\partial}{\partial x}\right)^\alpha=\left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha.$$

Also let

$$\langle x\rangle:=(1+|x|^2)^{1/2}$$

and for $s\in\mathbb{R}$ let

$$\nu_s(x)=(1+|x|)^s.$$

20.1. **Fourier Transform.**

Definition 20.2 (Fourier Transform)**.** For $f \in L^1$, let

$$\hat{f}(\xi) = \mathcal{F}f(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\mathbf{d}x \tag{20.2}$$

$$g^\vee(x) = \mathcal{F}^{-1}g(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\mathbf{d}\xi = \mathcal{F}g(-x) \tag{20.3}$$

The next theorem summarizes some more basic properties of the Fourier transform.

Theorem 20.3. *Suppose that $f, g \in L^1$. Then*

1. $\hat{f} \in C_0(\mathbb{R}^n)$ *and* $\left\|\hat{f}\right\|_u \le \|f\|_1$.
2. *For $y \in \mathbb{R}^n$, $(\tau_y f)\,\hat{}\,(\xi) = e^{-iy\cdot\xi}\hat{f}(\xi)$ where, as usual, $\tau_y f(x) := f(x-y)$.*
3. *The Fourier transform takes convolution to products, i.e. $(f \bigstar g)\hat{} = \hat{f}\hat{g}$.*
4. *For $f, g \in L^1$, $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$.*
5. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, then*
$$(f\circ T)^\wedge(\xi) = |\det T|^{-1} \hat{f}(\left(T^{-1}\right)^*\xi) \text{ and}$$
$$(f\circ T)^\vee(\xi) = |\det T|^{-1} f^\vee(\left(T^{-1}\right)^*\xi)$$
6. *If $(1+|x|)^k f(x) \in L^1$, then $\hat{f} \in C^k$ and $\partial^\alpha \hat{f} \in C_0$ for all $|\alpha| \le k$. Moreover,*
$$\partial_\xi^\alpha \hat{f}(\xi) = \mathcal{F}\left[(-ix)^\alpha f(x)\right](\xi) \tag{20.4}$$
for all $|\alpha| \le k$.
7. *If $f \in C^k$ and $\partial^\alpha f \in L^1$ for all $|\alpha| \le k$, then $(1+|\xi|)^k \hat{f}(\xi) \in C_0$ and*
$$(\partial^\alpha f)\hat{}\,(\xi) = (i\xi)^\alpha \hat{f}(\xi) \tag{20.5}$$
for all $|\alpha| \le k$.
8. *Suppose $g \in L^1(\mathbb{R}^k)$ and $h \in L^1(\mathbb{R}^{n-k})$ and $f = g \otimes h$, i.e.*
$$f(x) = g(x_1, \dots, x_k)h(x_{k+1}, \dots, x_n),$$
then $\hat{f} = \hat{g} \otimes \hat{h}$.

Proof. Item 1. is the Riemann Lebesgue Lemma 11.27. Items 2. − 5. are proved by the following straight forward computations:

$$(\tau_y f)\,\hat{}\,(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x-y)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-i(x+y)\cdot\xi} f(x)\mathbf{d}x = e^{-iy\cdot\xi}\hat{f}(\xi),$$

$$\begin{aligned}\langle \hat{f}, g\rangle &= \int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)\mathbf{d}\xi = \int_{\mathbb{R}^n} \mathbf{d}\xi g(\xi)\int_{\mathbb{R}^n} \mathbf{d}x e^{-ix\cdot\xi} f(x) \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{d}x\mathbf{d}\xi e^{-ix\cdot\xi} g(\xi)f(x) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{d}x \hat{g}(x)f(x) = \langle f, \hat{g}\rangle,\end{aligned}$$

$$\begin{aligned}(f\bigstar g)\hat{}\,(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f\bigstar g(x)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\left(\int_{\mathbb{R}^n} f(x-y)g(y)\mathbf{d}y\right)\mathbf{d}x \\ &= \int_{\mathbb{R}^n}\mathbf{d}y\int_{\mathbb{R}^n}\mathbf{d}x e^{-ix\cdot\xi} f(x-y)g(y) = \int_{\mathbb{R}^n}\mathbf{d}y\int_{\mathbb{R}^n}\mathbf{d}x e^{-i(x+y)\cdot\xi} f(x)g(y) \\ &= \int_{\mathbb{R}^n}\mathbf{d}y e^{-iy\cdot\xi} g(y)\int_{\mathbb{R}^n}\mathbf{d}x e^{-ix\cdot\xi} f(x) = \hat{f}(\xi)\hat{g}(\xi)\end{aligned}$$

and letting $y = Tx$ so that $\mathbf{d}x = |\det T|^{-1}\mathbf{d}y$

$$\begin{aligned}(f\circ T)^{\hat{}}(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(Tx)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-iT^{-1}y\cdot\xi} f(y)\,|\det T|^{-1}\,\mathbf{d}y \\ &= |\det T|^{-1}\,\hat{f}(\left(T^{-1}\right)^*\xi).\end{aligned}$$

Item 6. is simply a matter of differentiating under the integral sign which is easily justified because $(1+|x|)^k f(x) \in L^1$.

Item 7. follows by using Lemma 11.26 repeatedly (i.e. integration by parts) to find

$$\begin{aligned}(\partial^\alpha f)^{\hat{}}(\xi) &= \int_{\mathbb{R}^n} \partial_x^\alpha f(x) e^{-ix\cdot\xi}\mathbf{d}x = (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(x)\partial_x^\alpha e^{-ix\cdot\xi}\mathbf{d}x \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(x)(-i\xi)^\alpha e^{-ix\cdot\xi}\mathbf{d}x = (i\xi)^\alpha \hat{f}(\xi).\end{aligned}$$

Since $\partial^\alpha f \in L^1$ for all $|\alpha| \le k$, it follows that $(i\xi)^\alpha \hat{f}(\xi) = (\partial^\alpha f)^{\hat{}}(\xi) \in C_0$ for all $|\alpha| \le k$. Since

$$(1+|\xi|)^k \le \left(1+\sum_{i=1}^n |\xi_i|\right)^k = \sum_{|\alpha|\le k} c_\alpha\,|\xi^\alpha|$$

where $0 < c_\alpha < \infty$,

$$\left|(1+|\xi|)^k\,\hat{f}(\xi)\right| \le \sum_{|\alpha|\le k} c_\alpha\left|\xi^\alpha \hat{f}(\xi)\right| \to 0 \text{ as } \xi\to\infty.$$

Item 8. is a simple application of Fubini's theorem. ■

Example 20.4. If $f(x) = e^{-|x|^2/2}$ then $\hat{f}(\xi) = e^{-|\xi|^2/2}$, in short

$$\mathcal{F}e^{-|x|^2/2} = e^{-|\xi|^2/2} \text{ and } \mathcal{F}^{-1}e^{-|\xi|^2/2} = e^{-|x|^2/2}. \tag{20.6}$$

More generally, for $t>0$ let

$$p_t(x) := t^{-n/2}e^{-\frac{1}{2t}|x|^2} \tag{20.7}$$

then

$$\widehat{p_t}(\xi) = e^{-\frac{t}{2}|\xi|^2} \text{ and } (\widehat{p_t})^\vee(x) = p_t(x). \tag{20.8}$$

By Item 8. of Theorem 20.3, to prove Eq. (20.6) it suffices to consider the 1 − dimensional case because $e^{-|x|^2/2} = \prod_{i=1}^n e^{-x_i^2/2}$. Let $g(\xi) := \left(\mathcal{F}e^{-x^2/2}\right)(\xi)$, then by Eq. (20.4) and Eq. (20.5),

(20.9)

$$g'(\xi) = \mathcal{F}\left[(-ix)\,e^{-x^2/2}\right](\xi) = i\mathcal{F}\left[\frac{d}{dx}e^{-x^2/2}\right](\xi) = i(i\xi)\mathcal{F}\left[e^{-x^2/2}\right](\xi) = -\xi g(\xi).$$

Lemma 8.36 implies

$$g(0) = \int_{\mathbb{R}} e^{-x^2/2}\mathbf{d}x = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}dm(x) = 1,$$

and so solving Eq. (20.9) with $g(0)=1$ gives $\mathcal{F}\left[e^{-x^2/2}\right](\xi) = g(\xi) = e^{-\xi^2/2}$ as desired. The assertion that $\mathcal{F}^{-1}e^{-|\xi|^2/2} = e^{-|x|^2/2}$ follows similarly or by using Eq. (20.3) to conclude,

$$\mathcal{F}^{-1}\left[e^{-|\xi|^2/2}\right](x) = \mathcal{F}\left[e^{-|-\xi|^2/2}\right](x) = \mathcal{F}\left[e^{-|\xi|^2/2}\right](x) = e^{-|x|^2/2}.$$

The results in Eq. (20.8) now follow from Eq. (20.6) and item 5 of Theorem 20.3. For example, since $p_t(x) = t^{-n/2}p_1(x/\sqrt{t})$,

$$(\widehat{p}_t)(\xi) = t^{-n/2}\left(\sqrt{t}\right)^n \hat{p}_1(\sqrt{t}\xi) = e^{-\frac{t}{2}|\xi|^2}.$$

This may also be written as $(\widehat{p}_t)(\xi) = t^{-n/2}p_{\frac{1}{t}}(\xi)$. Using this and the fact that p_t is an even function,

$$(\widehat{p}_t)^\vee(x) = \mathcal{F}\widehat{p}_t(-x) = t^{-n/2}\mathcal{F}p_{\frac{1}{t}}(-x) = t^{-n/2}t^{n/2}p_t(-x) = p_t(x).$$

20.2. **Schwartz Test Functions.**

Definition 20.5. A function $f \in C(\mathbb{R}^n, \mathbb{C})$ is said to have **rapid decay** or **rapid decrease** if

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N|f(x)| < \infty \text{ for } N = 1, 2, \ldots.$$

Equivalently, for each $N \in \mathbb{N}$ there exists constants $C_N < \infty$ such that $|f(x)| \leq C_N(1+|x|)^{-N}$ for all $x \in \mathbb{R}^n$. A function $f \in C(\mathbb{R}^n, \mathbb{C})$ is said to have (at most) **polynomial growth** if there exists $N < \infty$ such

$$\sup (1+|x|)^{-N}|f(x)| < \infty,$$

i.e. there exists $N \in \mathbb{N}$ and $C < \infty$ such that $|f(x)| \leq C(1+|x|)^N$ for all $x \in \mathbb{R}^n$.

Definition 20.6 (Schwartz Test Functions)**.** Let $\mathcal{S}$ denote the space of functions $f \in C^\infty(\mathbb{R}^n)$ such that f and all of its partial derivatives have rapid decay and let

$$\|f\|_{N,\alpha} = \sup_{x\in\mathbb{R}^n}\left|(1+|x|)^N\partial^\alpha f(x)\right|$$

so that

$$\mathcal{S} = \left\{f \in C^\infty(\mathbb{R}^n) : \|f\|_{N,\alpha} < \infty \text{ for all } N \text{ and } \alpha\right\}.$$

Also let $\mathcal{P}$ denote those functions $g \in C^\infty(\mathbb{R}^n)$ such that g and all of its derivatives have at most polynomial growth, i.e. $g \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{P}$ iff for all multi-indices α, there exists $N_\alpha < \infty$ such

$$\sup (1+|x|)^{-N_\alpha}|\partial^\alpha g(x)| < \infty.$$

(Notice that any polynomial function on $\mathbb{R}^n$ is in $\mathcal{P}$.)

Remark 20.7. Since $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset L^2(\mathbb{R}^n)$, it follows that $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$.

Exercise 20.1. Let

$$L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha \tag{20.10}$$

with $a_\alpha \in \mathcal{P}$. Show $L(\mathcal{S}) \subset \mathcal{S}$ and in particular $\partial^\alpha f$ and $x^\alpha f$ are back in $\mathcal{S}$ for all multi-indices α.

Notation 20.8. Suppose that $p(x,\xi) = \Sigma_{|\alpha|\leq N}a_\alpha(x)\xi^\alpha$ where each function $a_\alpha(x)$ is a smooth function. We then set

$$p(x, D_x) := \Sigma_{|\alpha|\leq N}a_\alpha(x)D_x^\alpha$$

and if each $a_\alpha(x)$ is also a polynomial in x we will let

$$p(-D_\xi, \xi) := \Sigma_{|\alpha|\leq N}a_\alpha(-D_\xi)M_{\xi^\alpha}$$

where M_{ξ^α} is the operation of multiplication by ξ^α.

Proposition 20.9. *Let $p(x,\xi)$ be as above and assume each $a_\alpha(x)$ is a polynomial in x. Then for $f\in\mathcal{S}$,*

$$(p(x,D_x)f)^\wedge(\xi) = p(-D_\xi,\xi)\hat{f}(\xi) \tag{20.11}$$

and

$$p(\xi,D_\xi)\hat{f}(\xi) = [p(D_x,-x)f(x)]^\wedge(\xi). \tag{20.12}$$

Proof. The identities $(-D_\xi)^\alpha e^{-ix\cdot\xi} = x^\alpha e^{-ix\cdot\xi}$ and $D_x^\alpha e^{ix\cdot\xi} = \xi^\alpha e^{ix\cdot\xi}$ imply, for any polynomial function q on $\mathbb{R}^n$,

$$q(-D_\xi)e^{-ix\cdot\xi} = q(x)e^{-ix\cdot\xi} \text{ and } q(D_x)e^{ix\cdot\xi} = q(\xi)e^{ix\cdot\xi}. \tag{20.13}$$

Therefore using Eq. (20.13) repeatedly,

$$\begin{aligned}(p(x,D_x)f)^\wedge(\xi) &= \int_{\mathbb{R}^n}\sum_{|\alpha|\le N} a_\alpha(x)D_x^\alpha f(x)\cdot e^{-ix\cdot\xi}\mathbf{d}\xi\\ &= \int_{\mathbb{R}^n}\sum_{|\alpha|\le N} D_x^\alpha f(x)\cdot a_\alpha(-D_\xi)e^{-ix\cdot\xi}\mathbf{d}\xi\\ &= \int_{\mathbb{R}^n} f(x)\sum_{|\alpha|\le N}(-D_x)^\alpha\left[a_\alpha(-D_\xi)e^{-ix\cdot\xi}\right]\mathbf{d}\xi\\ &= \int_{\mathbb{R}^n} f(x)\sum_{|\alpha|\le N} a_\alpha(-D_\xi)\left[\xi^\alpha e^{-ix\cdot\xi}\right]\mathbf{d}\xi = p(-D_\xi,\xi)\hat{f}(\xi)\end{aligned}$$

wherein the third inequality we have used Lemma 11.26 to do repeated integration by parts, the fact that mixed partial derivatives commute in the fourth, and in the last we have repeatedly used Corollary 7.43 to differentiate under the integral. The proof of Eq. (20.12) is similar:

$$\begin{aligned}p(\xi,D_\xi)\hat{f}(\xi) &= p(\xi,D_\xi)\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\mathbf{d}x = \int_{\mathbb{R}^n} f(x)p(\xi,-x)e^{-ix\cdot\xi}\mathbf{d}x\\ &= \sum_{|\alpha|\le N}\int_{\mathbb{R}^n} f(x)(-x)^\alpha a_\alpha(\xi)e^{-ix\cdot\xi}\mathbf{d}x = \sum_{|\alpha|\le N}\int_{\mathbb{R}^n} f(x)(-x)^\alpha a_\alpha(-D_x)e^{-ix\cdot\xi}\mathbf{d}x\\ &= \sum_{|\alpha|\le N}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}a_\alpha(D_x)\left[(-x)^\alpha f(x)\right]\mathbf{d}x = [p(D_x,-x)f(x)]^\wedge(\xi).\end{aligned}$$

■

Corollary 20.10. *The Fourier transform preserves the space $\mathcal{S}$, i.e. $\mathcal{F}(\mathcal{S})\subset\mathcal{S}$.*

Proof. Let $p(x,\xi) = \Sigma_{|\alpha|\le N}a_\alpha(x)\xi^\alpha$ with each $a_\alpha(x)$ being a polynomial function in x. If $f\in\mathcal{S}$ then $p(D_x,-x)f\in\mathcal{S}\subset L^1$ and so by Eq. (20.12), $p(\xi,D_\xi)\hat{f}(\xi)$ is bounded in ξ, i.e.

$$\sup_{\xi\in\mathbb{R}^n}|p(\xi,D_\xi)\hat{f}(\xi)| \le C(p,f) < \infty.$$

Taking $p(x,\xi) = (1+|\xi|^2)^N\xi^\alpha$ with $N\in\mathbb{Z}_+$ in this estimate shows $\hat{f}(\xi)$ and all of its derivatives have rapid decay, i.e. $\hat{f}$ is in $\mathcal{S}$. ■

20.3. **Fourier Inversion Formula .**

Theorem 20.11 (Fourier Inversion Theorem)**.** *Suppose that $f \in L^1$ and $\hat{f} \in L^1$, then*

1. *there exists $f_0 \in C_0(\mathbb{R}^n)$ such that $f = f_0$ a.e.*
2. *$f_0 = \mathcal{F}^{-1}\mathcal{F}f$ and $f_0 = \mathcal{F}\mathcal{F}^{-1}f$,*
3. *f and $\hat{f}$ are in $L^1 \cap L^\infty$ and*
4. *$\|f\|_2 = \left\|\hat{f}\right\|_2$.*

In particular, $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a linear isomorphism of vector spaces.

Proof. First notice that $\hat{f} \in C_0(\mathbb{R}^n) \subset L^\infty$ and $\hat{f} \in L^1$ by assumption, so that $\hat{f} \in L^1 \cap L^\infty$. Let $p_t(x) \equiv t^{-n/2}e^{-\frac{1}{2t}|x|^2}$ be as in Example 20.4 so that $\widehat{p_t}(\xi) = e^{-\frac{t}{2}|\xi|^2}$ and $\widehat{p_t}^\vee = p_t$. Define $f_0 := \hat{f}^\vee \in C_0$ then

$$\begin{aligned} f_0(x) = (\hat{f})^\vee(x) &= \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}\mathbf{d}\xi = \lim_{t\downarrow 0}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}\widehat{p_t}(\xi)\mathbf{d}\xi \\ &= \lim_{t\downarrow 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)e^{i\xi\cdot(x-y)}\widehat{p_t}(\xi)\mathbf{d}\xi\ \mathbf{d}y \\ &= \lim_{t\downarrow 0}\int_{\mathbb{R}^n} f(y)p_t(y)\mathbf{d}y = f(x) \text{ a.e.} \end{aligned}$$

wherein we have used Theorem 11.21 in the last equality along with the observations that $p_t(y) = p_1(y/\sqrt{t})$ and $\int_{\mathbb{R}^n} p_1(y)\mathbf{d}y = 1$. In particular this shows that $f \in L^1 \cap L^\infty$. A similar argument shows that $\mathcal{F}^{-1}\mathcal{F}f = f_0$ as well.

Let us now compute the L^2 – norm of $\hat{f}$,

$$\begin{aligned} \|\hat{f}\|_2^2 = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi)}\mathbf{d}\xi &= \int_{\mathbb{R}^n}\mathbf{d}\xi\hat{f}(\xi)\int_{\mathbb{R}^n}\mathbf{d}x\overline{f(x)}e^{ix\cdot\xi} \\ &= \int_{\mathbb{R}^n}\mathbf{d}x\,\overline{f(x)}\int_{\mathbb{R}^n}\mathbf{d}\xi\hat{f}(\xi)e^{ix\cdot\xi} \\ &= \int_{\mathbb{R}^n}\mathbf{d}x\,\overline{f(x)}f(x) = \|f\|_2^2 \end{aligned}$$

because $\int_{\mathbb{R}^n}\mathbf{d}\xi\hat{f}(\xi)e^{ix\cdot\xi} = \mathcal{F}^{-1}\hat{f}(x) = f(x)$ a.e. ■

Corollary 20.12. *By the B.L.T. Theorem 4.1, the maps $\mathcal{F}|_{\mathcal{S}}$ and $\mathcal{F}^{-1}|_{\mathcal{S}}$ extend to bounded linear maps $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^{-1}$ from $L^2 \to L^2$. These maps satisfy the following properties:*

1. *$\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^{-1}$ are unitary and are inverses to one another as the notation suggests.*
2. *For $f \in L^2$ we may compute $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^{-1}$ by*

$$\bar{\mathcal{F}}f(\xi) = L^2\text{-}\lim_{R\to\infty}\int_{|x|\le R} f(x)e^{-ix\cdot\xi}\mathbf{d}x \text{ and} \tag{20.14}$$

$$\bar{\mathcal{F}}^{-1}f(\xi) = L^2\text{-}\lim_{R\to\infty}\int_{|x|\le R} f(x)e^{ix\cdot\xi}\mathbf{d}x. \tag{20.15}$$

3. *We may further extend $\bar{\mathcal{F}}$ to a map from $L^1 + L^2 \to C_0 + L^2$ (still denote by $\bar{\mathcal{F}}$) defined by $\bar{\mathcal{F}}f = \hat{h} + \bar{\mathcal{F}}g$ where $f = h + g \in L^1 + L^2$. For $f \in L^1 + L^2$, $\bar{\mathcal{F}}f$ may be characterized as the unique function $F \in L^1_{loc}(\mathbb{R}^n)$ such that*

$$\langle F, \phi\rangle = \langle f, \hat{\phi}\rangle \text{ for all } \phi \in C_c^\infty(\mathbb{R}^n). \tag{20.16}$$

Moreover if Eq. (20.16) holds then $F \in C_0 + L^2 \subset L^1_{loc}(\mathbb{R}^n)$ and Eq.(20.16) is valid for all $\phi \in \mathcal{S}$.

Proof. Item 1., If $f \in L^2$ and $\phi_n \in \mathcal{S}$ such that $\phi_n \to f$ in L^2, then $\bar{\mathcal{F}}f := \lim_{n\to\infty} \hat{\phi}_n$. Since $\hat{\phi}_n \in \mathcal{S} \subset L^1$, we may concluded that $\left\|\hat{\phi}_n\right\|_2 = \|\phi_n\|_2$ for all n. Thus

$$\left\|\bar{\mathcal{F}}f\right\|_2 = \lim_{n\to\infty} \left\|\hat{\phi}_n\right\|_2 = \lim_{n\to\infty} \|\phi_n\|_2 = \|f\|_2$$

which shows that $\bar{\mathcal{F}}$ is an isometry from L^2 to L^2 and similarly $\bar{\mathcal{F}}^{-1}$ is an isometry. Since $\bar{\mathcal{F}}^{-1}\bar{\mathcal{F}} = \mathcal{F}^{-1}\mathcal{F} = id$ on the dense set $\mathcal{S}$, it follows by continuity that $\bar{\mathcal{F}}^{-1}\bar{\mathcal{F}} = id$ on all of L^2. Hence $\bar{\mathcal{F}}\bar{\mathcal{F}}^{-1} = id$, and thus $\bar{\mathcal{F}}^{-1}$ is the inverse of $\bar{\mathcal{F}}$. This proves item 1.

Item 2. Let $f \in L^2$ and $R < \infty$ and set $f_R(x) := f(x)1_{|x|\le R}$. Then $f_R \in L^1 \cap L^2$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} \phi(x)\mathbf{d}x = 1$ and set $\phi_k(x) = k^n\phi(kx)$. Then $f_R \bigstar \phi_k \to f_R \in L^1 \cap L^2$ with $f_R \bigstar \phi_k \in C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}$. Hence

$$\bar{\mathcal{F}}f_R = L^2\text{–} \lim_{k\to\infty} \mathcal{F}\left(f_R \bigstar \phi_k\right) = \mathcal{F}f_R \text{ a.e.}$$

where in the second equality we used the fact that $\mathcal{F}$ is continuous on L^1. Hence $\int_{|x|\le R} f(x)e^{-ix\cdot\xi}\mathbf{d}x$ represents $\bar{\mathcal{F}}f_R(\xi)$ in L^2. Since $f_R \to f$ in L^2, Eq. (20.14) follows by the continuity of $\bar{\mathcal{F}}$ on L^2.

Item 3. If $f = h + g \in L^1 + L^2$ and $\phi \in \mathcal{S}$, then

$$\begin{aligned}\langle \hat{h} + \bar{\mathcal{F}}g, \phi\rangle &= \langle h, \phi\rangle + \langle \bar{\mathcal{F}}g, \phi\rangle = \langle h, \hat{\phi}\rangle + \lim_{R\to\infty} \langle \mathcal{F}\left(g1_{|\cdot|\le R}\right), \phi\rangle \\ &= \langle h, \hat{\phi}\rangle + \lim_{R\to\infty} \langle g1_{|\cdot|\le R}, \hat{\phi}\rangle = \langle h + g, \hat{\phi}\rangle. \end{aligned} \tag{20.17}$$

In particular if $h + g = 0$ a.e., then $\langle \hat{h} + \bar{\mathcal{F}}g, \phi\rangle = 0$ for all $\phi \in \mathcal{S}$ and since $\hat{h} + \bar{\mathcal{F}}g \in L^1_{loc}$ it follows from Corollary 11.28 that $\hat{h} + \bar{\mathcal{F}}g = 0$ a.e. This shows that $\bar{\mathcal{F}}f$ is well defined independent of how $f \in L^1 + L^2$ is decomposed into the sum of an L^1 and an L^2 function. Moreover Eq. (20.17) shows Eq. (20.16) holds with $F = \hat{h} + \bar{\mathcal{F}}g \in C_0 + L^2$ and $\phi \in \mathcal{S}$. Now suppose $G \in L^1_{loc}$ and $\langle G, \phi\rangle = \langle f, \hat{\phi}\rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Then by what we just proved, $\langle G, \phi\rangle = \langle F, \phi\rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$ and so an application of Corollary 11.28 shows $G = F \in C_0 + L^2$. ∎

Notation 20.13. Given the results of Corollary 20.12, there is little danger in writing $\hat{f}$ or $\mathcal{F}f$ for $\bar{\mathcal{F}}f$ when $f \in L^1 + L^2$.

Corollary 20.14. *If f and g are L^1 functions such that $\hat{f}, \hat{g} \in L^1$, then*

$$\mathcal{F}(fg) = \hat{f} \bigstar \hat{g} \text{ and } \mathcal{F}^{-1}(fg) = f^\vee \bigstar g^\vee.$$

Since $\mathcal{S}$ is closed under pointwise products and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism it follows that $\mathcal{S}$ is closed under convolution as well.

Proof. By Theorem 20.11, $f, g, \hat{f}, \hat{g} \in L^1 \cap L^\infty$ and hence $f \cdot g \in L^1 \cap L^\infty$ and $\hat{f} \bigstar \hat{g} \in L^1 \cap L^\infty$. Since

$$\mathcal{F}^{-1}\left(\hat{f} \bigstar \hat{g}\right) = \mathcal{F}^{-1}\left(\hat{f}\right) \cdot \mathcal{F}^{-1}\left(\hat{g}\right) = f \cdot g \in L^1$$

we may conclude from Theorem 20.11 that

$$\hat{f} \bigstar \hat{g} = \mathcal{F}\mathcal{F}^{-1}\left(\hat{f} \bigstar \hat{g}\right) = \mathcal{F}(f \cdot g).$$

Similarly one shows $\mathcal{F}^{-1}(fg) = f^\vee \bigstar g^\vee$. ∎

Corollary 20.15. *Let $p(x,\xi)$ and $p(x,D_x)$ be as in Notation 20.8 with each function $a_\alpha(x)$ being a smooth function of $x\in\mathbb{R}^n$. Then for $f\in\mathcal{S}$,*

$$p(x,D_x)f(x)=\int_{\mathbb{R}^n}p(x,\xi)\hat{f}(\xi)\,e^{ix\cdot\xi}\mathbf{d}\xi. \tag{20.18}$$

Proof. For $f\in\mathcal{S}$, we have

$$\begin{aligned}p(x,D_x)f(x)&=p(x,D_x)\left(\mathcal{F}^{-1}\hat{f}\right)(x)=p(x,D_x)\int_{\mathbb{R}^n}\hat{f}(\xi)\,e^{ix\cdot\xi}\mathbf{d}\xi\\&=\int_{\mathbb{R}^n}\hat{f}(\xi)\,p(x,D_x)e^{ix\cdot\xi}\mathbf{d}\xi=\int_{\mathbb{R}^n}\hat{f}(\xi)\,p(x,\xi)e^{ix\cdot\xi}\mathbf{d}\xi.\end{aligned}$$

If $p(x,\xi)$ is a more general function of (x,ξ) then that given in Notation 20.8, the right member of Eq. (20.18) may still make sense, in which case we may use it as a definition of $p(x,D_x)$. A linear operator defined this way is called a **pseudo differential operator** and they turn out to be a useful class of operators to study when working with partial differential equations. ■

Corollary 20.16. *Suppose $p(\xi)=\sum_{|\alpha|\le N}a_\alpha\xi^\alpha$ is a polynomial in $\xi\in\mathbb{R}^n$ and $f\in L^2$. Then $p(\partial)f$ exists in L^2 (see Notation 19.16) iff $\xi\to p(i\xi)\hat{f}(\xi)\in L^2$ in which case*

$$(p(\partial)f)\hat{}\,(\xi)=p(i\xi)\hat{f}(\xi)\text{ for a.e. }\xi.$$

In particular, if $g\in L^2$ then $f\in L^2$ solves the equation, $p(\partial)f=g$ iff $p(i\xi)\hat{f}(\xi)=\hat{g}(\xi)$ for a.e. ξ.

Proof. By definition $p(\partial)f=g$ in L^2 iff

$$\langle g,\phi\rangle=\langle f,p(-\partial)\phi\rangle\text{ for all }\phi\in C_c^\infty(\mathbb{R}^n). \tag{20.19}$$

If follows from repeated use of Lemma 19.14 that the previous equation is equivalent to

$$\langle g,\phi\rangle=\langle f,p(-\partial)\phi\rangle\text{ for all }\phi\in\mathcal{S}(\mathbb{R}^n). \tag{20.20}$$

This may also be easily proved directly as well as follows. Choose $\psi\in C_c^\infty(\mathbb{R}^n)$ such that $\psi(x)=1$ for $x\in B_0(1)$ and for $\phi\in\mathcal{S}(\mathbb{R}^n)$ let $\phi_n(x):=\psi(x/n)\phi(x)$. By the chain rule and the product rule (Eq. A.5 of Appendix A),

$$\partial^\alpha\phi_n(x)=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}n^{-|\beta|}\left(\partial^\beta\psi\right)(x/n)\cdot\partial^{\alpha-\beta}\phi(x)$$

along with the dominated convergence theorem shows $\phi_n\to\phi$ and $\partial^\alpha\phi_n\to\partial^\alpha\phi$ in L^2 as $n\to\infty$. Therefore if Eq. (20.19) holds, we find Eq. (20.20) holds because

$$\langle g,\phi\rangle=\lim_{n\to\infty}\langle g,\phi_n\rangle=\lim_{n\to\infty}\langle f,p(-\partial)\phi_n\rangle=\langle f,p(-\partial)\phi\rangle.$$

To complete the proof simply observe that $\langle g,\phi\rangle=\langle\hat{g},\phi^\vee\rangle$ and

$$\begin{aligned}\langle f,p(-\partial)\phi\rangle&=\langle\hat{f},[p(-\partial)\phi]^\vee\rangle=\langle\hat{f}(\xi),p(i\xi)\phi^\vee(\xi)\rangle\\&=\langle p(i\xi)\hat{f}(\xi),\phi^\vee(\xi)\rangle\end{aligned}$$

for all $\phi\in\mathcal{S}(\mathbb{R}^n)$. From these two observations and the fact that $\mathcal{F}$ is bijective on $\mathcal{S}$, one sees that Eq. (20.20) holds iff $\xi\to p(i\xi)\hat{f}(\xi)\in L^2$ and $\hat{g}(\xi)=p(i\xi)\hat{f}(\xi)$ for a.e. ξ. ■

20.4. **Summary of Basic Properties of $\mathcal{F}$ and $\mathcal{F}^{-1}$.** The following table summarizes some of the basic properties of the Fourier transform and its inverse.

f	$\longleftrightarrow$	$\hat{f}$ or $f^\vee$
Smoothness	$\longleftrightarrow$	Decay at infinity
∂^α	$\longleftrightarrow$	Multiplication by $(\pm i\xi)^\alpha$
$\mathcal{S}$	$\longleftrightarrow$	$\mathcal{S}$
$L^2(\mathbb{R}^n)$	$\longleftrightarrow$	$L^2(\mathbb{R}^n)$
Convolution	$\longleftrightarrow$	Products.

20.5. **Fourier Transforms of Measures and Bochner's Theorem.** To motivate the next definition suppose that μ is a finite measure on $\mathbb{R}^n$ which is absolutely continuous relative to Lebesgue measure, $d\mu(x) = \rho(x)\mathbf{d}x$. Then it is reasonable to require

$$\hat{\mu}(\xi) := \hat{\rho}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\rho(x)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} d\mu(x)$$

and

$$(\mu \bigstar g)(x) := \rho \bigstar g(x) = \int_{\mathbb{R}^n} g(x-y)\rho(x)dx = \int_{\mathbb{R}^n} g(x-y)d\mu(y)$$

when $g : \mathbb{R}^n \to \mathbb{C}$ is a function such that the latter integral is defined, for example assume g is bounded. These considerations lead to the following definitions.

Definition 20.17. The Fourier transform, $\hat{\mu}$, of a complex measure μ on $\mathcal{B}_{\mathbb{R}^n}$ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} d\mu(x) \tag{20.21}$$

and the convolution with a function g is defined by

$$(\mu \bigstar g)(x) = \int_{\mathbb{R}^n} g(x-y)d\mu(y)$$

when the integral is defined.

It follows from the dominated convergence theorem that $\hat{\mu}$ is continuous. Also by a variant of Exercise 11.11, if μ and ν are two complex measure on $\mathcal{B}_{\mathbb{R}^n}$ such that $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$. The reader is asked to give another proof of this fact in Exercise 20.3 below.

Example 20.18. Let σ_t be the surface measure on the sphere S_t of radius t centered at zero in $\mathbb{R}^3$. Then

$$\hat{\sigma}_t(\xi) = 4\pi t \frac{\sin t\,|\xi|}{|\xi|}.$$

Indeed,

$$\begin{aligned}
\hat{\sigma}_t(\xi) &= \int_{tS^2} e^{-ix\cdot\xi} d\sigma(x) = t^2 \int_{S^2} e^{-itx\cdot\xi} d\sigma(x) \\
&= t^2 \int_{S^2} e^{-itx_3|\xi|} d\sigma(x) = t^2 \int_0^{2\pi} d\theta \int_0^\pi d\phi \sin\phi e^{-it\cos\phi|\xi|} \\
&= 2\pi t^2 \int_{-1}^1 e^{-itu|\xi|} du = 2\pi t^2 \frac{1}{-it\,|\xi|} e^{-itu|\xi|}|_{u=-1}^{u=1} = 4\pi t^2 \frac{\sin t\,|\xi|}{t\,|\xi|}.
\end{aligned}$$

Definition 20.19. A function $\chi : \mathbb{R}^n \to \mathbb{C}$ is said to be **positive (semi) definite** iff the matrices $A := \{\chi(\xi_k - \xi_j)\}_{k,j=1}^m$ are positive definite for all $m \in \mathbb{N}$ and $\{\xi_j\}_{j=1}^m \subset \mathbb{R}^n$.

Lemma 20.20. *If $\chi \in C(\mathbb{R}^n, \mathbb{C})$ is a positive definite function, then*

(1) $\chi(0) \geq 0$.
(2) $\chi(-\xi) = \overline{\chi(\xi)}$ *for all* $\xi \in \mathbb{R}^n$.
(3) $|\chi(\xi)| \leq \chi(0)$ *for all* $\xi \in \mathbb{R}^n$.
(4) *For all* $f \in \mathbb{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \chi(\xi - \eta) f(\xi) \overline{f(\eta)} d\xi d\eta \geq 0. \tag{20.22}$$

Proof. Taking $m = 1$ and $\xi_1 = 0$ we learn $\chi(0)|\lambda|^2 \geq 0$ for all $\lambda \in \mathbb{C}$ which proves item 1. Taking $m = 2$, $\xi_1 = \xi$ and $\xi_2 = \eta$, the matrix

$$A := \begin{bmatrix} \chi(0) & \chi(\xi - \eta) \\ \chi(\eta - \xi) & \chi(0) \end{bmatrix}$$

is positive definite from which we conclude $\chi(\xi - \eta) = \overline{\chi(\eta - \xi)}$ (since $A = A^*$ by definition) and

$$0 \leq \det \begin{bmatrix} \chi(0) & \chi(\xi - \eta) \\ \chi(\eta - \xi) & \chi(0) \end{bmatrix} = |\chi(0)|^2 - |\chi(\xi - \eta)|^2.$$

and hence $|\chi(\xi)| \leq \chi(0)$ for all ξ. This proves items 2. and 3. Item 4. follows by approximating the integral in Eq. (20.22) by Riemann sums,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \chi(\xi - \eta) f(\xi) \overline{f(\eta)} d\xi d\eta = \lim_{mesh \to 0} \sum \chi(\xi_k - \xi_j) f(\xi_j) \overline{f(\xi_k)} \geq 0.$$

The details are left to the reader. ■

Lemma 20.21. *If μ is a finite positive measure on $\mathcal{B}_{\mathbb{R}^n}$, then $\chi := \hat{\mu} \in C(\mathbb{R}^n, \mathbb{C})$ is a positive definite function.*

Proof. As has already been observed after Definition 20.17, the dominated convergence theorem implies $\hat{\mu} \in C(\mathbb{R}^n, \mathbb{C})$. Since μ is a positive measure (and hence real),

$$\hat{\mu}(-\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} d\mu(x) = \overline{\int_{\mathbb{R}^n} e^{-i\xi \cdot x} d\mu(x)} = \overline{\hat{\mu}(-\xi)}.$$

From this it follows that for any $m \in \mathbb{N}$ and $\{\xi_j\}_{j=1}^m \subset \mathbb{R}^n$, the matrix $A := \{\hat{\mu}(\xi_k - \xi_j)\}_{k,j=1}^m$ is self-adjoint. Moreover if $\lambda \in \mathbb{C}^m$,

$$\sum_{k,j=1}^m \hat{\mu}(\xi_k - \xi_j) \lambda_k \bar{\lambda}_j = \int_{\mathbb{R}^n} \sum_{k,j=1}^m e^{-i(\xi_k - \xi_j) \cdot x} \lambda_k \bar{\lambda}_j d\mu(x) = \int_{\mathbb{R}^n} \sum_{k,j=1}^m e^{-i\xi_k \cdot x} \lambda_k \overline{e^{-i\xi_j \cdot x} \lambda_j} d\mu(x)$$
$$= \int_{\mathbb{R}^n} \left| \sum_{k=1}^m e^{-i\xi_k \cdot x} \lambda_k \right|^2 d\mu(x) \geq 0$$

showing A is positive definite. ■

Theorem 20.22 (Bochner's Theorem)**.** *Suppose $\chi \in C(\mathbb{R}^n, \mathbb{C})$ is positive definite function, then there exists a unique positive measure μ on $\mathcal{B}_{\mathbb{R}^n}$ such that $\chi = \hat{\mu}$.*

Proof. If $\chi(\xi) = \hat{\mu}(\xi)$, then for $f \in \mathcal{S}$ we would have

$$\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} \left(f^\vee\right)^\wedge d\mu = \int_{\mathbb{R}^n} f^\vee(\xi)\hat{\mu}(\xi)\mathbf{d}\xi.$$

This suggests that we define

$$I(f) := \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi)\mathbf{d}\xi \text{ for all } f \in \mathcal{S}.$$

We will now show I is positive in the sense if $f \in \mathcal{S}$ and $f \geq 0$ then $I(f) \geq 0$. For general $f \in \mathcal{S}$ we have

$$\begin{aligned} I(|f|^2) &= \int_{\mathbb{R}^n} \chi(\xi) \left(|f|^2\right)^\vee (\xi)\mathbf{d}\xi = \int_{\mathbb{R}^n} \chi(\xi) \left(f^\vee \bigstar \bar{f}^\vee\right)(\xi)\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi - \eta)\bar{f}^\vee(\eta)\mathbf{d}\eta\mathbf{d}\xi = \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi - \eta)\overline{f^\vee(-\eta)}\mathbf{d}\eta\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} \chi(\xi - \eta) f^\vee(\xi)\overline{f^\vee(\eta)}\mathbf{d}\eta\mathbf{d}\xi \geq 0. \end{aligned}$$

For $t > 0$ let $p_t(x) := t^{-n/2}e^{-|x|^2/2t} \in \mathcal{S}$ and define

$$I \bigstar p_t(x) := I(p_t(x - \cdot)) = I(\left|\sqrt{p_t(x - \cdot)}\right|^2)$$

which is non-negative by above computation and because $\sqrt{p_t(x - \cdot)} \in \mathcal{S}$.

Using

$$\begin{aligned} \left[p_t(x - \cdot)\right]^\vee(\xi) &= \int_{\mathbb{R}^n} p_t(x - y)e^{iy\cdot\xi}\mathbf{d}y = \int_{\mathbb{R}^n} p_t(y)e^{i(y+x)\cdot\xi}\mathbf{d}y \\ &= e^{ix\cdot\xi}p_t^\vee(\xi) = e^{ix\cdot\xi}e^{-t|\xi|^2/2}, \end{aligned}$$

$$\begin{aligned} \langle I \bigstar p_t, \psi\rangle &= \int_{\mathbb{R}^n} I(p_t(x - \cdot))\psi(x)\mathbf{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi(\xi)\left[p_t(x - \cdot)\right]^\vee(\xi)\psi(x)\mathbf{d}\xi\mathbf{d}x \\ &= \int_{\mathbb{R}^n} \chi(\xi)\psi^\vee(\xi)e^{-t|\xi|^2/2}\mathbf{d}\xi \end{aligned}$$

which coupled with the dominated convergence theorem shows

$$\langle I \bigstar p_t, \psi\rangle \to \int_{\mathbb{R}^n} \chi(\xi)\psi^\vee(\xi)\mathbf{d}\xi = I(\psi) \text{ as } t \downarrow 0.$$

Hence if $\psi \geq 0$, then $I(\psi) = \lim_{t\downarrow 0}\langle I \bigstar p_t, \psi\rangle \geq 0$.

Let $K \subset \mathbb{R}$ be a compact set and $\psi \in C_c(\mathbb{R}, [0, \infty))$ be a function such that $\psi = 1$ on K. If $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$ is a smooth function with $\text{supp}(f) \subset K$, then $0 \leq \|f\|_\infty \psi - f \in \mathcal{S}$ and hence

$$0 \leq \langle I, \|f\|_\infty \psi - f\rangle = \|f\|_\infty \langle I, \psi\rangle - \langle I, f\rangle$$

and therefore $\langle I, f\rangle \leq \|f\|_\infty \langle I, \psi\rangle$. Replacing f by $-f$ implies, $-\langle I, f\rangle \leq \|f\|_\infty \langle I, \psi\rangle$ and hence we have proved

$$|\langle I, f\rangle| \leq C(\text{supp}(f))\,\|f\|_\infty \tag{20.23}$$

for all $f \in \mathcal{D}_{\mathbb{R}^n} := C_c^\infty(\mathbb{R}^n, \mathbb{R})$ where $C(K)$ is a finite constant for each compact subset of $\mathbb{R}^n$. Because of the estimate in Eq. (20.23), it follows that $I|_{\mathcal{D}_{\mathbb{R}^n}}$ has a unique extension I to $C_c(\mathbb{R}^n, \mathbb{R})$ still satisfying the estimates in Eq. (20.23) and moreover this extension is still positive. So by the Riesz – Markov theorem, there exists a unique Radon – measure μ on $\mathbb{R}^n$ such that such that $\langle I, f\rangle = \mu(f)$ for all $f \in C_c(\mathbb{R}^n, \mathbb{R})$.

To finish the proof we must show $\hat{\mu}(\eta) = \chi(\eta)$ for all $\eta \in \mathbb{R}^n$ given

$$\mu(f) = \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi) \mathbf{d}\xi \text{ for all } f \in C_c^\infty(\mathbb{R}^n, \mathbb{R}).$$

Let $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R}_+)$ be a radial function such $f(0) = 1$ and $f(x)$ is decreasing as $|x|$ increases. Let $f_\epsilon(x) := f(\epsilon x)$, then by Theorem 20.3,

$$\mathcal{F}^{-1}\left[e^{-i\eta x} f_\epsilon(x)\right](\xi) = \epsilon^{-n} f^\vee(\frac{\xi - \eta}{\epsilon})$$

and therefore

$$\int_{\mathbb{R}^n} e^{-i\eta x} f_\epsilon(x) d\mu(x) = \int_{\mathbb{R}^n} \chi(\xi) \epsilon^{-n} f^\vee(\frac{\xi - \eta}{\epsilon}) \mathbf{d}\xi. \tag{20.24}$$

Because $\int_{\mathbb{R}^n} f^\vee(\xi)\mathbf{d}\xi = \mathcal{F} f^\vee(0) = f(0) = 1$, we may apply the approximate δ – function Theorem 11.21 to Eq. (20.24) to find

$$\int_{\mathbb{R}^n} e^{-i\eta x} f_\epsilon(x) d\mu(x) \to \chi(\eta) \text{ as } \epsilon \downarrow 0. \tag{20.25}$$

On the the other hand, when $\eta = 0$, the monotone convergence theorem implies $\mu(f_\epsilon) \uparrow \mu(1) = \mu(\mathbb{R}^n)$ and therefore $\mu(\mathbb{R}^n) = \mu(1) = \chi(0) < \infty$. Now knowing the μ is a finite measure we may use the dominated convergence theorem to concluded

$$\mu(e^{-i\eta x} f_\epsilon(x)) \to \mu(e^{-i\eta x}) = \hat{\mu}(\eta) \text{ as } \epsilon \downarrow 0$$

for all η. Combining this equation with Eq. (20.25) shows $\hat{\mu}(\eta) = \chi(\eta)$ for all $\eta \in \mathbb{R}^n$. ■

20.6. **Supplement: Heisenberg Uncertainty Principle.** Suppose that H is a Hilbert space and A, B are two densely defined symmetric operators on H. More explicitly, A is a densely defined symmetric linear operator on H means there is a dense subspace $\mathcal{D}_A \subset H$ and a linear map $A : \mathcal{D}_A \to H$ such that $(A\phi, \psi) = (\phi, A\psi)$ for all $\phi, \psi \in \mathcal{D}_A$. Let $\mathcal{D}_{AB} := \{\phi \in H : \phi \in \mathcal{D}_B \text{ and } B\phi \in \mathcal{D}_A\}$ and for $\phi \in \mathcal{D}_{AB}$ let $(AB)\phi = A(B\phi)$ with a similar definition of $\mathcal{D}_{BA}$ and BA. Moreover, let $\mathcal{D}_C := \mathcal{D}_{AB} \cap \mathcal{D}_{BA}$ and for $\phi \in \mathcal{D}_C$, let

$$C\phi = \frac{1}{i}[A, B]\phi = \frac{1}{i}(AB - BA)\phi.$$

Notice that for $\phi, \psi \in \mathcal{D}_C$ we have

$$\begin{aligned}(C\phi, \psi) &= \frac{1}{i}\{(AB\phi, \psi) - (BA\phi, \psi)\} = \frac{1}{i}\{(B\phi, A\psi) - (A\phi, B\psi)\} \\ &= \frac{1}{i}\{(\phi, BA\psi) - (\phi, AB\psi)\} = (\phi, C\psi),\end{aligned}$$

so that C is symmetric as well.

Theorem 20.23 (Heisenberg Uncertainty Principle)**.** *Continue the above notation and assumptions,*

$$\frac{1}{2}|(\psi, C\psi)| \le \sqrt{\|A\psi\|^2 - (\psi, A\psi)} \cdot \sqrt{\|B\psi\|^2 - (\psi, B\psi)} \tag{20.26}$$

for all $\psi \in \mathcal{D}_C$. Moreover if $\|\psi\| = 1$ and equality holds in Eq. (20.26), then

$$(A - (\psi, A\psi))\psi = i\lambda(B - (\psi, B\psi))\psi \text{ or}$$
$$(B - (\psi, B\psi)) = i\lambda\psi(A - (\psi, A\psi))\psi \tag{20.27}$$

for some $\lambda \in \mathbb{R}$.

Proof. By homogeneity (20.26) we may assume that $\|\psi\| = 1$. Let $a := (\psi, A\psi)$, $b = (\psi, B\psi)$, $\tilde{A} = A - aI$, and $\tilde{B} = B - bI$. Then we have still have

$$[\tilde{A}, \tilde{B}] = [A - aI, B - bI] = iC.$$

Now

$$\begin{aligned} i(\psi, C\psi) &= (\psi, iC\psi) = (\psi, [\tilde{A}, \tilde{B}]\psi) = (\psi, \tilde{A}\tilde{B}\psi) - (\psi, \tilde{B}\tilde{A}\psi) \\ &= (\tilde{A}\psi, \tilde{B}\psi) - (\tilde{B}\psi, \tilde{A}\psi) = 2i \operatorname{Im}(\tilde{A}\psi, \tilde{B}\psi) \end{aligned}$$

from which we learn

$$|(\psi, C\psi)| = 2\left|\operatorname{Im}(\tilde{A}\psi, \tilde{B}\psi)\right| \le 2\left|(\tilde{A}\psi, \tilde{B}\psi)\right| \le 2\left\|\tilde{A}\psi\right\|\left\|\tilde{B}\psi\right\|$$

with equality iff $\operatorname{Re}(\tilde{A}\psi, \tilde{B}\psi) = 0$ and $\tilde{A}\psi$ and $\tilde{B}\psi$ are linearly dependent, i.e. iff Eq. (20.27) holds.

The result follows from this equality and the identities

$$\begin{aligned} \left\|\tilde{A}\psi\right\|^2 &= \|A\psi - a\psi\|^2 = \|A\psi\|^2 + a^2\|\psi\|^2 - 2a\operatorname{Re}(A\psi, \psi) \\ &= \|A\psi\|^2 + a^2 - 2a^2 = \|A\psi\|^2 - (A\psi, \psi) \end{aligned}$$

and

$$\left\|\tilde{B}\psi\right\| = \|B\psi\|^2 - (B\psi, \psi).$$

■

Example 20.24. As an example, take $H = L^2(\mathbb{R})$, $A = \frac{1}{i}\partial_x$ and $B = M_x$ with $\mathcal{D}_A := \{f \in H : f' \in H\}$ (f' is the weak derivative) and $\mathcal{D}_B := \left\{f \in H : \int_{\mathbb{R}} |xf(x)|^2\, dx < \infty\right\}$. In this case,

$$\mathcal{D}_C = \{f \in H : f', xf \text{ and } xf' \text{ are in } H\}$$

and $C = -I$ on $\mathcal{D}_C$. Therefore for a **unit** vector $\psi \in \mathcal{D}_C$,

$$\frac{1}{2} \le \left\|\frac{1}{i}\psi' - a\psi\right\|_2 \cdot \|x\psi - b\psi\|_2$$

where $a = i\int_{\mathbb{R}} \psi\bar{\psi}' dm$ [41] and $b = \int_{\mathbb{R}} x\,|\psi(x)|^2\, dm(x)$. Thus we have

$$\frac{1}{4} = \frac{1}{4}\int_{\mathbb{R}} |\psi|^2\, dm \le \int_{\mathbb{R}} (k-a)^2\left|\hat{\psi}(k)\right|^2 dk \cdot \int_{\mathbb{R}} (x-b)^2\,|\psi(x)|^2\, dx. \tag{20.28}$$

[41] The constant a may also be described as

$$\begin{aligned} a &= i\int_{\mathbb{R}} \psi\bar{\psi}' dm = \sqrt{2\pi} i\int_{\mathbb{R}} \hat{\psi}(\xi)\overline{(\bar{\psi}')^{\wedge}}(\xi) d\xi \\ &= \int_{\mathbb{R}} \xi\left|\hat{\psi}(\xi)\right|^2 dm(\xi). \end{aligned}$$

Equality occurs if there exists $\lambda \in \mathbb{R}$ such that

$$i\lambda\,(x-b)\,\psi(x) = (\frac{1}{i}\partial_x - a)\psi(x) \text{ a.e.}$$

Working formally, this gives rise to the ordinary differential equation (in weak form),

$$\psi_x = \left[-\lambda(x-b) + ia\right]\psi \tag{20.29}$$

which has solutions (see Exercise 20.4 below)

$$\psi = C\exp\left(\int_{\mathbb{R}} \left[-\lambda(x-b)+ia\right]dx\right) = C\exp\left(-\frac{\lambda}{2}(x-b)^2 + iax\right). \tag{20.30}$$

Let $\lambda = \frac{1}{2t}$ and choose C so that $\|\psi\|_2 = 1$ to find

$$\psi_{t,a,b}(x) = \left(\frac{1}{2t}\right)^{1/4}\exp\left(-\frac{1}{4t}(x-b)^2 + iax\right)$$

are the functions which saturate the Heisenberg uncertainty principle in Eq. (20.28).

20.6.1. *Exercises.*

Exercise 20.2. Let $f \in L^2(\mathbb{R}^n)$ and α be a multi-index. If $\partial^\alpha f$ exists in $L^2(\mathbb{R}^n)$ then $\mathcal{F}(\partial^\alpha f) = (i\xi)^\alpha\,\hat{f}(\xi)$ in $L^2(\mathbb{R}^n)$ and conversely if $\left(\xi \to \xi^\alpha \hat{f}(\xi)\right) \in L^2(\mathbb{R}^n)$ then $\partial^\alpha f$ exists.

Exercise 20.3. Suppose μ is a complex measure on $\mathbb{R}^n$ and $\hat{\mu}(\xi)$ is its Fourier transform as defined in Definition 20.17. Show μ satisfies,

$$\langle \hat{\mu}, \phi\rangle := \int_{\mathbb{R}^n} \hat{\mu}(\xi)\phi(\xi)d\xi = \mu(\hat{\phi}) := \int_{\mathbb{R}^n} \hat{\phi}d\mu \text{ for all } \phi \in \mathcal{S}$$

and use this to show if μ is a complex measure such that $\hat{\mu} \equiv 0$, then $\mu \equiv 0$.

Exercise 20.4. Show that ψ described in Eq. (20.30) is the general solution to Eq. (20.29). **Hint:** Suppose that ϕ is any solution to Eq. (20.29) and ψ is given as in Eq. (20.30) with $C = 1$. Consider the weak – differential equation solved by ϕ/ψ.

20.6.2. *More Proofs of the Fourier Inversion Theorem.*

Exercise 20.5. Suppose that $f \in L^1(\mathbb{R})$ and assume that f continuously differentiable in a neighborhood of 0, show

$$\lim_{M\to\infty}\int_{-\infty}^{\infty} \frac{\sin Mx}{x} f(x)dx = \pi f(0) \tag{20.31}$$

using the following steps.

(1) Use Example 8.26 to deduce,

$$\lim_{M\to\infty}\int_{-1}^{1} \frac{\sin Mx}{x}dx = \lim_{M\to\infty}\int_{-M}^{M}\frac{\sin x}{x}dx = \pi.$$

(2) Explain why

$$0 = \lim_{M\to\infty}\int_{|x|\ge 1} \sin Mx \cdot \frac{f(x)}{x}dx \text{ and}$$

$$0 = \lim_{M\to\infty}\int_{|x|\le 1} \sin Mx \cdot \frac{f(x)-f(0)}{x}dx.$$

(3) Add the previous two equations and use part (1) to prove Eq. (20.31).

Exercise 20.6 (Fourier Inversion Formula)**.** Suppose that $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$.

(1) Further assume that f is continuously differentiable in a neighborhood of 0. Show that
$$\Lambda := \int_{\mathbb{R}} \hat{f}(\xi)\mathbf{d}\xi = f(0).$$
Hint: by the dominated convergence theorem, $\Lambda := \lim_{M\to\infty} \int_{|\xi|\le M} \hat{f}(\xi)\mathbf{d}\xi$. Now use the definition of $\hat{f}(\xi)$, Fubini's theorem and Exercise 20.5.
(2) Apply part 1. of this exercise with f replace by $\tau_y f$ for some $y \in \mathbb{R}$ to prove
$$f(y) = \int_{\mathbb{R}} \hat{f}(\xi)e^{iy\cdot\xi}\mathbf{d}\xi \tag{20.32}$$
provided f is now continuously differentiable near y.

The goal of the next exercises is to give yet another proof of the Fourier inversion formula.

Notation 20.25. For $L > 0$, let $C_L^k(\mathbb{R})$ denote the space of $C^k - 2\pi L$ periodic functions:
$$C_L^k(\mathbb{R}) := \left\{ f \in C^k(\mathbb{R}) : f(x + 2\pi L) = f(x) \text{ for all } x \in \mathbb{R} \right\}.$$
Also let $\langle \cdot, \cdot \rangle_L$ denote the inner product on the Hilbert space $H_L := L^2([-\pi L, \pi L])$ given by
$$(f, g)_L := \frac{1}{2\pi L} \int_{[-\pi L,\pi L]} f(x)\bar{g}(x)dx.$$

Exercise 20.7. Recall that $\left\{\chi_k^L(x) := e^{ikx/L} : k \in \mathbb{Z}\right\}$ is an orthonormal basis for H_L and in particular for $f \in H_L$,
$$f = \sum_{k\in\mathbb{Z}} \langle f, \chi_k^L \rangle_L \chi_k^L \tag{20.33}$$
where the convergence takes place in $L^2([-\pi L, \pi L])$. Suppose now that $f \in C_L^2(\mathbb{R})$[42]. Show (by two integration by parts)
$$\left|(f_L, \chi_k^L)_L\right| \le \frac{L^2}{k^2} \|f''\|_u$$
where $\|g\|_u$ denote the uniform norm of a function g. Use this to conclude that the sum in Eq. (20.33) is uniformly convergent and from this conclude that Eq. (20.33) holds pointwise.

Exercise 20.8 (Fourier Inversion Formula on $\mathcal{S}$)**.** Let $f \in \mathcal{S}(\mathbb{R})$, $L > 0$ and
$$f_L(x) := \sum_{k\in\mathbb{Z}} f(x + 2\pi kL). \tag{20.34}$$
Show:

(1) The sum defining f_L is convergent and moreover that $f_L \in C_L^\infty(\mathbb{R})$.
(2) Show $(f_L, \chi_k^L)_L = \frac{1}{\sqrt{2\pi}L}\hat{f}(k/L)$.

[42]We view $C_L^2(\mathbb{R})$ as a subspace of H_L by identifying $f \in C_L^2(\mathbb{R})$ with $f|_{[-\pi L,\pi L]} \in H_L$.

(3) Conclude from Exercise 20.7 that

$$f_L(x) = \frac{1}{\sqrt{2\pi}L}\sum_{k\in\mathbb{Z}} \hat{f}(k/L)e^{ikx/L} \text{ for all } x \in \mathbb{R}. \tag{20.35}$$

(4) Show, by passing to the limit, $L \to \infty$, in Eq. (20.35) that Eq. (20.32) holds for all $x \in \mathbb{R}$. **Hint:** Recall that $\hat{f} \in \mathcal{S}$.

Exercise 20.9. Folland 8.13 on p. 254.

Exercise 20.10. Folland 8.14 on p. 254. (Wirtinger's inequality.)

Exercise 20.11. Folland 8.15 on p. 255. (The sampling Theorem. Modify to agree with notation in notes, see Solution F.20 below.)

Exercise 20.12. Folland 8.16 on p. 255.

Exercise 20.13. Folland 8.17 on p. 255.

Exercise 20.14. .Folland 8.19 on p. 256. (The Fourier transform of a function whose support has finite measure.)

Exercise 20.15. Folland 8.22 on p. 256. (Bessel functions.)

Exercise 20.16. Folland 8.23 on p. 256. (Hermite Polynomial problems and Harmonic oscillators.)

Exercise 20.17. Folland 8.31 on p. 263. (Poisson Summation formula problem.)