

4. CAUCHY – KOVALEVSKAYA THEOREM

As a warm up we will start with the corresponding result for ordinary differential equations.

Theorem 4.1 (ODE Version of Cauchy – Kovalevskaya, I.). *Suppose $a > 0$ and $f : (-a, a) \rightarrow \mathbb{R}$ is real analytic near 0 and $u(t)$ is the unique solution to the ODE*

$$(4.1) \quad \dot{u}(t) = f(u(t)) \text{ with } u(0) = 0.$$

Then u is also real analytic near 0.

We will give four proofs. However it is the last proof that the reader should focus on for understanding the PDE version of Theorem 4.1.

Proof. (First Proof.) If $f(0) = 0$, then $u(t) = 0$ for all t is the unique solution to Eq. (4.1) which is clearly analytic. So we may now assume that $f(0) \neq 0$. Let $G(z) := \int_0^z \frac{1}{f(u)} du$, another real analytic function near 0. Then as usual we have

$$\frac{d}{dt} G(u(t)) = \frac{1}{f(u(t))} \dot{u}(t) = 1$$

and hence $G(u(t)) = t$. We then have $u(t) = G^{-1}(t)$ which is real analytic near $t = 0$ since $G'(0) = \frac{1}{f(0)} \neq 0$. ■

Proof. (Second Proof.) For $z \in \mathbb{C}$ let $u_z(t)$ denote the solution to the ODE

$$(4.2) \quad \dot{u}_z(t) = z f(u_z(t)) \text{ with } u_z(0) = 0.$$

Notice that if $u(t)$ is analytic, then $t \rightarrow u(tz)$ satisfies the same equation as u_z . Since $G(z, u) = z f(u)$ is holomorphic in z and u , it follows that u_z in Eq. (4.2) depends holomorphically on z as can be seen by showing $\bar{\partial}_z u_z = 0$, i.e. showing $z \rightarrow u_z$ satisfies the Cauchy Riemann equations. Therefore if $\epsilon > 0$ is chosen small enough such that Eq. (4.2) has a solution for $|t| < \epsilon$ and $|z| < 2$, then

$$(4.3) \quad u(t) = u_1(t) = \sum_{n=0}^{\infty} \frac{1^n}{n!} \partial_z^n u_z(t)|_{z=0}.$$

Now when $z \in \mathbb{R}$, $u_z(t) = u(tz)$ and therefore

$$\partial_z^n u_z(t)|_{z=0} = \partial_z^n u(tz)|_{z=0} = u^{(n)}(0) t^n.$$

Putting this back in Eq. (4.3) shows

$$u(t) = \sum_{n=0}^{\infty} \frac{1}{n!} u^{(n)}(0) t^n$$

which shows $u(t)$ is analytic for t near 0. ■

Proof. (Third Proof.) Go back to the original proof of existence of solutions, but now replace t by $z \in \mathbb{C}$ and $\int_0^t f(u(\tau)) d\tau$ by $\int_0^z f(u(\xi)) d\xi = \int_0^1 f(u(tz)) z dt$. Then the usual Picard iterates proof work in the class of holomorphic functions to give a holomorphic function $u(z)$ solving Eq. (4.1). ■

Proof. (Fourth Proof: Method of Majorants) Suppose for the moment we have an analytic solution to Eq. (4.1). Then by repeatedly differentiating Eq. (4.1) we

learn

$$\begin{aligned} \ddot{u}(t) &= f'(u(t))\dot{u}(t) = f'(u(t))f(u(t)) \\ u^{(3)}(t) &= f''(u(t))f^2(u(t)) + [f'(u(t))]^2 f(u(t)) \\ &\vdots \\ u^{(n)}(t) &= p_n \left(f(u(t)), \dots, f^{(n-1)}(u(t)) \right) \end{aligned}$$

where p_n is a polynomial in n variables with all non-negative integer coefficients. The first few polynomials are $p_1(x) = x$, $p_2(x, y) = xy$, $p_3(x, y, z) = x^2z + xy^2$. Notice that these polynomials are universal, i.e. are independent of the function f and

$$\begin{aligned} \left| u^{(n)}(0) \right| &= \left| p_n \left(f(0), \dots, f^{(n-1)}(0) \right) \right| \\ &\leq p_n \left(|f(0)|, \dots, |f^{(n-1)}(0)| \right) \leq p_n \left(g(0), \dots, g^{(n-1)}(0) \right) \end{aligned}$$

where g is any analytic function such that $|f^{(k)}(0)| \leq g^{(k)}(0)$ for all $k \in \mathbb{Z}_+$. (We will abbreviate this last condition as $f \ll g$.) Now suppose that $v(t)$ is a solution to

$$(4.4) \quad \dot{v}(t) = g(v(t)) \text{ with } v(0) = 0,$$

then we know from above that

$$v^{(n)}(0) = p_n \left(g(0), \dots, g^{(n-1)}(0) \right) \geq \left| u^{(n)}(0) \right| \text{ for all } n.$$

Hence if knew that v were analytic with radius of convergence larger than some $\rho > 0$, then by comparison we would find

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left| u^{(n)}(0) \right| \rho^n \leq \sum_{n=0}^{\infty} \frac{1}{n!} v^{(n)}(0) \rho^n < \infty$$

and this would show

$$u(t) := \sum_{n=0}^{\infty} \frac{1}{n!} p_n \left(f(0), \dots, f^{(n-1)}(0) \right) t^n$$

is a well defined analytic function for $|t| < \rho$.

I now claim that $u(t)$ solves Eq. (4.1). Indeed, both sides of Eq. (4.1) are analytic in t , so it suffices to show the derivatives of each side of Eq. (4.1) agree at $t = 0$. For example $\dot{u}(0) = f(0)$, $\ddot{u}(0) = \frac{d}{dt}|_0 f(u(t))$, etc. However this is the case by the very definition of $u^{(n)}(0)$ for all n .

So to finish the proof, it suffices to find an analytic function g such that $|f^{(k)}(0)| \leq g^{(k)}(0)$ for all $k \in \mathbb{Z}_+$ and for which we know the solution to Eq. (4.4) is analytic about $t = 0$. To this end, suppose that the power series expansion for $f(t)$ at $t = 0$ has radius of convergence larger than $r > 0$, then $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) r^n$ is convergent and in particular,

$$C := \max_n \left| \frac{1}{n!} f^{(n)}(0) r^n \right| < \infty$$

from which we conclude

$$\max_n \left| \frac{1}{n!} f^{(n)}(0) \right| \leq C r^{-n}.$$

Let

$$g(u) := \sum_{n=0}^{\infty} Cr^{-n}u^n = C \frac{1}{1-u/r} = C \frac{r}{r-u}.$$

Then clearly $f \ll g$. To conclude the proof, we will explicitly solve Eq. (4.4) with this function $g(t)$,

$$\dot{v}(t) = C \frac{r}{r-v(t)} \text{ with } v(0) = 0.$$

By the usual separation of variables methods we find $rv(t) - \frac{1}{2}v^2(t) = Crt$, i.e.

$$2Crt - 2rv(t) + v^2(t) = 0$$

which has solutions, $v(t) = r \pm \sqrt{r^2 - 2Crt}$. We must take the negative sign to get the correct initial condition, so that

$$(4.5) \quad v(t) = r - \sqrt{r^2 - 2Crt} = r - r\sqrt{1 - 2Ct/r}$$

which is real analytic for $|t| < \rho := r/C$. ■

Let us now Jazz up this theorem to that case of a system of ordinary differential equations. For this we will need the following lemma.

Lemma 4.2. *Suppose $h : (-a, a)^d \rightarrow \mathbb{R}^d$ is real analytic near $0 \in (-a, a)^d$, then*

$$h \ll \frac{Cr}{r - z_1 - \cdots - z_d}$$

for some constants C and r .

Proof. By definition, there exists $\rho > 0$ such that

$$h(z) = \sum_{\alpha} h_{\alpha} z^{\alpha} \text{ for } |z| < \rho$$

where $h_{\alpha} = \frac{1}{\alpha!} \partial^{\alpha} h(0)$. Taking $z = r(1, 1, \dots, 1)$ with $r < \rho$ implies there exists $C < \infty$ such that $|h_{\alpha}| r^{|\alpha|} \leq C$ for all α , i.e.

$$|h_{\alpha}| \leq Cr^{-|\alpha|} \leq C \frac{|\alpha|!}{\alpha!} r^{-|\alpha|}.$$

This completes the proof since

$$\begin{aligned} \sum_{\alpha} C \frac{|\alpha|!}{\alpha!} r^{-|\alpha|} z^{\alpha} &= C \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} \left(\frac{z}{r}\right)^{\alpha} = C \sum_{n=0}^{\infty} \left(\frac{z_1 + \cdots + z_d}{r}\right)^n \\ &= C \frac{1}{1 - \left(\frac{z_1 + \cdots + z_d}{r}\right)} = \frac{Cr}{r - z_1 - \cdots - z_d} \end{aligned}$$

all of which is valid provided $|z| := |z_1| + \cdots + |z_d| < r$. ■

Theorem 4.3 (ODE Version of Cauchy – Kovalevskaya, II.). *Suppose $a > 0$ and $f : (-a, a)^d \rightarrow \mathbb{R}^d$ be real analytic near $0 \in (-a, a)^d$ and $u(t)$ is the unique solution to the ODE*

$$(4.6) \quad \dot{u}(t) = f(u(t)) \text{ with } u(0) = 0.$$

Then u is also real analytic near 0.

Proof. All but the first proof of Theorem 4.1 may be adapted to cover this case. The only proof which perhaps needs a little more comment is the fourth proof. By Lemma 4.2, we can find $C, r > 0$ such that

$$f_j(z) \ll g_j(z) := \frac{Cr}{r - z_1 - \dots - z_d}$$

for all j . Let $v(t)$ denote the solution to the ODE,

$$(4.7) \quad \dot{v}(t) = g(v(t)) = \frac{Cr}{r - v_1(t) - \dots - v_d(t)}(1, 1, \dots, 1)$$

with $v(0) = 0$. By symmetry, $v_j(t) = v_1(t) =: w(t)$ for each j so Eq. (4.7) implies

$$\dot{w}(t) = \frac{Cr}{r - dw(t)} = \frac{C(r/d)}{(r/d) - w(t)} \text{ with } w(0) = 0.$$

We have already solved this equation (see Eq. (4.5) with r replaced by r/d) to find

$$(4.8) \quad w(t) = r/d - \sqrt{r^2/d^2 - 2Crt/d} = r/d \left(1 - \sqrt{1 - 2Cdt/r}\right).$$

Thus $v(t) = w(t)(1, 1, \dots, 1)$ is a real analytic function which is convergent for $|t| < r/(2Cd)$.

Now suppose that u is a real analytic solution to Eq. (4.6). Then by repeatedly differentiating Eq. (4.6) we learn

$$(4.9) \quad \begin{aligned} \ddot{u}_j(t) &= \partial_i f_j(u(t)) \dot{u}_i(t) = \partial_i f_j(u(t)) f_i(u(t)) \\ u_j^{(3)}(t) &= \partial_k \partial_i f_j(u(t)) \dot{u}_k(t) \dot{u}_i(t) + \partial_i f_j(u(t)) \ddot{u}_i(t) \\ &\vdots \\ u_j^{(n)}(t) &= p_n \left(\{ \partial^\alpha f_j(u(t)) \}_{|\alpha| < n}, \{ u_i^{(k)}(t) \}_{k < n, 1 \leq i \leq d} \right) \end{aligned}$$

where p_n is a polynomial with all non-negative integer coefficients. We now define $u_j^{(n)}(0)$ inductively so that

$$u_j^{(n)}(0) = p_n \left(\{ \partial^\alpha f_j(u(0)) \}_{|\alpha| < n}, \{ u_i^{(k)}(0) \}_{k < n, 1 \leq i \leq d} \right)$$

for all n and j and we will attempt to define

$$(4.10) \quad u(t) = \sum_{n=0}^{\infty} \frac{1}{n!} u^{(n)}(0) t^n.$$

To see this sum is convergent we make use of the fact that the polynomials p_n are universal i.e. are independent of the function f_j and have non-negative coefficients so that by induction

$$\begin{aligned} |u_j^{(n)}(0)| &\leq p_n \left(\{ |\partial^\alpha f_j(u(0))| \}_{|\alpha| < n}, \{ |u_i^{(k)}(0)| \}_{k < n, 1 \leq i \leq d} \right) \\ &\leq p_n \left(\{ \partial^\alpha g_j(u(0)) \}_{|\alpha| < n}, \{ v_i^{(k)}(0) \}_{k < n, 1 \leq i \leq d} \right) = v_j^{(n)}(0). \end{aligned}$$

Notice the when $n = 0$ that $|u_j(0)| = 0 = v_j(0)$.¹ Thus we have shown $u \ll v$ and so by comparison the sum in Eq. (4.10) is convergent for t near 0. As before $u(t)$ solves Eq. (4.6) since both functions $\dot{u}(t)$ and $f(u(t))$ are analytic functions of t which have common values for all derivatives in t at $t = 0$. ■

4.1. PDE Cauchy Kovalevskaya Theorem. In this section we will consider the following general quasi-linear system of partial differential equations

$$(4.11) \quad \sum_{|\alpha|=k} a_\alpha(x, J^{k-1}u) \partial_x^\alpha u(x) + c(x, J^{k-1}u) = 0$$

where

$$J^l u(x) = (u(x), Du(x), D^2u(x), \dots, D^l u(x))$$

is the “ l – jet” of u . Here $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $a_\alpha(J^{k-1}u, x)$ is an $m \times m$ matrix. As usual we will want to give boundary data on some hypersurface $\Sigma \subset \mathbb{R}^n$. Let ν denote a smooth vector field along Σ such that $\nu(x) \notin T_x \Sigma$ ($T_x \Sigma$ is the tangent space to Σ at x) for $x \in \Sigma$. For example we might take $\nu(x)$ to be orthogonal to $T_x \Sigma$ for all $x \in \Sigma$. To hope to get a unique solution to Eq. (4.11) we will further assume there are smooth functions g_l on Σ for $l = 0, \dots, k-1$ and we will require

$$(4.12) \quad D^l u(x)(\nu(x), \dots, \nu(x)) = g_l(x) \text{ for } x \in \Sigma \text{ and } l = 0, \dots, k-1.$$

Proposition 4.4. *Given a smooth function u on a neighborhood of Σ satisfying Eq. (4.12), we may calculate $D^l u(x)$ for $x \in \Sigma$ and $l < k$ in terms of the functions g_l and their tangential derivatives.*

Proof. Let us begin by choosing a coordinate system y on \mathbb{R}^n such that $\Sigma \cap D(y) = \{y_n = 0\}$ and let us extend ν to a neighborhood of Σ by requiring $\frac{\partial \nu}{\partial y_n} = 0$. To complete the proof, we are going to show by induction on k that we may compute

$$\left(\frac{\partial}{\partial y} \right)^\alpha u(x) \text{ for all } x \in \Sigma \text{ and } |\alpha| < k$$

from Eq. (4.12).

The claim is clear when $k = 1$, since $u = g_0$ on Σ . Now suppose that $k = 2$ and let $\nu_i = \nu_i(y_1, \dots, y_{n-1})$ such that

$$\nu = \sum_{i=1}^n \nu_i \frac{\partial}{\partial y_i} \text{ in a neighborhood of } \Sigma.$$

Then

$$g_1 = (Du)\nu = \nu u = \sum_{i=1}^n \nu_i \frac{\partial u}{\partial y_i} = \sum_{i < n} \nu_i \frac{\partial g_0}{\partial y_i} + \nu_n \frac{\partial u}{\partial y_n}.$$

Since ν is not tangential to $\Sigma = \{y_n = 0\}$, it follows that $\nu_n \neq 0$ and hence

$$(4.13) \quad \frac{\partial u}{\partial y_n} = \frac{1}{\nu_n} \left(g_1 - \sum_{i < n} \nu_i \frac{\partial g_0}{\partial y_i} \right) \text{ on } \Sigma.$$

¹The argument shows that $v_j^{(n)}(0) \geq 0$ for all n . This is also easily seen directly by induction using Eq. (4.9) with f replaced by g and the fact that $\partial^\alpha g_j(0) \geq 0$ for all α .

For $k = 3$, first observe from the equality $u = g_0$ on Σ and Eq. (4.13) we may compute all derivatives of u of the form $\frac{\partial^\alpha u}{\partial y^\alpha}$ on Σ provided $\alpha_n \leq 1$. From Eq. (4.12) for $l = 2$, we have

$$g_2 = (D^2 u)(v, v) = v^2 u + \text{l.o.ts.} = \sum \nu_j \frac{\partial}{\partial y_j} \left(\nu_i \frac{\partial u}{\partial y_i} \right) + \text{l.o.ts.} = \nu_n^2 \frac{\partial^2 u}{\partial y_n^2} + \text{l.o.ts.}$$

where l.o.ts. denotes terms involving $\frac{\partial^\alpha u}{\partial y^\alpha}$ with $\alpha_n \leq 1$. From this result, it follows that we may compute $\frac{\partial^2 u}{\partial y_n^2}$ in terms of derivatives of g_0 , g_1 and g_2 . The reader is asked to finish the full inductive argument of the proof. ■

Remark 4.5. The above argument shows that from Eq. (4.12) we may compute $\frac{\partial^\alpha u}{\partial y^\alpha}$ for any α such that $\alpha_n < k$.

To study Eq. (4.11) in more detail, let us rewrite Eq. (4.11) in the y -coordinates. Using the product and the chain rule repeatedly Eq. (4.11) may be written as

$$(4.14) \quad \sum_{|\alpha|=k} b_\alpha(y, J^{k-1}u) \partial_y^\alpha u(y) + c(y, J^{k-1}u) = 0$$

where

$$J^l u(y) = (u(y), Du(y), D^2 u(y), \dots, D^l u(y)).$$

We will be especially concerned with the $b_{(0,0,\dots,0,k)}$ coefficient which can be determined as follows:

$$\begin{aligned} \sum_{|\alpha|=k} a_\alpha \left(\frac{\partial}{\partial x} \right)^\alpha &= \sum_{|\alpha|=k} a_\alpha \left(\sum_{j=1}^n \frac{\partial y_j}{\partial x} \frac{\partial}{\partial y_j} \right)^\alpha = \sum_{|\alpha|=k} a_\alpha \left(\frac{\partial y_n}{\partial x} \frac{\partial}{\partial y_n} \right)^\alpha + \text{l.o.ts.} \\ &= \sum_{|\alpha|=k} a_\alpha \left(\frac{\partial y_n}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial y_n} \right)^k + \text{l.o.ts.} \end{aligned}$$

where l.o.ts. now denotes terms involving $\frac{\partial^\alpha u}{\partial y^\alpha}$ with $\alpha_n < k$. From this equation we learn that

$$b_{(0,0,\dots,0,k)}(y, J^{k-1}u) = \sum_{|\alpha|=k} a_\alpha \left(\frac{\partial y_n}{\partial x} \right)^\alpha = \sum_{|\alpha|=k} a_\alpha \left(dy_n \left(\frac{\partial}{\partial x} \right) \right)^\alpha.$$

Definition 4.6. We will say that boundary data $(\Sigma, g_0, \dots, g_{k-1})$ is non-characteristic for Eq. (4.11) at $x \in \Sigma$ if

$$b_{(0,0,\dots,0,k)}(y, J^{k-1}u) = \sum_{|\alpha|=k} a_\alpha(x, J^{k-1}u(x)) \left(dy_n \left(\frac{\partial}{\partial x} \right) \right)^\alpha$$

is invertible at x .

Notice that this condition is independent of the choice of coordinate system y . To see this, for $\xi \in (\mathbb{R}^n)^*$ let

$$\sigma(\xi) = \sum_{|\alpha|=k} a_\alpha(x, J^{k-1}u(x)) \left(\xi \left(\frac{\partial}{\partial x} \right) \right)^\alpha$$

which is k - linear form on $(\mathbb{R}^n)^*$. This form is coordinate independent since if f is a smooth function such that $f(x) = 0$ and $df_x = \xi$, then

$$\sigma(\xi) = \frac{1}{k!} \sum_{|\alpha|=k} a_\alpha(x, J^{k-1}u(x)) \left(\frac{\partial}{\partial x} \right)^\alpha f^k|_x.$$

Noting that

$$b_{(0,0,\dots,0,k)}(y, J^{k-1}u) = \sigma(dy_n)$$

our non-characteristic condition becomes, $\sigma(dy_n)$ is invertible. Finally dy_n is the unique element ξ of $(\mathbb{R}^n)^* \setminus \{0\}$ up to scaling such that $\xi|_{T_x\Sigma} \equiv 0$. So the non-characteristic condition may be written invariantly as $\sigma(\xi)$ is invertible for all (or any) $\xi \in (\mathbb{R}^n)^* \setminus \{0\}$ such that $\xi|_{T_x\Sigma} \equiv 0$.

Assuming the given boundary data is non-characteristic, Eq. (4.11) may be put into "standard form,"

$$(4.15) \quad \sum_{|\alpha|=k} b_\alpha(y, J^{k-1}u) \partial_y^\alpha u(y) + c(y, J^{k-1}u) = 0$$

with

$$\frac{\partial^l u}{\partial y_n^l} = g_l \text{ on } y_n = 0 \text{ for } l < k$$

where $b_{(0,0,\dots,0,k)}(y, J^{k-1}u) = Id$ - matrix and

$$J^l u(y) = (u(y), Du(y), D^2u(y), \dots, D^l u(y)).$$

By adding new dependent variables and possible a new independent variable for y_n one may reduce the problem to solving the system in Eq. (4.20) below. The resulting theorem may be stated as follows.

Theorem 4.7 (Cauchy Kovalevskaya). *Suppose all the coefficients in Eq. (4.11) are real analytic and the boundary data in Eq. (4.12) are also real analytic and non-characteristic near some point $a \in \Sigma$. Then there is a unique real analytic solution to Eqs. (4.11) and (4.12). (The boundary data in Eq. (4.12) is said to be real analytic if there exists coordinates y as above which are real analytic and the functions ν and g_l for $l = 0, \dots, k-1$ are real analytic functions in the y - coordinate system.)*

Example 4.8. Suppose a, b, C, r are positive constants. We wish to show the solution to the quasi-linear PDE

$$(4.16) \quad w_t = \frac{Cr}{r-y-aw} [bw_y + 1] \text{ with } w(0, y) = 0$$

is real analytic near $(t, y) = (0, 0)$. To do this we will solve the equation using the method of characteristics. Let $g(y, z) := \frac{Cr}{r-y-az}$, then the characteristic equations are

$$\begin{aligned} t' &= 0 \text{ with } t(0) = 0 \\ y' &= -bg(y, z) \text{ with } y(0) = y_0 \text{ and} \\ z' &= g(y, z) \text{ with } z(0) = 0. \end{aligned}$$

From these equations we see that we may identify t with s and that $y + bz = y_0$. Thus $z(t) = w(t, y(t))$ satisfies

$$\begin{aligned}\dot{z} &= g(y_0 - bz, z) = \frac{Cr}{r - y_0 + bz - az} \\ &= \frac{Cr}{r - y_0 + (b - a)z} \text{ with } z(0) = 0.\end{aligned}$$

Integrating this equation gives

$$\begin{aligned}Crt &= \int_0^t (r - y_0 + (b - a)z(\tau)) \dot{z}(\tau) d\tau = (r - y_0)z - \frac{1}{2}(a - b)z^2 \\ &= (r - y - bz)z - \frac{1}{2}(a - b)z^2 = (r - y)z - \frac{1}{2}(a + b)z^2,\end{aligned}$$

i.e.

$$\frac{1}{2}(a + b)z^2 - (r - y)z + Crt = 0.$$

The quadratic formula gives

$$w(t, y) = \frac{1}{a + b} \left[(r - y) \pm \sqrt{(r - y)^2 - 2(a + b)Crt} \right]$$

and using $w(0, y) = 0$ we conclude

$$(4.17) \quad w(t, y) = \frac{1}{a + b} \left[(r - y) - \sqrt{(r - y)^2 - 2(a + b)Crt} \right].$$

Notice the w is real analytic for (t, y) near $(0, 0)$.

In general we could use the method of characteristics and ODE properties (as in Example 4.8) to show

$$u_t = a(x, u)u_x + b(x, u) \text{ with } u(0, x) = g(x)$$

has local real analytic solutions if a, b and g are real analytic. The method would also work for the fully non-linear case as well. However, the method of characteristics fails for systems while the method we will present here works in this generality.

Exercise 4.1. Verify w in Eq. (4.17) solves Eq. (4.16).

Solution 1 (4.1). Let $\rho := \sqrt{(r - y)^2 - 2(a + b)Crt}$, then

$$\begin{aligned}w(t, y) &= \frac{1}{a + b} [r - y - \rho] = \frac{r - y}{a + b} - \frac{1}{a + b}\rho, \\ w_t &= Cr/\rho, \quad \rho = r - y - (a + b)w \text{ and}\end{aligned}$$

$$bw_y + 1 = \frac{b}{a + b} [-1 + (r - y)/\rho] + 1 = \frac{1}{a + b} [a + b(r - y)/\rho].$$

Hence

$$\begin{aligned}\frac{bw_y + 1}{w_t} &= \frac{1}{(a + b)Cr} [\rho a + b(r - y)] \\ &= \frac{1}{(a + b)Cr} [(r - y - (a + b)w)a + b(r - y)] \\ &= \frac{1}{Cr} [r - y - aw]\end{aligned}$$

as desired.

Example 4.9. Now let us solve for

$$v(t, x) = (v^1, \dots, v^m)(t, x_1, \dots, x_n)$$

where v satisfies

$$v_t^j = \frac{Cr}{r - x_1 - \dots - x_n - \sum_{k=1}^m v^k} \left[1 + \sum_{i=1}^n \sum_{k=1}^m \partial_i v^k \right] \text{ with } v(0, x) = 0.$$

By symmetry, $v^j = v^1 =: w(t, y)$ for all j where $y = x_1 + \dots + x_n$. Since $\partial_i v^j = w_y$, the above equations all may be written as

$$w_t = \frac{Cr}{r - y - mw} [mnw_y + 1] \text{ with } w(0, y) = 0.$$

Therefore from Example 4.8 with $a = m$ and $b = mn$, we find

$$(4.18) \quad w(t, y) = \frac{1}{m(n+1)} \left[(r-y) - \sqrt{(r-y)^2 - 2m(n+1)Crt} \right].$$

and hence that

$$(4.19) \quad v(t, x) = w(t, x_1 + \dots + x_n) (1, 1, 1, \dots, 1) \in \mathbb{R}^m.$$

4.2. Proof of Theorem 4.7. As is outlined in Evans, Theorem 4.7 may be reduced to the following theorem.

Theorem 4.10. *Let $(t, x, z) = (t, x_1, \dots, x_n, z_1, \dots, z_m) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ and assume $(t, x, z) \rightarrow B_j(t, x, z) \in \{m \times m \text{ - matrices}\}$ (for $j = 1, \dots, n$) and $(t, x, z) \rightarrow c(t, x, z) \in \mathbb{R}^m$ are real analytic functions near $(0, 0, 0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ and $x \rightarrow f(x) \in \mathbb{R}^m$ is real analytic near $0 \in \mathbb{R}^n$. Then there exists, in a neighborhood of $(t, x) = (0, 0) \in \mathbb{R} \times \mathbb{R}^n$, a unique real analytic solution $u(t, x) \in \mathbb{R}^m$ to the quasi-linear system*

$$(4.20) \quad u_t(t, x) = \sum_{j=1}^n B_j(t, x, u(t, x)) \partial_j u(t, x) + c(t, x, u(t, x)) \text{ with } u(0, x) = f(x).$$

Proof. (Sketch.)

Step 0. By replacing $u(t, x)$ by $u(t, x) - f(x)$, we may assume $f \equiv 0$. By letting $u^{m+1}(t, x) = t$ if necessary, we may assume B_j and c do not depend on t . With these reductions we are left to solve

$$(4.21) \quad u_t(t, x) = \sum_{j=1}^n B_j(x, u(t, x)) \partial_j u(t, x) + c(x, u(t, x)) \text{ with } u(0, x) = 0.$$

Step 1. Let

$$g(x, z) := \frac{Cr}{r - x_1 - \dots - x_n - z_1 - \dots - z_m}$$

where C and r are positive constants such that

$$(B_j)_{kl} \ll g \text{ and } c_k \ll g$$

for all k, l, j . For this choice of C and r , let v denote the solution constructed in Example 4.9 above.

Step 2. By repeatedly differentiating Eq. (4.20), show that if u solves Eq. (4.20) then $\partial_x^\alpha \partial_t^k u^j(0, 0)$ is a **universal** polynomial in the derivatives $\{\partial_t^l \partial_x^\alpha\}_{\alpha, l < k}$.

of the entries of B_j and c and u with all coefficients being non-negative. Use this fact and induction to conclude

$$|\partial_x^\alpha \partial_t^k u^j(0,0)| \leq \partial_x^\alpha \partial_t^k v^j(0,0) \text{ for all } \alpha, k \text{ and } l.$$

Step 3. Use the computation in Step 2. to define $\partial_x^\alpha \partial_t^k u^j(0,0)$ for all α and k and then defined

$$(4.22) \quad u(t,x) := \sum_{\alpha,k} \frac{\partial_x^\alpha \partial_t^k u(0,0)}{\alpha!k!} t^k x^\alpha.$$

Because of step 2. and Example 4.9, this series is convergent for (t,x) sufficiently close to zero.

Step 4. The function u defined in Step 3. solves Eq. (4.20) because both

$$u_t(t,x) \text{ and } \sum_{j=1}^n B_j(x, u(t,x)) \partial_j u(t,x) + c(x, u(t,x))$$

are both real analytic functions in (t,x) each having, by construction, the same derivatives at $(0,0)$. ■

4.3. Examples.

Corollary 4.11 (Isothermal Coordinates). *Suppose that we are given a metric $ds^2 = E dx^2 + 2F dx dy + G dy^2$ on \mathbb{R}^2 such that G/E and F/E are real analytic near $(0,0)$. Then there exists a complex function u and a positive function ρ such that $Du(0,0)$ is invertible and $ds^2 = \rho |du|^2$ where $du = u_x dx + u_y dy$.*

Proof. Working out $|du|^2$ gives

$$|du|^2 = |u_x|^2 dx^2 + 2 \operatorname{Re}(u_x \bar{u}_y) dx dy + |u_y|^2 dy^2.$$

Writing $u_y = \lambda u_x$, the previous equation becomes

$$|du|^2 = |u_x|^2 \left(dx^2 + 2 \operatorname{Re}(\lambda) dx dy + |\lambda|^2 dy^2 \right).$$

Hence we must have

$$E = \rho |u_x|^2, \quad F = \rho |u_x|^2 \operatorname{Re} \lambda \text{ and } G = \rho |u_x|^2 |\lambda|^2$$

or equivalently

$$F/E = \operatorname{Re} \lambda \text{ and } G/E = |\lambda|^2.$$

Writing $\lambda = a + ib$, we find $a = F/E$ and $a^2 + b^2 = G/E$ so that

$$\lambda = \frac{F}{E} \pm i \sqrt{G/E - (F/E)^2} = \frac{1}{E} \left(F \pm i \sqrt{GE - F^2} \right).$$

We make a choice of the sign above, then we are looking for $u(x,y) \in \mathbb{C}$ such that $u_y = \lambda u_x$. Letting $u = \alpha + i\beta$, the equation $u_y = \lambda u_x$ may be written as the system of real equations

$$\begin{aligned} \alpha_y &= \operatorname{Re} [(a + ib)(\alpha_x + i\beta_x)] = a\alpha_x - b\beta_x \text{ and} \\ \beta_y &= \operatorname{Im} [(a + ib)(\alpha_x + i\beta_x)] = a\beta_x + b\alpha_x \end{aligned}$$

which is equivalent to

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_y = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_x.$$

So we may apply the Cauchy Kovalevskaya theorem 4.10 with $t = y$ to find a real analytic solution to this equation with (say) $u(x, 0) = x$, i.e. $\alpha(x, 0) = x$ and $\beta(x, 0) = 0$. (We could take $u(x, 0) = f(x)$ for any real analytic function f such that $f'(0) \neq 0$.) The only thing that remains to check is that $Du(0, 0)$ is invertible. But

$$Du(0, 0) = \begin{pmatrix} \operatorname{Re} u_x & \operatorname{Re} u_y \\ \operatorname{Im} u_x & \operatorname{Im} u_y \end{pmatrix} = \begin{pmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{pmatrix} = \begin{pmatrix} \alpha_x & a\alpha_x - b\beta_x \\ \beta_x & a\beta_x + b\alpha_x \end{pmatrix}$$

so that

$$\det [Du] = b(\alpha_x^2 + \beta_x^2) = \operatorname{Im} \lambda |u_x|^2.$$

Thus

$$\det [Du(0, 0)] = \operatorname{Im} \lambda(0, 0) = \pm \sqrt{G/E - (F/E)^2}|_{(0,0)} \neq 0.$$

■

Example 4.12. Consider the linear PDE,

$$(4.23) \quad u_y = u_x \text{ with } u(x, 0) = f(x)$$

where $f(x) = \sum_{m=0}^{\infty} a_m x^m$ as real analytic function near $x = 0$ with radius of convergence ρ . (So for any $r < \rho$, $|a_m| \leq Cr^{-n}$.) Formally the solution to Eq. (4.23) should be given by

$$u(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_y^n u(x, y)|_{y=0} y^n.$$

Now using the PDE (4.23),

$$\partial_y^n u(x, y)|_{y=0} = \partial_x^n u(x, 0) = f^{(n)}(x).$$

Thus we get

$$(4.24) \quad u(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) y^n.$$

By the Cauchy estimates,

$$|f^{(n)}(x)| \leq \frac{n! \rho}{(\rho - |x|)^{n+1}}$$

and so

$$\sum_{n=0}^{\infty} \frac{1}{n!} |f^{(n)}(x) y^n| \leq \rho \sum_{n=0}^{\infty} \frac{|y|^n}{(\rho - |x|)^{n+1}}$$

which is finite provided $|y| < \rho - |x|$, i.e. $|x| + |y| < \rho$. This of course makes sense because we know the solution to Eq. (4.23) is given by

$$u(x, y) = f(x + y).$$

Now we can expand Eq. (4.24) out to find

$$(4.25) \quad \begin{aligned} u(x, y) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m \geq n} m(m-1) \dots (m-n+1) a_m x^{m-n} \right) y^n \\ &= \sum_{m \geq n \geq 0} \binom{m}{n} a_m x^{m-n} y^n. \end{aligned}$$

Since

$$\sum_{m \geq n \geq 0} \binom{m}{n} |a_m x^{m-n} y^n| \leq C \sum_{m \geq n \geq 0} \binom{m}{n} |r^{-m} x^{m-n} y^n| = C \sum_{m \geq 0} r^{-m} (|x| + |y|)^m < \infty$$

provided $|x| + |y| < r$. Since $r < \rho$ was arbitrary, it follows that Eq. (4.25) is convergent for $|x| + |y| < \rho$.

Let us redo this example. By the PDE in Eq. (4.23), $\partial_y^m \partial_x^n u(x, y) = \partial_x^{n+m} u(x, y)$ and hence

$$\partial_y^m \partial_x^n u(0, 0) = f^{(m+n)}(0).$$

Written another way

$$D^\alpha u(0, 0) = f^{(|\alpha|)}(0)$$

and so the power series expansion for u must be given by

$$(4.26) \quad u(x, y) = \sum_{\alpha} \frac{f^{(|\alpha|)}(0)}{\alpha!} (x, y)^\alpha.$$

Using $f^{(m)}(0)/m! \leq Cr^{-m}$ we learn

$$\begin{aligned} \sum_{\alpha} \left| \frac{f^{(|\alpha|)}(0)}{\alpha!} (x, y)^\alpha \right| &\leq C \sum_{\alpha} \frac{|f^{(|\alpha|)}(0)|}{\alpha!} |x|^{\alpha_1} |y|^{\alpha_2} = C \sum_{m=0}^{\infty} \frac{|f^{(m)}(0)|}{m!} \sum_{|\alpha|=m} \frac{m!}{\alpha!} |x|^{\alpha_1} |y|^{\alpha_2} \\ &\leq C \sum_{m=0}^{\infty} r^{-m} (|x| + |y|)^m = C \frac{r}{r - (|x| + |y|)} < \infty \end{aligned}$$

if $|x| + |y| < r$. Since $r < \rho$ was arbitrary, it follows that the series in Eq. (4.26) converges for $|x| + |y| < \rho$.

Now it is easy to check directly that Eq. (4.26) solves the PDE. However this is necessary since by construction $D^\alpha u_y(0, 0) = D^\alpha u_x(0, 0)$ for all α . This implies, because u_y and u_x are both real analytic, that $u_x = u_y$.