5. A very short introduction to generalized functions

Let U be an open subset of $\mathbb{R}^n$ and

$$C_c^\infty(U) = \cup_{K \sqsubset\sqsubset U} C^\infty(K) \tag{5.1}$$

denote the set of smooth functions on U with compact support in U.

Definition 5.1. A sequence $\{\phi_k\}_{k=1}^\infty \subset \mathcal{D}(U)$ converges to $\phi \in \mathcal{D}(U)$, iff there is a compact set $K \sqsubset\sqsubset U$ such that $\text{supp}(\phi_k) \subset K$ for all k and $\phi_k \to \phi$ in $C^\infty(K)$.

Definition 5.2 (Distributions on $U \subset_o \mathbb{R}^n$)**.** A generalized function T on $U \subset_o \mathbb{R}^n$ is a continuous linear functional on $\mathcal{D}(U)$, i.e. $T : \mathcal{D}(U) \to \mathbb{C}$ is linear and $\lim_{n\to\infty}\langle T, \phi_k\rangle = 0$ for all $\{\phi_k\} \subset \mathcal{D}(U)$ such that $\phi_k \to 0$ in $\mathcal{D}(U)$. Here we have written $\langle T, \phi\rangle$ for $T(\phi)$. We denote the space of generalized functions by $\mathcal{D}'(U)$.

Example 5.3. Here are a couple of examples of distributions.

1. For $f \in L^1_{loc}(U)$ define $T_f \in \mathcal{D}'(U)$ by $\langle T_f, \phi\rangle = \int_U \phi f dm$ for all $\phi \in \mathcal{D}(U)$. This is called the distribution associated to f.
2. More generally let μ be a complex measure on U, then $\langle \mu, \phi\rangle := \int_U \phi d\mu$ is a distribution. For example if $x \in U$, and $\mu = \delta_x$ then $\langle \delta_x, \phi\rangle = \phi(x)$ for all $\phi \in \mathcal{D}$.

Lemma 5.4. *Let $a_\alpha \in C^\infty(U)$ and $L = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$ – a m^{th} order linear differential operator on $\mathcal{D}(U)$. Then for $f \in C^m(U)$ and $\phi \in \mathcal{D}(U)$,*

$$\langle Lf, \phi\rangle := \langle T_{Lf}, \phi\rangle = \langle T, L^*\phi\rangle$$

*where L^* is the **formal adjoint** of L defined by*

$$L^*\phi = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\, \partial^\alpha \left[a_\alpha \phi\right].$$

Proof. This is simply repeated integration by parts. No boundary terms arise since ϕ has compact support. ■

Definition 5.5 (Multiplication by smooth functions)**.** Suppose that $g \in C^\infty(U)$ and $T \in \mathcal{D}'(U)$ then we define $gT \in \mathcal{D}'(U)$ by

$$\langle gT, \phi\rangle = \langle T, g\phi\rangle \text{ for all } \phi \in \mathcal{D}(U).$$

It is easily checked that gT is continuous.

Definition 5.6 (Differentiation)**.** For $T \in \mathcal{D}'(U)$ and $i \in \{1, 2, \ldots, n\}$ let $\partial_i T \in \mathcal{D}'(U)$ be the distribution defined by

$$\langle \partial_i T, \phi\rangle = -\langle T, \partial_i \phi\rangle \text{ for all } \phi \in \mathcal{D}(U).$$

Again it is easy to check that $\partial_i T$ is a distribution.

Definition 5.7. More generally if L is as in Lemma 5.4 and $T \in \mathcal{D}'$ we define $LT \in \mathcal{D}'$ by

$$\langle LT, \phi\rangle = \langle T, L^*\phi\rangle.$$

Example 5.8. Suppose that $f \in L^1_{loc}$ and $g \in C^\infty(U)$, then $gT_f = T_{gf}$. If further $f \in C^1(U)$, then $\partial_i T_f = T_{\partial_i f}$. More generally if $f \in C^m(U)$ then, by Lemma 5.4, $LT_f = T_{Lf}$.

Because of Definition 5.7 we may now talk about distributional or generalized solutions T to PDEs of the form $LT = S$ where $S \in \mathcal{D}'$.

Example 5.9. For the moment let us also assume that $U = \mathbb{R}$. $\langle T_f, \phi\rangle = \int_U \phi f dm$. Then we have

(1) $\lim_{M\to\infty} T_{\sin Mx} = 0$
(2) $\lim_{M\to\infty} T_{M^{-1}\sin Mx} = \pi\delta_0$ where δ_0 is the point measure at 0.
(3) If $f \in L^1(\mathbb{R}^n, dm)$ with $\int_{\mathbb{R}^n} f dm = 1$ and $f_\epsilon(x) = \epsilon^{-n} f(x/\epsilon)$, then $\lim_{\epsilon\downarrow 0} T_{f_\epsilon} = \delta_0$. Indeed,

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\langle T_{f_\epsilon}, \phi\rangle &= \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n}\epsilon^{-n} f(x/\epsilon)\phi(x)dx \\ &= \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n} f(x)\phi(\epsilon x)dx \overset{\text{D.C.T.}}{=} \int_{\mathbb{R}^n} f(x)\lim_{\epsilon\downarrow 0}\phi(\epsilon x)dx \\ &= \phi(0)\int_{\mathbb{R}^n} f(x)dx = \phi(0) = \langle\delta_0, \phi\rangle.\end{aligned}$$

As a concrete example we have

$$\lim_{\epsilon\downarrow 0}\frac{\epsilon}{\pi(x^2+\epsilon^2)} = \delta_0 \text{ on } \mathbb{R},$$

i.e.

$$\lim_{\epsilon\downarrow 0} T_{\frac{\epsilon}{\pi(x^2+\epsilon^2)}} = \delta_0.$$

Example 5.10. Suppose that $a \in U$, then

$$\langle\partial_i\delta_a, \phi\rangle = -\partial_i\phi(a)$$

and more generally we have

$$\langle L\delta_a, \phi\rangle = (L^*\phi)(a).$$

Lemma 5.11. *Suppose $f \in C^1([a,b])$ and $g \in PC^1([a,b])$, i.e. $g \in C^1([a,b]\setminus\Lambda)$ where Λ is a finite subset of (a,b) and $g(\alpha+)$, $g(\alpha-)$ exists for $\alpha \in \Lambda$. Then*

(5.2)
$$\int_a^b f'(x)g(x)dx = [f'(x)g(x)]\,|_a^b - \int_a^b f(x)g'(x)dx - \sum_{\alpha\in\Lambda} f(\alpha)\left(g(\alpha+) - g(\alpha-)\right).$$

In particular

$$\frac{d}{dx}T_g = T_{g'} + \sum_{\alpha\in\Lambda}\left(g(\alpha+) - g(\alpha-)\right)\delta_\alpha$$

Proof. Write $\Lambda \cup \{a,b\}$ as $\{a = \alpha_0 < \alpha_1 < \cdots < \alpha_n = b\}$, then

$$\begin{aligned}\int_a^b f'(x)g(x)dx &= \sum_{k=0}^{n-1}\int_{\alpha_k}^{\alpha_{k+1}} f'(x)g(x)dx = \sum_{k=0}^{n-1}\left[[f(x)g(x)]\,|_{\alpha_k+}^{\alpha_{k+1}-} - \int_{\alpha_k}^{\alpha_{k+1}} f(x)g'(x)dx\right] \\ &= [f'(x)g(x)]\,|_a^b - \int_a^b f(x)g'(x)dx - \sum_{k=1}^{n-1}[f(x)g(x)]\,|_{\alpha_k-}^{\alpha_k+}\end{aligned}$$

which is the same as Eq. (5.2). ■