19. Weak and Strong Derivatives

For this section, let Ω be an open subset of $\mathbb{R}^d$, $p,q,r \in [1,\infty]$, $L^p(\Omega) = L^p(\Omega, \mathcal{B}_\Omega, m)$ and $L^p_{loc}(\Omega) = L^p_{loc}(\Omega, \mathcal{B}_\Omega, m)$, where m is Lebesgue measure on $\mathcal{B}_{\mathbb{R}^d}$ and $\mathcal{B}_\Omega$ is the Borel $\sigma-$algebra on Ω. If $\Omega = \mathbb{R}^d$, we will simply write L^p and L^p_{loc} for $L^p(\mathbb{R}^d)$ and $L^p_{loc}(\mathbb{R}^d)$ respectively. Also let

$$\langle f, g\rangle := \int_\Omega f g dm$$

for any pair of measurable functions $f, g : \Omega \to \mathbb{C}$ such that $fg \in L^1(\Omega)$. For example, by Hölder's inequality, if $\langle f, g\rangle$ is defined for $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ when $q = \frac{p}{p-1}$.

Definition 19.1. A sequence $\{u_n\}_{n=1}^\infty \subset L^p_{loc}(\Omega)$ is said to converge to $u \in L^p_{loc}(\Omega)$ if $\lim_{n\to\infty} \|u - u_n\|_{L^q(K)} = 0$ for all compact subsets $K \subset \Omega$.

The following simple but useful remark will be used (typically without further comment) in the sequel.

Remark 19.2. Suppose $r, p, q \in [1,\infty]$ are such that $r^{-1} = p^{-1} + q^{-1}$ and $f_t \to f$ in $L^p(\Omega)$ and $g_t \to g$ in $L^q(\Omega)$ as $t \to 0$, then $f_t g_t \to fg$ in $L^r(\Omega)$. Indeed,

$$\begin{aligned}\|f_t g_t - fg\|_r &= \|(f_t - f)\, g_t + f\,(g_t - g)\|_r \\ &\le \|f_t - f\|_p \|g_t\|_q + \|f\|_p \|g_t - g\|_q \to 0 \text{ as } t \to 0\end{aligned}$$

19.1. **Basic Definitions and Properties.**

Definition 19.3 (Weak Differentiability)**.** Let $v \in \mathbb{R}^d$ and $u \in L^p(\Omega)$ ($u \in L^p_{loc}(\Omega)$) then $\partial_v u$ is said to **exist weakly** in $L^p(\Omega)$ ($L^p_{loc}(\Omega)$) if there exists a function $g \in L^p(\Omega)$ ($g \in L^p_{loc}(\Omega)$) such that

$$\langle u, \partial_v \phi\rangle = -\langle g, \phi\rangle \text{ for all } \phi \in C_c^\infty(\Omega). \tag{19.1}$$

The function g if it exists will be denoted by $\partial_v^{(w)} u$. Similarly if $\alpha \in \mathbb{N}_0^d$ and ∂^α is as in Notation 11.10, we say $\partial^\alpha u$ **exists weakly** in $L^p(\Omega)$ ($L^p_{loc}(\Omega)$) iff there exists $g \in L^p(\Omega)$ ($L^p_{loc}(\Omega)$) such that

$$\langle u, \partial^\alpha \phi\rangle = (-1)^{|\alpha|} \langle g, \phi\rangle \text{ for all } \phi \in C_c^\infty(\Omega).$$

More generally if $p(\xi) = \sum_{|\alpha|\le N} a_\alpha \xi^\alpha$ is a polynomial in $\xi \in \mathbb{R}^n$, then $p(\partial)u$ **exists weakly** in $L^p(\Omega)$ ($L^p_{loc}(\Omega)$) iff there exists $g \in L^p(\Omega)$ ($L^p_{loc}(\Omega)$) such that

$$\langle u, p(-\partial)\phi\rangle = \langle g, \phi\rangle \text{ for all } \phi \in C_c^\infty(\Omega) \tag{19.2}$$

and we denote g by w$-p(\partial)u$.

By Corollary 11.28, there is at most one $g \in L^1_{loc}(\Omega)$ such that Eq. (19.2) holds, so w$-p(\partial)u$ is well defined.

Lemma 19.4. *Let $p(\xi)$ be a polynomial on $\mathbb{R}^d$, $k = \deg(p) \in \mathbb{N}$, and $u \in L^1_{loc}(\Omega)$ such that $p(\partial)u$ exists weakly in $L^1_{loc}(\Omega)$. Then*

1. $\operatorname{supp}_m(\text{w}-p(\partial)u) \subset \operatorname{supp}_m(u)$, *where* $\operatorname{supp}_m(u)$ *is the essential support of u relative to Lebesgue measure, see Definition 11.14.*
2. *If $\deg p = k$ and $u|_U \in C^k(U, \mathbb{C})$ for some open set $U \subset \Omega$, then* w$-p(\partial)u = p(\partial)\, u$ *a.e. on U.*

Proof.

(1) Since
$$\langle \text{w}{-}p(\partial)u, \phi\rangle = -\langle u, p(-\partial)\phi\rangle = 0 \text{ for all } \phi \in C_c^\infty(\Omega \setminus \operatorname{supp}_m(u)),$$
an application of Corollary 11.28 shows $\text{w}{-}p(\partial)u = 0$ a.e. on $\Omega \setminus \operatorname{supp}_m(u)$. So by Lemma 11.15, $\Omega \setminus \operatorname{supp}_m(u) \subset \Omega \setminus \operatorname{supp}_m(\text{w}{-}p(\partial)u)$, i.e. $\operatorname{supp}_m(\text{w}{-}p(\partial)u) \subset \operatorname{supp}_m(u)$.

(2) Suppose that $u|_U$ is C^k and let $\psi \in C_c^\infty(U)$. (We view ψ as a function in $C_c^\infty(\mathbb{R}^d)$ by setting $\psi \equiv 0$ on $\mathbb{R}^d \setminus U$.) By Corollary 11.25, there exists $\gamma \in C_c^\infty(\Omega)$ such that $0 \le \gamma \le 1$ and $\gamma = 1$ in a neighborhood of $\operatorname{supp}(\psi)$. Then by setting $\gamma u = 0$ on $\mathbb{R}^d \setminus \operatorname{supp}(\gamma)$ we may view $\gamma u \in C_c^k(\mathbb{R}^d)$ and so by standard integration by parts (see Lemma 11.26) and the ordinary product rule,
$$\begin{aligned}\langle \text{w}{-}p(\partial)u, \psi\rangle &= \langle u, p(-\partial)\psi\rangle = -\langle \gamma u, p(-\partial)\psi\rangle \\ &= \langle p(\partial)(\gamma u), \psi\rangle = \langle p(\partial)u, \psi\rangle \end{aligned} \tag{19.3}$$
wherein the last equality we have γ is constant on $\operatorname{supp}(\psi)$. Since Eq. (19.3) is true for all $\psi \in C_c^\infty(U)$, an application of Corollary 11.28 with $h = \text{w}{-}p(\partial)u - p(\partial)u$ and $\mu = m$ shows $\text{w}{-}p(\partial)u = p(\partial)u$ a.e. on U.

■

Notation 19.5. In light of Lemma 19.4 there is no danger in simply writing $p(\partial)u$ for $\text{w}{-}p(\partial)u$. So in the sequel we will always interpret $p(\partial)u$ in the weak or "distributional" sense.

Example 19.6. Suppose $u(x) = |x|$ for $x \in \mathbb{R}$, then $\partial u(x) = \operatorname{sgn}(x)$ in $L^1_{loc}(\mathbb{R})$ while $\partial^2 u(x) = 2\delta(x)$ so $\partial^2 u(x)$ does not exist weakly in $L^1_{loc}(\mathbb{R})$.

Example 19.7. Suppose $d = 2$ and $u(x,y) = 1_{y>x}$. Then $u \in L^1_{loc}(\mathbb{R}^2)$, while $\partial_x 1_{y>x} = -\delta(y-x)$ and $\partial_y 1_{y>x} = \delta(y-x)$ and so that neither $\partial_x u$ or $\partial_y u$ exists weakly. On the other hand $(\partial_x + \partial_y)u = 0$ weakly. To prove these assertions, notice $u \in C^\infty(\mathbb{R}^2 \setminus \Delta)$ where $\Delta = \{(x,x) : x \in \mathbb{R}^2\}$. So by Lemma 19.4, for any polynomial $p(\xi)$ without constant term, if $p(\partial)u$ exists weakly then $p(\partial)u = 0$. However,
$$\begin{aligned}\langle u, -\partial_x\phi\rangle &= -\int_{y>x} \phi_x(x,y)dxdy = -\int_{\mathbb{R}} \phi(y,y)dy, \\ \langle u, -\partial_y\phi\rangle &= -\int_{y>x} \phi_y(x,y)dxdy = \int_{\mathbb{R}} \phi(x,x)dx \text{ and} \\ \langle u, -(\partial_x + \partial_y)\phi\rangle &= 0\end{aligned}$$
from which it follows that $\partial_x u$ and $\partial_y u$ can not be zero while $(\partial_x + \partial_y)u = 0$.

On the other hand if $p(\xi)$ and $q(\xi)$ are two polynomials and $u \in L^1_{loc}(\Omega)$ is a function such that $p(\partial)u$ exists weakly in $L^1_{loc}(\Omega)$ and $q(\partial)[p(\partial)u]$ exists weakly in $L^1_{loc}(\Omega)$ then $(qp)(\partial)u$ exists weakly in $L^1_{loc}(\Omega)$. This is because
$$\begin{aligned}\langle u, (qp)(-\partial)\phi\rangle &= \langle u, p(-\partial)q(-\partial)\phi\rangle \\ &= \langle p(\partial)u, q(-\partial)\phi\rangle = \langle q(\partial)p(\partial)u, \phi\rangle \text{ for all } \phi \in C_c^\infty(\Omega).\end{aligned}$$

Example 19.8. Let $u(x,y) = 1_{x>0} + 1_{y>0}$ in $L^1_{loc}(\mathbb{R}^2)$. Then $\partial_x u(x,y) = \delta(x)$ and $\partial_y u(x,y) = \delta(y)$ so $\partial_x u(x,y)$ and $\partial_y u(x,y)$ do **not** exist weakly in $L^1_{loc}(\mathbb{R}^2)$. However $\partial_y\partial_x u$ does exists weakly and is the zero function. This shows $\partial_y\partial_x u$ may exists weakly despite the fact both $\partial_x u$ and $\partial_y u$ do not exists weakly in $L^1_{loc}(\mathbb{R}^2)$.

Lemma 19.9. *Suppose $u \in L^1_{loc}(\Omega)$ and $p(\xi)$ is a polynomial of degree k such that $p(\partial)u$ exists weakly in $L^1_{loc}(\Omega)$ then*

$$\langle p(\partial)u, \phi\rangle = \langle u, p(-\partial)\phi\rangle \text{ for all } \phi \in C_c^k(\Omega). \tag{19.4}$$

Note: *The point here is that Eq. (19.4) holds for all $\phi \in C_c^k(\Omega)$ not just $\phi \in C_c^\infty(\Omega)$.*

Proof. Let $\phi \in C_c^k(\Omega)$ and choose $\eta \in C_c^\infty(B(0,1))$ such that $\int_{\mathbb{R}^d}\eta(x)dx = 1$ and let $\eta_\epsilon(x) := \epsilon^{-d}\eta(x/\epsilon)$. Then $\eta_\epsilon * \phi \in C_c^\infty(\Omega)$ for ϵ sufficiently small and $p(-\partial)[\eta_\epsilon * \phi] = \eta_\epsilon * p(-\partial)\phi \to p(-\partial)\phi$ and $\eta_\epsilon * \phi \to \phi$ uniformly on compact sets as $\epsilon \downarrow 0$. Therefore by the dominated convergence theorem,

$$\langle p(\partial)u, \phi\rangle = \lim_{\epsilon\downarrow 0}\langle p(\partial)u, \eta_\epsilon * \phi\rangle = \lim_{\epsilon\downarrow 0}\langle u, p(-\partial)(\eta_\epsilon * \phi)\rangle = \langle u, p(-\partial)\phi\rangle.$$

■

Lemma 19.10 (Product Rule)**.** *Let $u \in L^1_{loc}(\Omega)$, $v \in \mathbb{R}^d$ and $\phi \in C^1(\Omega)$. If $\partial_v^{(w)}u$ exists in $L^1_{loc}(\Omega)$, then $\partial_v^{(w)}(\phi u)$ exists in $L^1_{loc}(\Omega)$ and*

$$\partial_v^{(w)}(\phi u) = \partial_v\phi \cdot u + \phi\partial_v^{(w)}u \text{ a.e.}$$

Moreover if $\phi \in C_c^1(\Omega)$ and $F := \phi u \in L^1$ (here we define F on $\mathbb{R}^d$ by setting $F = 0$ on $\mathbb{R}^d \setminus \Omega$), then $\partial^{(w)}F = \partial_v\phi \cdot u + \phi\partial_v^{(w)}u$ exists weakly in $L^1(\mathbb{R}^d)$.

Proof. Let $\psi \in C_c^\infty(\Omega)$, then using Lemma 19.9,

$$\begin{aligned}
-\langle \phi u, \partial_v\psi\rangle &= -\langle u, \phi\partial_v\psi\rangle = -\langle u, \partial_v(\phi\psi) - \partial_v\phi \cdot \psi\rangle = \langle \partial_v^{(w)}u, \phi\psi\rangle + \langle \partial_v\phi \cdot u, \psi\rangle \\
&= \langle \phi\partial_v^{(w)}u, \psi\rangle + \langle \partial_v\phi \cdot u, \psi\rangle.
\end{aligned}$$

This proves the first assertion. To prove the second assertion let $\gamma \in C_c^\infty(\Omega)$ such that $0 \le \gamma \le 1$ and $\gamma = 1$ on a neighborhood of $\mathrm{supp}(\phi)$. So for $\psi \in C_c^\infty(\mathbb{R}^d)$, using $\partial_v\gamma = 0$ on $\mathrm{supp}(\phi)$ and $\gamma\psi \in C_c^\infty(\Omega)$, we find

$$\begin{aligned}
\langle F, \partial_v\psi\rangle &= \langle \gamma F, \partial_v\psi\rangle = \langle F, \gamma\partial_v\psi\rangle = \langle (\phi u), \partial_v(\gamma\psi) - \partial_v\gamma \cdot \psi\rangle \\
&= \langle (\phi u), \partial_v(\gamma\psi)\rangle = -\langle \partial_v^{(w)}(\phi u), (\gamma\psi)\rangle \\
&= -\langle \partial_v\phi \cdot u + \phi\partial_v^{(w)}u, \gamma\psi\rangle = -\langle \partial_v\phi \cdot u + \phi\partial_v^{(w)}u, \psi\rangle.
\end{aligned}$$

This show $\partial_v^{(w)}F = \partial_v\phi \cdot u + \phi\partial_v^{(w)}u$ as desired. ■

Lemma 19.11. *Suppose $q \in [1,\infty)$, $p(\xi)$ is a polynomial in $\xi \in \mathbb{R}^d$ and $u \in L^q_{loc}(\Omega)$. If there exists $\{u_m\}_{m=1}^\infty \subset L^q_{loc}(\Omega)$ such that $p(\partial)u_m$ exists in $L^q_{loc}(\Omega)$ for all m and there exists $g \in L^q_{loc}(\Omega)$ such that for all $\phi \in C_c^\infty(\Omega)$,*

$$\lim_{m\to\infty}\langle u_m, \phi\rangle = \langle u, \phi\rangle \text{ and } \lim_{m\to\infty}\langle p(\partial)u_m, \phi\rangle = \langle g, \phi\rangle$$

then $p(\partial)u$ exists in $L^q_{loc}(\Omega)$ and $p(\partial)u = g$.

Proof. Since

$$\langle u, p(\partial)\phi\rangle = \lim_{m\to\infty}\langle u_m, p(\partial)\phi\rangle = -\lim_{m\to\infty}\langle p(\partial)u_m, \phi\rangle = \langle g, \phi\rangle$$

for all $\phi \in C_c^\infty(\Omega)$, $p(\partial)u$ exists and is equal to $g \in L^q_{loc}(\Omega)$. ■

Conversely we have the following proposition.

Proposition 19.12 (Mollification)**.** *Suppose $q \in [1,\infty)$, $p_1(\xi),\dots,p_N(\xi)$ is a collection of polynomials in $\xi \in \mathbb{R}^d$ and $u \in L^q_{loc}(\Omega)$ such that $p_l(\partial)u$ exists weakly in $L^q_{loc}(\Omega)$ for $l = 1,2,\dots,N$. Then there exists $u_n \in C_c^\infty(\Omega)$ such that $u_n \to u$ in $L^q_{loc}(\Omega)$ and $p_l(\partial)u_n \to p_l(\partial)u$ in $L^q_{loc}(\Omega)$ for $l = 1,2,\dots,N$.*

Proof. Let $\eta \in C_c^\infty(B(0,1))$ such that $\int_{\mathbb{R}^d} \eta dm = 1$ and $\eta_\epsilon(x) := \epsilon^{-d}\eta(x/\epsilon)$ be as in the proof of Lemma 19.9. For any function $f \in L^1_{loc}(\Omega)$, $\epsilon > 0$ and $x \in \Omega_\epsilon := \{y \in \Omega : dist(y,\Omega^c) > \epsilon\}$, let

$$f_\epsilon(x) := f * \eta_\epsilon(x) := 1_\Omega f * \eta_\epsilon(x) = \int_\Omega f(y)\eta_\epsilon(x-y)dy.$$

Notice that $f_\epsilon \in C^\infty(\Omega_\epsilon)$ and $\Omega_\epsilon \uparrow \Omega$ as $\epsilon \downarrow 0$.

Given a compact set $K \subset \Omega$ let $K_\epsilon := \{x \in \Omega : \mathrm{dist}(x,K) \le \epsilon\}$. Then $K_\epsilon \downarrow K$ as $\epsilon \downarrow 0$, there exists $\epsilon_0 > 0$ such that $K_0 := K_{\epsilon_0}$ is a compact subset of $\Omega_0 := \Omega_{\epsilon_0} \subset \Omega$ (see Figure 38) and for $x \in K$,

$$f * \eta_\epsilon(x) := \int_\Omega f(y)\eta_\epsilon(x-y)dy = \int_{K_\epsilon} f(y)\eta_\epsilon(x-y)dy.$$

Therefore, using Theorem 11.21,

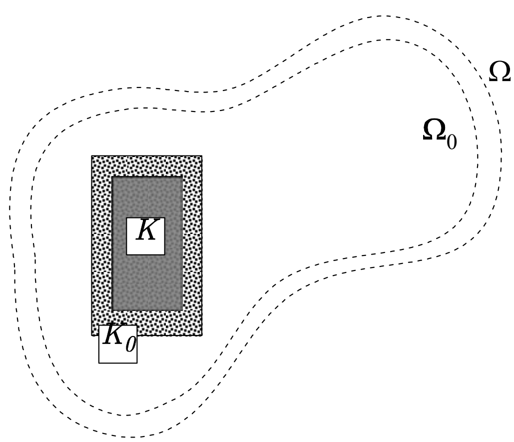

FIGURE 38. The geomentry of $K \subset K_0 \subset \Omega_0 \subset \Omega$.

$$\|f * \eta_\epsilon - f\|_{L^p(K)} = \|(1_{K_0}f) * \eta_\epsilon - 1_{K_0}f\|_{L^p(K)} \le \|(1_{K_0}f) * \eta_\epsilon - 1_{K_0}f\|_{L^p(\mathbb{R}^d)} \to 0 \text{ as } \epsilon \downarrow 0.$$

Hence, for all $f \in L^q_{loc}(\Omega)$, $f * \eta_\epsilon \in C^\infty(\Omega_\epsilon)$ and

$$\lim_{\epsilon \downarrow 0} \|f * \eta_\epsilon - f\|_{L^p(K)} = 0. \tag{19.5}$$

Now let $p(\xi)$ be a polynomial on $\mathbb{R}^d$, $u \in L^q_{loc}(\Omega)$ such that $p(\partial)u \in L^q_{loc}(\Omega)$ and $v_\epsilon := \eta_\epsilon * u \in C^\infty(\Omega_\epsilon)$ as above. Then for $x \in K$ and $\epsilon < \epsilon_0$,

$$\begin{aligned} p(\partial)v_\epsilon(x) &= \int_\Omega u(y)p(\partial_x)\eta_\epsilon(x-y)dy = \int_\Omega u(y)p(-\partial_y)\eta_\epsilon(x-y)dy \\ &= \int_\Omega u(y)p(-\partial_y)\eta_\epsilon(x-y)dy = \langle u, p(\partial)\eta_\epsilon(x-\cdot)\rangle \\ &= \langle p(\partial)u, \eta_\epsilon(x-\cdot)\rangle = (p(\partial)u)_\epsilon(x). \end{aligned} \tag{19.6}$$

From Eq. (19.6) we may now apply Eq. (19.5) with $f = u$ and $f = p_l(\partial)u$ for $1 \le l \le N$ to find

$$\|v_\epsilon - u\|_{L^p(K)} + \sum_{l=1}^{N} \|p_l(\partial)v_\epsilon - p_l(\partial)u\|_{L^p(K)} \to 0 \text{ as } \epsilon \downarrow 0.$$

For $n \in \mathbb{N}$, let

$$K_n := \{x \in \Omega : |x| \le n \text{ and } d(x, \Omega^c) \ge 1/n\}$$

(so $K_n \subset K_{n+1}^o \subset K_{n+1}$ for all n and $K_n \uparrow \Omega$ as $n \to \infty$ or see Lemma 10.10) and choose $\psi_n \in C_c^\infty(K_{n+1}^o, [0,1])$, using Corollary 11.25, so that $\psi_n = 1$ on a neighborhood of K_n. Choose $\epsilon_n \downarrow 0$ such that $K_{n+1} \subset \Omega_{\epsilon_n}$ and

$$\|v_{\epsilon_n} - u\|_{L^p(K_n)} + \sum_{l=1}^{N} \|p_l(\partial)v_{\epsilon_n} - p_l(\partial)u\|_{L^p(K_n)} \le 1/n.$$

Then $u_n := \psi_n \cdot v_{\epsilon_n} \in C_c^\infty(\Omega)$ and since $u_n = v_{\epsilon_n}$ on K_n we still have

$$\|u_n - u\|_{L^p(K_n)} + \sum_{l=1}^{N} \|p_l(\partial)u_n - p_l(\partial)u\|_{L^p(K_n)} \le 1/n. \tag{19.7}$$

Since any compact set $K \subset \Omega$ is contained in K_n^o for all n sufficiently large, Eq. (19.7) implies

$$\lim_{n\to\infty} \left[\|u_n - u\|_{L^p(K)} + \sum_{l=1}^{N} \|p_l(\partial)u_n - p_l(\partial)u\|_{L^p(K)} \right] = 0.$$

■

The following proposition is another variant of Proposition 19.12 which the reader is asked to prove in Exercise 19.2 below.

Proposition 19.13. *Suppose $q \in [1,\infty)$, $p_1(\xi), \dots, p_N(\xi)$ is a collection of polynomials in $\xi \in \mathbb{R}^d$ and $u \in L^q = L^q(\mathbb{R}^d)$ such that $p_l(\partial)u \in L^q$ for $l = 1, 2, \dots, N$. Then there exists $u_n \in C_c^\infty(\mathbb{R}^d)$ such that*

$$\lim_{n\to\infty} \left[\|u_n - u\|_{L^p} + \sum_{l=1}^{N} \|p_l(\partial)u_n - p_l(\partial)u\|_{L^p} \right] = 0.$$

Notation 19.14 (Difference quotients)**.** For $v \in \mathbb{R}^d$ and $h \in \mathbb{R}\setminus\{0\}$ and a function $u : \Omega \to \mathbb{C}$, let

$$\partial_v^h u(x) := \frac{u(x+hv) - u(x)}{h}$$

for those $x \in \Omega$ such that $x + hv \in \Omega$. When v is one of the standard basis elements, e_i for $1 \le i \le d$, we will write $\partial_i^h u(x)$ rather than $\partial_{e_i}^h u(x)$. Also let

$$\nabla^h u(x) := \left(\partial_1^h u(x), \dots, \partial_n^h u(x)\right)$$

be the difference quotient approximation to the gradient.

Definition 19.15 (Strong Differentiability)**.** Let $v \in \mathbb{R}^d$ and $u \in L^p$, then $\partial_v u$ is said to exist **strongly** in L^p if the $\lim_{h\to 0} \partial_v^h u$ exists in L^p. We will denote the limit by $\partial_v^{(s)} u$.

It is easily verified that if $u \in L^p$, $v \in \mathbb{R}^d$ and $\partial_v^{(s)} u \in L^p$ exists then $\partial_v^{(w)} u$ exists and $\partial_v^{(w)} u = \partial_v^{(s)} u$. The key to checking this assetion is the identity,

$$\begin{aligned} \langle \partial_v^h u, \phi \rangle &= \int_{\mathbb{R}^d} \frac{u(x+hv) - u(x)}{h} \phi(x) dx \\ &= \int_{\mathbb{R}^d} u(x) \frac{\phi(x-hv) - \phi(x)}{h} dx = \langle u, \partial_{-v}^h \phi \rangle. \end{aligned} \tag{19.8}$$

Hence if $\partial_v^{(s)} u = \lim_{h \to 0} \partial_v^h u$ exists in L^p and $\phi \in C_c^\infty(\mathbb{R}^d)$, then

$$\langle \partial_v^{(s)} u, \phi \rangle = \lim_{h \to 0} \langle \partial_v^h u, \phi \rangle = \lim_{h \to 0} \langle u, \partial_{-v}^h \phi \rangle = \frac{d}{dh}|_0 \langle u, \phi(\cdot - hv) \rangle = -\langle u, \partial_v \phi \rangle$$

wherein Corollary 7.43 has been used in the last equality to bring the derivative past the integral. This shows $\partial_v^{(w)} u$ exists and is equal to $\partial_v^{(s)} u$. What is somewhat more surprising is that the converse assertion that if $\partial_v^{(w)} u$ exists then so does $\partial_v^{(s)} u$. Theorem 19.18 is a generalization of Theorem 12.36 from L^2 to L^p. For the reader's convenience, let us give a self-contained proof of the version of the Banach - Alaoglu's Theorem which will be used in the proof of Theorem 19.18. (This is the same as Theorem 18.27 above.)

Proposition 19.16 (Weak-$*$ Compactness: Banach - Alaoglu's Theorem)**.** *Let X be a separable Banach space and $\{f_n\} \subset X^*$ be a bounded sequence, then there exist a subsequence $\{\tilde{f}_n\} \subset \{f_n\}$ such that $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$ with $f \in X^*$.*

Proof. Let $D \subset X$ be a countable linearly independent subset of X such that $\overline{\operatorname{span}(D)} = X$. Using Cantor's diagonal trick, choose $\{\tilde{f}_n\} \subseteq \{f_n\}$ such that $\lambda_x := \lim_{n \to \infty} \tilde{f}_n(x)$ exist for all $x \in D$. Define $f : \operatorname{span}(D) \to \mathbb{R}$ by the formula

$$f(\sum_{x \in D} a_x x) = \sum_{x \in D} a_x \lambda_x$$

where by assumption $\#(\{x \in D : a_x \neq 0\}) < \infty$. Then $f : \operatorname{span}(D) \to \mathbb{R}$ is linear and moreover $\tilde{f}_n(y) \to f(y)$ for all $y \in \operatorname{span}(D)$. Now

$$|f(y)| = \lim_{n \to \infty} |\tilde{f}_n(y)| \le \limsup_{n \to \infty} \|\tilde{f}_n\| \, \|y\| \le C\|y\| \text{ for all } y \in \operatorname{span}(D).$$

Hence by the B.L.T. Theorem 4.1, f extends uniquely to a bounded linear functional on X. We still denote the extension of f by $f \in X^*$. Finally, if $x \in X$ and $y \in \operatorname{span}(D)$

$$\begin{aligned} |f(x) - \tilde{f}_n(x)| &\le |f(x) - f(y)| + |f(y) - \tilde{f}_n(y)| + |\tilde{f}_n(y) - \tilde{f}_n(x)| \\ &\le \|f\| \, \|x-y\| + \|\tilde{f}_n\| \, \|x-y\| + |f(y) - \tilde{f}_n(y)\| \\ &\le 2C\|x-y\| + |f(y) - \tilde{f}_n(y)| \to 2C\|x-y\| \text{ as } n \to \infty. \end{aligned}$$

Therefore $\limsup_{n \to \infty} |f(x) - \tilde{f}_n(x)| \le 2C\|x-y\| \to 0$ as $y \to x$. ■

Corollary 19.17. *Let $p \in (1, \infty]$ and $q = \frac{p}{p-1}$. Then to every bounded sequence $\{u_n\}_{n=1}^\infty \subset L^p(\Omega)$ there is a subsequence $\{\tilde{u}_n\}_{n=1}^\infty$ and an element $u \in L^p(\Omega)$ such that*

$$\lim_{n \to \infty} \langle \tilde{u}_n, g \rangle = \langle u, g \rangle \text{ for all } g \in L^q(\Omega).$$

Proof. By Theorem 15.14, the map

$$v \in L^p(\Omega) \to \langle v, \cdot\rangle \in (L^q(\Omega))^*$$

is an isometric isomorphism of Banach spaces. By Theorem 11.3, $L^q(\Omega)$ is separable for all $q \in [1,\infty)$ and hence the result now follows from Proposition 19.16. ∎

Theorem 19.18 (Weak and Strong Differentiability)**.** *Suppose $p \in [1,\infty)$, $u \in L^p(\mathbb{R}^d)$ and $v \in \mathbb{R}^d \setminus \{0\}$. Then the following are equivalent:*

(1) *There exists $g \in L^p(\mathbb{R}^d)$ and $\{h_n\}_{n=1}^\infty \subset \mathbb{R}\setminus\{0\}$ such that $\lim_{n\to\infty} h_n = 0$ and*
$$\lim_{n\to\infty}\langle \partial_v^{h_n} u, \phi\rangle = \langle g, \phi\rangle \text{ for all } \phi \in C_c^\infty(\mathbb{R}^d).$$
(2) *$\partial_v^{(w)} u$ exists and is equal to $g \in L^p(\mathbb{R}^d)$, i.e. $\langle u, \partial_v\phi\rangle = -\langle g, \phi\rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$.*
(3) *There exists $g \in L^p(\mathbb{R}^d)$ and $u_n \in C_c^\infty(\mathbb{R}^d)$ such that $u_n \overset{L^p}{\to} u$ and $\partial_v u_n \overset{L^p}{\to} g$ as $n \to \infty$.*
(4) *$\partial_v^{(s)} u$ exists and is is equal to $g \in L^p(\mathbb{R}^d)$, i.e. $\partial_v^h u \to g$ in L^p as $h \to 0$.*

Moreover if $p \in (1,\infty)$ any one of the equivalent conditions 1. – 4. above are implied by the following condition.

1′. *There exists $\{h_n\}_{n=1}^\infty \subset \mathbb{R}\setminus\{0\}$ such that $\lim_{n\to\infty} h_n = 0$ and $\sup_n \left\|\partial_v^{h_n} u\right\|_p < \infty$.*

Proof. 4. $\Longrightarrow$ 1. is simply the assertion that strong convergence implies weak convergence.

1. $\Longrightarrow$ 2. For $\phi \in C_c^\infty(\mathbb{R}^d)$, Eq. (19.8) and the dominated convergence theorem implies

$$\langle g, \phi\rangle = \lim_{n\to\infty}\langle \partial_v^{h_n} u, \phi\rangle = \lim_{n\to\infty}\langle u, \partial_{-v}^{h_n}\phi\rangle = -\langle u, \partial_v\phi\rangle.$$

2. $\Longrightarrow$ 3. Let $\eta \in C_c^\infty(\mathbb{R}^d,\mathbb{R})$ such that $\int_{\mathbb{R}^d}\eta(x)dx = 1$ and let $\eta_m(x) = m^d\eta(mx)$, then by Proposition 11.24, $h_m := \eta_m * u \in C^\infty(\mathbb{R}^d)$ for all m and

$$\begin{aligned}\partial_v h_m(x) = \partial_v\eta_m * u(x) &= \int_{\mathbb{R}^d}\partial_v\eta_m(x-y)u(y)dy = \langle u, -\partial_v\left[\eta_m(x-\cdot)\right]\rangle \\ &= \langle g, \eta_m(x-\cdot)\rangle = \eta_m * g(x).\end{aligned}$$

By Theorem 11.21, $h_m \to u \in L^p(\mathbb{R}^d)$ and $\partial_v h_m = \eta_m * g \to g$ in $L^p(\mathbb{R}^d)$ as $m \to \infty$. This shows 3. holds except for the fact that h_m need not have compact support. To fix this let $\psi \in C_c^\infty(\mathbb{R}^d,[0,1])$ such that $\psi = 1$ in a neighborhood of 0 and let $\psi_\epsilon(x) = \psi(\epsilon x)$ and $(\partial_v\psi)_\epsilon(x) := (\partial_v\psi)(\epsilon x)$. Then

$$\partial_v(\psi_\epsilon h_m) = \partial_v\psi_\epsilon h_m + \psi_\epsilon\partial_v h_m = \epsilon(\partial_v\psi)_\epsilon h_m + \psi_\epsilon\partial_v h_m$$

so that $\psi_\epsilon h_m \to h_m$ in L^p and $\partial_v(\psi_\epsilon h_m) \to \partial_v h_m$ in L^p as $\epsilon \downarrow 0$. Let $u_m = \psi_{\epsilon_m} h_m$ where ϵ_m is chosen to be greater than zero but small enough so that

$$\|\psi_{\epsilon_m} h_m - h_m\|_p + \|\partial_v(\psi_{\epsilon_m} h_m) \to \partial_v h_m\|_p < 1/m.$$

Then $u_m \in C_c^\infty(\mathbb{R}^d)$, $u_m \to u$ and $\partial_v u_m \to g$ in L^p as $m \to \infty$.

3. $\Longrightarrow$ 4. By the fundamental theorem of calculus

$$\begin{aligned}\partial_v^h u_m(x) &= \frac{u_m(x+hv) - u_m(x)}{h} \\ &= \frac{1}{h}\int_0^1 \frac{d}{ds}u_m(x+shv)ds = \int_0^1(\partial_v u_m)(x+shv)ds. \qquad (19.9)\end{aligned}$$

and therefore,

$$\partial_v^h u_m(x) - \partial_v u_m(x) = \int_0^1 \left[(\partial_v u_m)(x + shv) - \partial_v u_m(x) \right] ds.$$

So by Minkowski's inequality for integrals, Theorem 9.27,

$$\left\| \partial_v^h u_m(x) - \partial_v u_m \right\|_p \le \int_0^1 \left\| (\partial_v u_m)(\cdot + shv) - \partial_v u_m \right\|_p ds$$

and letting $m \to \infty$ in this equation then implies

$$\left\| \partial_v^h u - g \right\|_p \le \int_0^1 \left\| g(\cdot + shv) - g \right\|_p ds.$$

By the dominated convergence theorem and Proposition 11.13, the right member of this equation tends to zero as $h \to 0$ and this shows item 4. holds.

($1'$. $\Longrightarrow$ 1. when $p > 1$) This is a consequence of Corollary 19.17 (or see Theorem 18.27 above) which asserts, by passing to a subsequence if necessary, that $\partial_v^{h_n} u \overset{w}{\to} g$ for some $g \in L^p(\mathbb{R}^d)$. ■

Example 19.19. The fact that ($1'$) does not imply the equivalent conditions 1 – 4 in Theorem 19.18 when $p = 1$ is demonstrated by the following example. Let $u := 1_{[0,1]}$, then

$$\int_{\mathbb{R}} \left| \frac{u(x+h) - u(x)}{h} \right| dx = \frac{1}{|h|} \int_{\mathbb{R}} \left| 1_{[-h,1-h]}(x) - 1_{[0,1]}(x) \right| dx = 2$$

for $|h| < 1$. On the other hand the distributional derivative of u is $\partial u(x) = \delta(x) - \delta(x-1)$ which is not in L^1.

Alternatively, if there exists $g \in L^1(\mathbb{R}, dm)$ such that

$$\lim_{n\to\infty} \frac{u(x+h_n) - u(x)}{h_n} = g(x) \text{ in } L^1$$

for some sequence $\{h_n\}_{n=1}^\infty$ as above. Then for $\phi \in C_c^\infty(\mathbb{R})$ we would have on one hand,

$$\begin{aligned}\int_{\mathbb{R}} \frac{u(x+h_n) - u(x)}{h_n} \phi(x) dx &= \int_{\mathbb{R}} \frac{\phi(x-h_n) - \phi(x)}{h_n} u(x) dx \\ &\to -\int_0^1 \phi'(x) dx = (\phi(0) - \phi(1)) \text{ as } n \to \infty,\end{aligned}$$

while on the other hand,

$$\int_{\mathbb{R}} \frac{u(x+h_n) - u(x)}{h_n} \phi(x) dx \to \int_{\mathbb{R}} g(x)\phi(x) dx.$$

These two equations imply

$$\int_{\mathbb{R}} g(x)\phi(x) dx = \phi(0) - \phi(1) \text{ for all } \phi \in C_c^\infty(\mathbb{R}) \tag{19.10}$$

and in particular that $\int_{\mathbb{R}} g(x)\phi(x)dx = 0$ for all $\phi \in C_c(\mathbb{R} \setminus \{0,1\})$. By Corollary 11.28, $g(x) = 0$ for m – a.e. $x \in \mathbb{R} \setminus \{0,1\}$ and hence $g(x) = 0$ for m – a.e. $x \in \mathbb{R}$. But this clearly contradicts Eq. (19.10). This example also shows that the unit ball in $L^1(\mathbb{R}, dm)$ is not weakly sequentially compact. Compare with Example 18.24.

Proposition 19.20 (A weak form of Weyls Lemma)**.** *If $u \in L^2(\mathbb{R}^d)$ such that $f := \triangle u \in L^2(\mathbb{R}^d)$ then $\partial^\alpha u \in L^2\left(\mathbb{R}^d\right)$ for $|\alpha| \le 2$. Furthermore if $k \in \mathbb{N}_0$ and $\partial^\beta f \in L^2\left(\mathbb{R}^d\right)$ for all $|\beta| \le k$, then $\partial^\alpha u \in L^2\left(\mathbb{R}^d\right)$ for $|\alpha| \le k+2$.*

Proof. By Proposition 19.13, there exists $u_n \in C_c^\infty\left(\mathbb{R}^d\right)$ such that $u_n \to u$ and $\Delta u_n \to \Delta u = f$ in $L^2\left(\mathbb{R}^d\right)$. By integration by parts we find

$$\int_{\mathbb{R}^d} |\nabla(u_n - u_m)|^2 \, dm = (-\Delta(u_n - u_m), (u_n - u_m))_{L^2} \to -(f-f, u-u) = 0 \text{ as } m,n \to \infty$$

and hence by item 3. of Theorem 19.18, $\partial_i u \in L^2$ for each i. Since

$$\|\nabla u\|_{L^2}^2 = \lim_{n\to\infty} \int_{\mathbb{R}^d} |\nabla u_n|^2 \, dm = (-\Delta u_n, u_n)_{L^2} \to -(f,u) \text{ as } n \to \infty$$

we also learn that

$$\|\nabla u\|_{L^2}^2 = -(f,u) \le \|f\|_{L^2} \cdot \|u\|_{L^2}.$$

Let us now consider

$$\begin{aligned}\sum_{i,j=1}^d \int_{\mathbb{R}^d} |\partial_i \partial_j u_n|^2 \, dm &= -\sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_j u_n \partial_i^2 \partial_j u_n dm \\ &= -\sum_{j=1}^d \int_{\mathbb{R}^d} \partial_j u_n \partial_j \Delta u_n dm = \sum_{j=1}^d \int_{\mathbb{R}^d} \partial_j^2 u_n \Delta u_n dm \\ &= \int_{\mathbb{R}^d} |\Delta u_n|^2 \, dm = \|\Delta u_n\|_{L^2}^2.\end{aligned}$$

Replacing u_n by $u_n - u_m$ in this calculation shows

$$\sum_{i,j=1}^d \int_{\mathbb{R}^d} |\partial_i \partial_j (u_n - u_m)|^2 \, dm = \|\Delta(u_n - u_m)\|_{L^2}^2 \to 0 \text{ as } m,n \to \infty$$

and therefore by Lemma 19.4 (also see Exercise 19.3), $\partial_i \partial_j u \in L^2\left(\mathbb{R}^d\right)$ for all i,j and

$$\sum_{i,j=1}^d \int_{\mathbb{R}^d} |\partial_i \partial_j u|^2 \, dm = \|\Delta u\|_{L^2}^2 = \|f\|_{L^2}^2.$$

Let us now further assume $\partial_i f \in L^2\left(\mathbb{R}^d\right)$. Then for $h \in \mathbb{R} \setminus \{0\}$, $\partial_i^h u \in L^2(\mathbb{R}^d)$ and $\Delta \partial_i^h u = \partial_i^h \Delta u = \partial_i^h f \in L^2(\mathbb{R}^d)$ and hence by what we have just proved, $\partial^\alpha \partial_i^h u = \partial_i^h \partial^\alpha u \in L^2$ and

$$\begin{aligned}\sum_{|\alpha|\le 2} \left\|\partial_i^h \partial^\alpha u\right\|_{L^2(\mathbb{R}^d)}^2 &\le C\left[\left\|\partial_i^h f\right\|_{L^2}^2 + \left\|\partial_i^h f\right\|_{L^2} \cdot \left\|\partial_i^h u\right\|_{L^2}\right] \\ &\le C\left[\|\partial_i f\|_{L^2}^2 + \|\partial_i f\|_{L^2} \cdot \|\partial_i u\|_{L^2}\right]\end{aligned}$$

with the last bound being independent of $h \ne 0$. Therefore applying Theorem 19.18 again we learn that $\partial_i \partial^\alpha u \in L^2(\mathbb{R}^d)$ for all $|\alpha| \le 2$. The remainder of the proof, which is now an induction argument using the above ideas, is left as an exercise to the reader. ■

Theorem 19.21. *Suppose that Ω is a precompact open subset of $\mathbb{R}^d$ and V is an open precompact subset of Ω.*

(1) *If* $1 \le p < \infty$, $u \in L^p(\Omega)$ *and* $\partial_i u \in L^p(\Omega)$, *then* $\|\partial_i^h u\|_{L^p(V)} \le \|\partial_i u\|_{L^p(\Omega)}$ *for all* $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \Omega^c)$.

(2) *Suppose that* $1 < p \le \infty$, $u \in L^p(\Omega)$ *and assume there exists a constants* $C_V < \infty$ *and* $\epsilon_V \in (0, \frac{1}{2}\mathrm{dist}(V, \Omega^c))$ *such that*

$$\|\partial_i^h u\|_{L^p(V)} \le C_V \text{ for all } 0 < |h| < \epsilon_V.$$

Then $\partial_i u \in L^p(V)$ *and* $\|\partial_i u\|_{L^p(V)} \le C_V$. *Moreover if* $C := \sup_{V \subset\subset \Omega} C_V < \infty$ *then in fact* $\partial_i u \in L^p(\Omega)$ *and* $\|\partial_i u\|_{L^p(\Omega)} \le C$.

Proof. 1. Let $U \subset_o \Omega$ such that $\bar{V} \subset U$ and $\bar{U}$ is a compact subset of Ω. For $u \in C^1(\Omega) \cap L^p(\Omega)$, $x \in B$ and $0 < |h| < \frac{1}{2}\mathrm{dist}(V, U^c)$,

$$\partial_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} = \int_0^1 \partial_i u(x + the_i)\, dt$$

and in particular,

$$|\partial_i^h u(x)| \le \int_0^1 |\partial u(x + the_i)| dt.$$

Therefore by Minikowski's inequality for integrals,

$$\|\partial_i^h u\|_{L^p(V)} \le \int_0^1 \|\partial u(\cdot + the_i)\|_{L^p(V)} dt \le \|\partial_i u\|_{L^p(U)}. \tag{19.11}$$

For general $u \in L^p(\Omega)$ with $\partial_i u \in L^p(\Omega)$, by Proposition 19.12, there exists $u_n \in C_c^\infty(\Omega)$ such that $u_n \to u$ and $\partial_i u_n \to \partial_i u$ in $L^p_{loc}(\Omega)$. Therefore we may replace u by u_n in Eq. (19.11) and then pass to the limit to find

$$\|\partial_i^h u\|_{L^p(V)} \le \|\partial_i u\|_{L^p(U)} \le \|\partial_i u\|_{L^p(\Omega)}.$$

2. If $\|\partial_i^h u\|_{L^p(V)} \le C_V$ for all h sufficiently small then by Corollary 19.17 there exists $h_n \to 0$ such that $\partial_i^{h_n} u \overset{w}{\to} v \in L^p(V)$. Hence if $\varphi \in C_c^\infty(V)$,

$$\int_V v\varphi dm = \lim_{n\to\infty} \int_\Omega \partial_i^{h_n} u \varphi dm = \lim_{n\to\infty} \int_\Omega u \partial_i^{-h_n} \varphi dm$$
$$= -\int_\Omega u \partial_i \varphi \, dm = -\int_V u \partial_i \varphi \, dm.$$

Therefore $\partial_i u = v \in L^p(V)$ and $\|\partial_i u\|_{L^p(V)} \le \|v\|_{L^p(V)} \le C_V$. Finally if $C := \sup_{V \subset\subset \Omega} C_V < \infty$, then by the dominated convergence theorem,

$$\|\partial_i u\|_{L^p(\Omega)} = \lim_{V \uparrow \Omega} \|\partial_i u\|_{L^p(V)} \le C.$$

■

We will now give a couple of applications of Theorem 19.18.

Lemma 19.22. *Let* $v \in \mathbb{R}^d$.

(1) *If* $h \in L^1$ *and* $\partial_v h$ *exists in* L^1, *then* $\int_{\mathbb{R}^d} \partial_v h(x) dx = 0$.

(2) *If* $p, q, r \in [1, \infty)$ *satisfy* $r^{-1} = p^{-1} + q^{-1}$, $f \in L^p$ *and* $g \in L^q$ *are functions such that* $\partial_v f$ *and* $\partial_v g$ *exists in* L^p *and* L^q *respectively, then* $\partial_v(fg)$ *exists in* L^r *and* $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$. *Moreover if* $r = 1$ *we have the integration by parts formula,*

$$\langle \partial_v f, g \rangle = -\langle f, \partial_v g \rangle. \tag{19.12}$$

(3) *If $p=1$, $\partial_v f$ exists in L^1 and $g \in BC^1(\mathbb{R}^d)$ (i.e. $g \in C^1(\mathbb{R}^d)$ with g and its first derivatives being bounded) then $\partial_v(gf)$ exists in L^1 and $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$ and again Eq. (19.12) holds.*

Proof. 1) By item 3. of Theorem 19.18 there exists $h_n \in C_c^\infty(\mathbb{R}^d)$ such that $h_n \to h$ and $\partial_v h_n \to \partial_v h$ in L^1. Then

$$\int_{\mathbb{R}^d} \partial_v h_n(x)dx = \frac{d}{dt}|_0 \int_{\mathbb{R}^d} h_n(x+hv)dx = \frac{d}{dt}|_0 \int_{\mathbb{R}^d} h_n(x)dx = 0$$

and letting $n \to \infty$ proves the first assertion.

2) Similarly there exists $f_n, g_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $\partial_v f_n \to \partial_v f$ in L^p and $g_n \to g$ and $\partial_v g_n \to \partial_v g$ in L^q as $n \to \infty$. So by the standard product rule and Remark 19.2, $f_n g_n \to fg \in L^r$ as $n \to \infty$ and

$$\partial_v(f_n g_n) = \partial_v f_n \cdot g_n + f_n \cdot \partial_v g_n \to \partial_v f \cdot g + f \cdot \partial_v g \text{ in } L^r \text{ as } n \to \infty.$$

It now follows from another application of Theorem 19.18 that $\partial_v(fg)$ exists in L^r and $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$. Eq. (19.12) follows from this product rule and item 1. when $r=1$.

3) Let $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $\partial_v f_n \to \partial_v f$ in L^1 as $n \to \infty$. Then as above, $gf_n \to gf$ in L^1 and $\partial_v(gf_n) \to \partial_v g \cdot f + g\partial_v f$ in L^1 as $n \to \infty$. In particular if $\phi \in C_c^\infty(\mathbb{R}^d)$, then

$$\begin{aligned} \langle gf, \partial_v \phi\rangle &= \lim_{n\to\infty} \langle gf_n, \partial_v \phi\rangle = -\lim_{n\to\infty} \langle \partial_v (gf_n), \phi\rangle \\ &= -\lim_{n\to\infty} \langle \partial_v g \cdot f_n + g\partial_v f_n, \phi\rangle = -\langle \partial_v g \cdot f + g\partial_v f, \phi\rangle. \end{aligned}$$

This shows $\partial_v(fg)$ exists (weakly) and $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$. Again Eq. (19.12) holds in this case by item 1. already proved. ∎

Lemma 19.23. *Let $p, q, r \in [1, \infty]$ satisfy $p^{-1} + q^{-1} = 1 + r^{-1}$, $f \in L^p$, $g \in L^q$ and $v \in \mathbb{R}^d$.*

(1) *If $\partial_v f$ exists strongly in L^r, then $\partial_v(f*g)$ exists strongly in L^p and*
$$\partial_v(f*g) = (\partial_v f) * g.$$
(2) *If $\partial_v g$ exists strongly in L^q, then $\partial_v(f*g)$ exists strongly in L^r and*
$$\partial_v(f*g) = f * \partial_v g.$$
(3) *If $\partial_v f$ exists weakly in L^p and $g \in C_c^\infty(\mathbb{R}^d)$, then $f*g \in C^\infty(\mathbb{R}^d)$, $\partial_v(f*g)$ exists strongly in L^r and*
$$\partial_v(f*g) = f*\partial_v g = (\partial_v f)*g.$$

Proof. Items 1 and 2. By Young's inequality (Theorem 11.19) and simple computations:

$$\begin{aligned} \left\| \frac{\tau_{-hv}(f*g) - f*g}{h} - (\partial_v f)*g \right\|_r &= \left\| \frac{\tau_{-hv} f * g - f*g}{h} - (\partial_v f)*g \right\|_r \\ &= \left\| \left[\frac{\tau_{-hv} f - f}{h} - (\partial_v f)\right] * g \right\|_r \\ &\le \left\| \frac{\tau_{-hv} f - f}{h} - (\partial_v f)\right\|_p \|g\|_q \end{aligned}$$

which tends to zero as $h \to 0$. The second item is proved analogously, or just make use of the fact that $f*g = g*f$ and apply Item 1.

Using the fact that $g(x-\cdot)\in C_c^\infty(\mathbb{R}^d)$ and the definition of the weak derivative,

$$\begin{aligned} f*\partial_v g(x) &= \int_{\mathbb{R}^d} f(y)\left(\partial_v g\right)(x-y)dy = -\int_{\mathbb{R}^d} f(y)\left(\partial_v g(x-\cdot)\right)(y)dy \\ &= \int_{\mathbb{R}^d} \partial_v f(y) g(x-y)dy = \partial_v f * g(x). \end{aligned}$$

Item 3. is a consequence of this equality and items 1. and 2. ■

19.2. **The connection of Weak and pointwise derivatives.**

Proposition 19.24. *Let $\Omega=(\alpha,\beta)\subset\mathbb{R}$ be an open interval and $f\in L^1_{loc}(\Omega)$ such that $\partial^{(w)}f=0$ in $L^1_{loc}(\Omega)$. Then there exists $c\in\mathbb{C}$ such that $f=c$ a.e. More generally, suppose $F:C_c^\infty(\Omega)\to\mathbb{C}$ is a linear functional such that $F(\phi')=0$ for all $\phi\in C_c^\infty(\Omega)$, where $\phi'(x)=\frac{d}{dx}\phi(x)$, then there exists $c\in\mathbb{C}$ such that*

$$F(\phi)=\langle c,\phi\rangle=\int_\Omega c\phi(x)dx \text{ for all } \phi\in C_c^\infty(\Omega). \tag{19.13}$$

Proof. Before giving a proof of the second assertion, let us show it includes the first. Indeed, if $F(\phi):=\int_\Omega \phi f dm$ and $\partial^{(w)}f=0$, then $F(\phi')=0$ for all $\phi\in C_c^\infty(\Omega)$ and therefore there exists $c\in\mathbb{C}$ such that

$$\int_\Omega \phi f dm = F(\phi)=c\langle\phi,1\rangle = c\int_\Omega \phi f dm.$$

But this implies $f=c$ a.e. So it only remains to prove the second assertion.

Let $\eta\in C_c^\infty(\Omega)$ such that $\int_\Omega \eta dm=1$. Given $\phi\in C_c^\infty(\Omega)\subset C_c^\infty(\mathbb{R})$, let $\psi(x)=\int_{-\infty}^x\left(\phi(y)-\eta(y)\langle\phi,1\rangle\right)dy$. Then $\psi'(x)=\phi(x)-\eta(x)\langle\phi,1\rangle$ and $\psi\in C_c^\infty(\Omega)$ as the reader should check. Therefore,

$$0=F(\psi)=F(\phi-\langle\phi,\eta\rangle\eta)=F(\phi)-\langle\phi,1\rangle F(\eta)$$

which shows Eq. (19.13) holds with $c=F(\eta)$. This concludes the proof, however it will be instructive to give another proof of the first assertion.

Alternative proof of first assertion. Suppose $f\in L^1_{loc}(\Omega)$ and $\partial^{(w)}f=0$ and $f_m:=f*\eta_m$ as is in the proof of Lemma 19.9. Then $f_m'=\partial^{(w)}f*\eta_m=0$, so $f_m=c_m$ for some constant $c_m\in\mathbb{C}$. By Theorem 11.21, $f_m\to f$ in $L^1_{loc}(\Omega)$ and therefore if $J=[a,b]$ is a compact subinterval of Ω,

$$|c_m-c_k|=\frac{1}{b-a}\int_J |f_m-f_k|\,dm\to 0 \text{ as } m,k\to\infty.$$

So $\{c_m\}_{m=1}^\infty$ is a Cauchy sequence and therefore $c:=\lim_{m\to\infty}c_m$ exists and $f=\lim_{m\to\infty}f_m=c$ a.e. ■

Theorem 19.25. *Suppose $f\in L^1_{loc}(\Omega)$. Then there exists a complex measure μ on $\mathcal{B}_\Omega$ such that*

$$-\langle f,\phi'\rangle=\mu(\phi):=\int_\Omega \phi d\mu \text{ for all } \phi\in C_c^\infty(\Omega) \tag{19.14}$$

iff there exists a right continuous function F of bounded variation such that $F=f$ a.e. In this case $\mu=\mu_F$, i.e. $\mu((a,b])=F(b)-F(a)$ for all $-\infty<a<b<\infty$.

Proof. Suppose $f = F$ a.e. where F is as above and let $\mu = \mu_F$ be the associated measure on $\mathcal{B}_\Omega$. Let $G(t) = F(t) - F(-\infty) = \mu((-\infty, t])$, then using Fubini's theorem and the fundamental theorem of calculus,

$$\begin{aligned} -\langle f, \phi'\rangle = -\langle F, \phi'\rangle = -\langle G, \phi'\rangle &= -\int_\Omega \phi'(t)\left[\int_\Omega 1_{(-\infty,t]}(s)d\mu(s)\right]dt \\ &= -\int_\Omega\int_\Omega \phi'(t)1_{(-\infty,t]}(s)dtd\mu(s) = \int_\Omega \phi(s)d\mu(s) = \mu(\phi). \end{aligned}$$

Conversely if Eq. (19.14) holds for some measure μ, let $F(t) := \mu((-\infty, t])$ then working backwards from above,

$$-\langle f, \phi'\rangle = \mu(\phi) = \int_\Omega \phi(s)d\mu(s) = -\int_\Omega\int_\Omega \phi'(t)1_{(-\infty,t]}(s)dtd\mu(s) = -\int_\Omega \phi'(t)F(t)dt.$$

This shows $\partial^{(w)}(f - F) = 0$ and therefore by Proposition 19.24, $f = F + c$ a.e. for some constant $c \in \mathbb{C}$. Since $F + c$ is right continuous with bounded variation, the proof is complete. ■

Proposition 19.26. *Let $\Omega \subset \mathbb{R}$ be an open interval and $f \in L^1_{loc}(\Omega)$. Then $\partial^w f$ exists in $L^1_{loc}(\Omega)$ iff f has a continuous version $\tilde{f}$ which is absolutely continuous on all compact subintervals of Ω. Moreover, $\partial^w f = \tilde{f}'$ a.e., where $\tilde{f}'(x)$ is the usual pointwise derivative.*

Proof. If f is locally absolutely continuous and $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset [a, b] \subset \Omega$, then by integration by parts, Corollary 16.32,

$$\int_\Omega f'\phi dm = \int_a^b f'\phi dm = -\int_a^b f\phi' dm + f\phi|_a^b = -\int_\Omega f\phi' dm.$$

This shows $\partial^w f$ exists and $\partial^w f = f' \in L^1_{loc}(\Omega)$.

Now suppose that $\partial^w f$ exists in $L^1_{loc}(\Omega)$ and $a \in \Omega$. Define $F \in C(\Omega)$ by $F(x) := \int_a^x \partial^w f(y)dy$. Then F is absolutely continuous on compacts and therefore by fundamental theorem of calculus for absolutely continuous functions (Theorem 16.31), $F'(x)$ exists and is equal to $\partial^w f(x)$ for a.e. $x \in \Omega$. Moreover, by the first part of the argument, $\partial^w F$ exists and $\partial^w F = \partial^w f$, and so by Proposition 19.24 there is a constant c such that

$$\tilde{f}(x) := F(x) + c = f(x) \text{ for a.e. } x \in \Omega.$$

■

Definition 19.27. Let X and Y be metric spaces. A function $u : X \to Y$ is said to be **Lipschitz** if there exists $C < \infty$ such that

$$d^Y(u(x), u(x')) \le Cd^X(x, x') \text{ for all } x, x' \in X$$

and said to be **locally Lipschitz** if for all compact subsets $K \subset X$ there exists $C_K < \infty$ such that

$$d^Y(u(x), u(x')) \le C_K d^X(x, x') \text{ for all } x, x' \in K.$$

Proposition 19.28. *Let $u \in L^1_{loc}(\Omega)$. Then there exists a locally Lipschitz function $\tilde{u} : \Omega \to \mathbb{C}$ such that $\tilde{u} = u$ a.e. iff $\partial_i u \in L^1_{loc}(\Omega)$ exists and is locally (essentially) bounded for $i = 1, 2, \ldots, d$.*

Proof. Suppose $u = \tilde{u}$ a.e. and $\tilde{u}$ is Lipschitz and let $p \in (1, \infty)$ and V be a precompact open set such that $\bar{V} \subset W$ and let $V_\epsilon := \left\{x \in \Omega : \operatorname{dist}(x, \bar{V}) \le \epsilon\right\}$. Then for $\epsilon < \operatorname{dist}(\bar{V}, \Omega^c)$, $V_\epsilon \subset \Omega$ and therefore there is constant $C(V, \epsilon) < \infty$ such that $|\tilde{u}(y) - \tilde{u}(x)| \le C(V, \epsilon)\,|y - x|$ for all $x, y \in V_\epsilon$. So for $0 < |h| \le 1$ and $v \in \mathbb{R}^d$ with $|v| = 1$,

$$\int_V \left|\frac{u(x + hv) - u(x)}{h}\right|^p dx = \int_V \left|\frac{\tilde{u}(x + hv) - \tilde{u}(x)}{h}\right|^p dx \le C(V, \epsilon)\,|v|^p.$$

Therefore Theorem 19.18 may be applied to conclude $\partial_v u$ exists in L^p and moreover,

$$\lim_{h \to 0} \frac{\tilde{u}(x + hv) - \tilde{u}(x)}{h} = \partial_v u(x) \text{ for } m - \text{a.e. } x \in V.$$

Since there exists $\{h_n\}_{n=1}^\infty \subset \mathbb{R} \setminus \{0\}$ such that $\lim_{n \to \infty} h_n = 0$ and

$$|\partial_v u(x)| = \lim_{n \to \infty} \left|\frac{\tilde{u}(x + h_n v) - \tilde{u}(x)}{h_n}\right| \le C(V) \text{ for a.e. } x \in V,$$

it follows that $\|\partial_v u\|_\infty \le C(V)$ where $C(V) := \lim_{\epsilon \downarrow 0} C(V, \epsilon)$.

Conversely, let $\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > \epsilon\}$ and $\eta \in C_c^\infty(B(0,1), [0, \infty))$ such that $\int_{\mathbb{R}^n} \eta(x) dx = 1$, $\eta_m(x) = m^n \eta(mx)$ and $u_m := u * \eta_m$ as in the proof of Theorem 19.18. Suppose $V \subset_o \Omega$ with $\bar{V} \subset \Omega$ and ϵ is sufficiently small. Then $u_m \in C^\infty(\Omega_\epsilon)$, $\partial_v u_m = \partial_v u * \eta_m$, $|\partial_v u_m(x)| \le \|\partial_v u\|_{L^\infty(V_{m^{-1}})} =: C(V, m) < \infty$ and therefore,

$$\begin{aligned}
|u_m(y) - u_m(x)| &= \left|\int_0^1 \frac{d}{dt} u_m(x + t(y - x)) dt\right| \\
&= \left|\int_0^1 (y - x) \cdot \nabla u_m(x + t(y - x)) dt\right| \\
&\le \int_0^1 |y - x| \cdot |\nabla u_m(x + t(y - x))|\, dt \\
&\le C(V, m)\,|y - x| \text{ for all } x, y \in V. \qquad (19.15)
\end{aligned}$$

By passing to a subsequence if necessary, we may assume that $\lim_{m \to \infty} u_m(x) = u(x)$ for m – a.e. $x \in V$ and then letting $m \to \infty$ in Eq. (19.15) implies

$$|u(y) - u(x)| \le C(V)\,|y - x| \text{ for all } x, y \notin E \qquad (19.16)$$

where $E \subset V$ is a m – null set. Define $\tilde{u}_V : V \to \mathbb{C}$ by $\tilde{u}_V = u$ on $V \setminus E^c$ and $\tilde{u}_V(x) = \lim_{\substack{y \to x \\ y \notin E}} u(y)$ if $x \in E$. Then clearly $\tilde{u}_V = u$ a.e. on V and it is easy to show $\tilde{u}_V$ is well defined and $\tilde{u}_V : V \to \mathbb{C}$ is Lipschitz continuous. To complete the proof, choose precompact open sets V_n such that $V_n \subset \bar{V}_n \subset V_{n+1} \subset \Omega$ for all n and for $x \in V_n$ let $\tilde{u}(x) := \tilde{u}_{V_n}(x)$. ■

Here is an alternative way to construct the function $\tilde{u}_V$ above. For $x \in V \setminus E$,

$$\begin{aligned}
|u_m(x) - u(x)| &= \left|\int_V u(x - y)\eta(my)m^n dy - u(x)\right| = \left|\int_V \left[u(x - y/m) - u(x)\right] \eta(y) dy\right| \\
&\le \int_V |u(x - y/m) - u(x)|\, \eta(y) dy \le \frac{C}{m} \int_V |y|\, \eta(y) dy
\end{aligned}$$

wherein the last equality we have used Eq. (19.16) with V replaced by V_ϵ for some small $\epsilon > 0$. Letting $K := C \int_V |y|\, \eta(y) dy < \infty$ we have shown

$$\|u_m - u\|_\infty \le K/m \to 0 \text{ as } m \to \infty$$

and consequently

$$\|u_m - u_n\|_u = \|u_m - u_n\|_\infty \le 2K/m \to 0 \text{ as } m \to \infty.$$

Therefore, u_n converges uniformly to a continuous function $\tilde{u}_V$.

The next theorem is from Chapter 1. of Maz'ja [2].

Theorem 19.29. *Let $p \ge 1$ and Ω be an open subset of $\mathbb{R}^d$, $x \in \mathbb{R}^d$ be written as $x = (y,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$,*

$$Y := \left\{y \in \mathbb{R}^{d-1} : (\{y\} \times \mathbb{R}) \cap \Omega \ne \emptyset\right\}$$

and $u \in L^p(\Omega)$. Then $\partial_t u$ exists weakly in $L^p(\Omega)$ iff there is a version $\tilde{u}$ of u such that for a.e. $y \in Y$ the function $t \to \tilde{u}(y,t)$ is absolutely continuous, $\partial_t u(y,t) = \frac{\partial \tilde{u}(y,t)}{\partial t}$ a.e., and $\left\|\frac{\partial \tilde{u}}{\partial t}\right\|_{L^p(\Omega)} < \infty$.

Proof. For the proof of Theorem 19.29, it suffices to consider the case where $\Omega = (0,1)^d$. Write $x \in \Omega$ as $x = (y,t) \in Y \times (0,1) = (0,1)^{d-1} \times (0,1)$ and $\partial_t u$ for the weak derivative $\partial_{e_d} u$. By assumption

$$\int_\Omega |\partial_t u(y,t)|\, dydt = \|\partial_t u\|_1 \le \|\partial_t u\|_p < \infty$$

and so by Fubini's theorem there exists a set of full measure, $Y_0 \subset Y$, such that

$$\int_0^1 |\partial_t u(y,t)|\, dt < \infty \text{ for } y \in Y_0.$$

So for $y \in Y_0$, the function $v(y,t) := \int_0^t \partial_t u(y,\tau) d\tau$ is well defined and absolutely continuous in t with $\frac{\partial}{\partial t} v(y,t) = \partial_t u(y,t)$ for a.e. $t \in (0,1)$. Let $\xi \in C_c^\infty(Y)$ and $\eta \in C_c^\infty((0,1))$, then integration by parts for absolutely functions implies

$$\int_0^1 v(y,t)\dot{\eta}(t) dt = -\int_0^1 \frac{\partial}{\partial t} v(y,t)\eta(t) dt \text{ for all } y \in Y_0.$$

Multiplying both sides of this equation by $\xi(y)$ and integrating in y shows

$$\int_\Omega v(x)\dot{\eta}(t)\xi(y) dydt = -\int_\Omega \frac{\partial}{\partial t} v(y,t)\eta(t)\xi(y) dydt = -\int_\Omega \partial_t u(y,t)\eta(t)\xi(y) dydt.$$

Using the definition of the weak derivative, this equation may be written as

$$\int_\Omega u(x)\dot{\eta}(t)\xi(y) dydt = -\int_\Omega \partial_t u(x)\eta(t)\xi(y) dydt$$

and comparing the last two equations shows

$$\int_\Omega [v(x) - u(x)]\,\dot{\eta}(t)\xi(y) dydt = 0.$$

Since $\xi \in C_c^\infty(Y)$ is arbitrary, this implies there exists a set $Y_1 \subset Y_0$ of full measure such that

$$\int_\Omega [v(y,t) - u(y,t)]\,\dot{\eta}(t) dt = 0 \text{ for all } y \in Y_1$$

from which we conclude, using Proposition 19.24, that $u(y,t) = v(y,t) + C(y)$ for $t \in J_y$ where $m_{d-1}(J_y) = 1$, here m_k denotes k – dimensional Lebesgue measure. In conclusion we have shown that

$$u(y,t) = \tilde{u}(y,t) := \int_0^t \partial_t u(y,\tau) d\tau + C(y) \text{ for all } y \in Y_1 \text{ and } t \in J_y. \tag{19.17}$$

We can be more precise about the formula for $\tilde{u}(y,t)$ by integrating both sides of Eq. (19.17) on t we learn

$$C(y) = \int_0^1 dt \int_0^t \partial_\tau u(y,\tau)d\tau - \int_0^1 u(y,t)dt = \int_0^1 (1-\tau)\,\partial_\tau u(y,\tau)d\tau - \int_0^1 u(y,t)dt$$
$$= \int_0^1 \left[(1-t)\,\partial_t u(y,t) - u(y,t)\right] dt$$

and hence

$$\tilde{u}(y,t) := \int_0^t \partial_\tau u(y,\tau)d\tau + \int_0^1 \left[(1-\tau)\,\partial_\tau u(y,\tau) - u(y,\tau)\right] d\tau$$

which is well defined for $y \in Y_0$.

For the converse suppose that such a $\tilde{u}$ exists, then for $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega u(y,t)\partial_t\phi(y,t)dydt = \int_\Omega \tilde{u}(y,t)\partial_t\phi(y,t)dtdy = -\int_\Omega \frac{\partial\tilde{u}(y,t)}{\partial t}\phi(y,t)dtdy$$

wherein we have used integration by parts for absolutely continuous functions. From this equation we learn the weak derivative $\partial_t u(y,t)$ exists and is given by $\frac{\partial\tilde{u}(y,t)}{\partial t}$ a.e. ■

19.3. **Exercises.**

Exercise 19.1. Give another proof of Lemma 19.10 base on Proposition 19.12.

Exercise 19.2. Prove Proposition 19.13. **Hints:** 1. Use u_ϵ as defined in the proof of Proposition 19.12 to show it suffices to consider the case where $u \in C^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\partial^\alpha u \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$. 2. Then let $\psi \in C_c^\infty(B(0,1),[0,1])$ such that $\psi = 1$ on a neighborhood of 0 and let $u_n(x) := u(x)\psi(x/n)$.

Exercise 19.3. Let $p \in [1,\infty)$, α be a multi index (if $\alpha = 0$ let ∂^0 be the identity operator on L^p),

$$D(\partial^\alpha) := \{f \in L^p(\mathbb{R}^n) : \partial^\alpha f \text{ exists weakly in } L^p(\mathbb{R}^n)\}$$

and for $f \in D(\partial^\alpha)$ (the domain of ∂^α) let $\partial^\alpha f$ denote the α – weak derivative of f. (See Definition 19.3.)

(1) Show ∂^α is a densely defined operator on L^p, i.e. $D(\partial^\alpha)$ is a dense linear subspace of L^p and $\partial^\alpha : D(\partial^\alpha) \to L^p$ is a linear transformation.
(2) Show $\partial^\alpha : D(\partial^\alpha) \to L^p$ is a closed operator, i.e. the graph,
$$\Gamma(\partial^\alpha) := \{(f, \partial^\alpha f) \in L^p \times L^p : f \in D(\partial^\alpha)\},$$
is a closed subspace of $L^p \times L^p$.
(3) Show $\partial^\alpha : D(\partial^\alpha) \subset L^p \to L^p$ is not bounded unless $\alpha = 0$. (The norm on $D(\partial^\alpha)$ is taken to be the L^p – norm.)

Exercise 19.4. Let $p \in [1,\infty)$, $f \in L^p$ and α be a multi index. Show $\partial^\alpha f$ exists weakly (see Definition 19.3) in L^p iff there exists $f_n \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^p$ such that $f_n \to f$ and $\partial^\alpha f_n \to g$ in L^p as $n \to \infty$. **Hints:** See exercises 19.2 and 19.3.

Exercise 19.5. Folland 8.8 on p. 246.

Exercise 19.6. Assume $n = 1$ and let $\partial = \partial_{e_1}$ where $e_1 = (1) \in \mathbb{R}^1 = \mathbb{R}$.

(1) Let $f(x) = |x|$, show ∂f exists weakly in $L^1_{loc}(\mathbb{R})$ and $\partial f(x) = \operatorname{sgn}(x)$ for m – a.e. x.

(2) Show $\partial(\partial f)$ does **not** exists weakly in $L^1_{loc}(\mathbb{R})$.
(3) Generalize item 1. as follows. Suppose $f \in C(\mathbb{R}, \mathbb{R})$ and there exists a finite set $\Lambda := \{t_1 < t_2 < \cdots < t_N\} \subset \mathbb{R}$ such that $f \in C^1(\mathbb{R} \setminus \Lambda, \mathbb{R})$. Assuming $\partial f \in L^1_{loc}(\mathbb{R})$, show ∂f exists weakly and $\partial^{(w)} f(x) = \partial f(x)$ for m – a.e. x.

Exercise 19.7. Suppose that $f \in L^1_{loc}(\Omega)$ and $v \in \mathbb{R}^d$ and $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^d$. If $\partial_j f := \partial_{e_j} f$ exists weakly in $L^1_{loc}(\Omega)$ for all $j = 1, 2, \ldots, n$ then $\partial_v f$ exists weakly in $L^1_{loc}(\Omega)$ and $\partial_v f = \sum_{j=1}^n v_j \partial_j f$.

Exercise 19.8. Suppose, $f \in L^1_{loc}(\mathbb{R}^d)$ and $\partial_v f$ exists weakly and $\partial_v f = 0$ in $L^1_{loc}(\mathbb{R}^d)$ for all $v \in \mathbb{R}^d$. Then there exists $\lambda \in \mathbb{C}$ such that $f(x) = \lambda$ for m – a.e. $x \in \mathbb{R}^d$. **Hint:** See steps 1. and 2. in the outline given in Exercise 19.9 below.

Exercise 19.9 (A generalization of Exercise 19.8)**.** Suppose Ω is a connected open subset of $\mathbb{R}^d$ and $f \in L^1_{loc}(\Omega)$. If $\partial^\alpha f = 0$ weakly for $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = N+1$, then $f(x) = p(x)$ for m – a.e. x where $p(x)$ is a polynomial of degree at most N. Here is an outline.

(1) Suppose $x_0 \in \Omega$ and $\epsilon > 0$ such that $C := C_{x_0}(\epsilon) \subset \Omega$ and let η_n be a sequence of approximate δ – functions such $\text{supp}(\eta_n) \subset B_0(1/n)$ for all n. Then for n large enough, $\partial^\alpha(f * \eta_n) = (\partial^\alpha f) * \eta_n$ on C for $|\alpha| = N+1$. Now use Taylor's theorem to conclude there exists a polynomial p_n of degree at most N such that $f_n = p_n$ on C.
(2) Show $p := \lim_{n\to\infty} p_n$ exists on C and then let $n \to \infty$ in step 1. to show there exists a polynomial p of degree at most N such that $f = p$ a.e. on C.
(3) Use Taylor's theorem to show if p and q are two polynomials on $\mathbb{R}^d$ which agree on an open set then $p = q$.
(4) Finish the proof with a connectedness argument using the results of steps 2. and 3. above.

Exercise 19.10. Suppose $\Omega \subset_o \mathbb{R}^d$ and $v, w \in \mathbb{R}^d$. Assume $f \in L^1_{loc}(\Omega)$ and that $\partial_v \partial_w f$ exists weakly in $L^1_{loc}(\Omega)$, show $\partial_w \partial_v f$ also exists weakly and $\partial_w \partial_v f = \partial_v \partial_w f$.

Exercise 19.11. Let $d = 2$ and $f(x, y) = 1_{x \geq 0}$. Show $\partial^{(1,1)} f = 0$ weakly in L^1_{loc} despite the fact that $\partial_1 f$ does not exist weakly in L^1_{loc}!