# 4. Algebras, $\sigma$ – Algebras and Measurability

## 4.1. Introduction: What are measures and why "measurable" sets.

**Definition 4.1** (Preliminary). Suppose that $X$ is a set and $\mathcal{P}(X)$ denotes the collection of all subsets of $X$. A measure $\mu$ on $X$ is a function $\mu : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. If $\{A_i\}_{i=1}^N$ is a finite $(N < \infty)$ or countable $(N = \infty)$ collection of subsets of $X$ which are pair-wise disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$\mu(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mu(A_i).$$

**Example 4.2.** Suppose that $X$ is any set and $x \in X$ is a point. For $A \subset X$, let

$$\delta_x(A) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu = \delta_x$ is a measure on $X$ called the at $x$.

**Example 4.3.** Suppose that $\mu$ is a measure on $X$ and $\lambda > 0$, then $\lambda\mu$ is also a measure on $X$. Moreover, if $\{\mu_\alpha : \alpha \in J\}$ are all measures on $X$, then $\mu = \sum_{\alpha \in J} \mu_\alpha$, i.e.

$$\mu(A) = \sum_{\alpha \in J} \mu_\alpha(A) \text{ for all } A \subset X$$

is a measure on $X$. (See Section 2 for the meaning of this sum.) We must show that $\mu$ is countably additive. Suppose that $\{A_i\}_{i=1}^\infty$ is a collection of pair-wise disjoint subsets of $X$, then

$$\begin{aligned}
\mu(\cup_{i=1}^\infty A_i) &= \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \sum_{\alpha \in J} \mu_\alpha(A_i) \\
&= \sum_{\alpha \in J} \sum_{i=1}^\infty \mu_\alpha(A_i) = \sum_{\alpha \in J} \mu_\alpha(\cup_{i=1}^\infty A_i) \\
&= \mu(\cup_{i=1}^\infty A_i)
\end{aligned}$$

where in the third equality we used Theorem 2.21 below and in the fourth we used that fact that $\mu_\alpha$ is a measure.

**Example 4.4.** Suppose that $X$ is a set $\lambda : X \to [0, \infty]$ is a function. Then

$$\mu := \sum_{x \in X} \lambda(x) \delta_x$$

is a measure, explicitly

$$\mu(A) = \sum_{x \in A} \lambda(x)$$

for all $A \subset X$.

### 4.2. **The problem with Lebesgue "measure".**

**Question 1.** Does there exist a measure $\mu : \mathcal{P}(\mathbb{R}) \rightarrow [0,\infty]$ such that

1. $\mu([a,b)) = (b-a)$ for all $a < b$ and
2. $\mu(A+x) = \mu(A)$ for all $x \in \mathbb{R}$?

The unfortunate answer is no which we now demonstrate. In fact the answer is no even if we replace (1) by the condition that $0 < \mu((0,1]) < \infty$.

Let us identify $[0,1)$ with the unit circle $S := \{z \in \mathbb{C} : |z| = 1\}$ by the map $\phi(t) = e^{i2\pi t} \in S$ for $t \in [0,1)$. Using this identification we may use $\mu$ to define a function $\nu$ on $\mathcal{P}(S)$ by $\nu(\phi(A)) = \mu(A)$ for all $A \subset [0,1)$. This new function is a measure on $S$ with the property that $0 < \nu((0,1]) < \infty$. For $z \in S$ and $N \subset S$ let

$$zN := \{zn \in S : n \in N\}, \tag{4.1}$$

that is to say $e^{i\theta}N$ is $N$ rotated counter clockwise by angle $\theta$. We now claim that $\nu$ is invariant under these rotations, i.e.

$$\nu(zN) = \nu(N) \tag{4.2}$$

for all $z \in S$ and $N \subset S$. To verify this, write $N = \phi(A)$ and $z = \phi(t)$ for some $t \in [0,1)$ and $A \subset [0,1)$. Then

$$\phi(t)\phi(A) = \phi(t + A \bmod 1)$$

where For $N \subset [0,1)$ and $\alpha \in [0,1)$, let

$$\begin{aligned} t + A \bmod 1 &= \{a + t \bmod 1 \in [0,1) : a \in N\} \\ &= (a + A \cap \{a < 1-t\}) \cup ((t-1) + A \cap \{a \geq 1-t\}). \end{aligned}$$

Thus

$$\begin{aligned} \nu(\phi(t)\phi(A)) &= \mu(t + A \bmod 1) \\ &= \mu\left((a + A \cap \{a < 1-t\}) \cup ((t-1) + A \cap \{a \geq 1-t\})\right) \\ &= \mu\left((a + A \cap \{a < 1-t\})\right) + \mu\left(((t-1) + A \cap \{a \geq 1-t\})\right) \\ &= \mu\left(A \cap \{a < 1-t\}\right) + \mu\left(A \cap \{a \geq 1-t\}\right) \\ &= \mu\left((A \cap \{a < 1-t\}) \cup (A \cap \{a \geq 1-t\})\right) \\ &= \mu(A) = \nu(\phi(A)). \end{aligned}$$

Therefore it suffices to prove that no finite measure $\nu$ on $S$ such that Eq. (4.2) holds. To do this we will "construct" a non-measurable set $N = \phi(A)$ for some $A \subset [0,1)$.

To do this let $R$ be the countable set

$$R := \{z = e^{i2\pi t} : t \in [0,1) \cap \mathbb{Q}\}.$$

As above $R$ acts on $S$ by rotations and divides $S$ up into equivalence classes, where $z, w \in S$ are equivalent if $z = rw$ for some $r \in R$. Choose (using the axiom of choice) one representative point $n$ from each of these equivalence classes and let $N \subset S$ be the set of these representative points. Then every point $z \in S$ may be uniquely written as $z = nr$ with $n \in N$ and $r \in R$. That is to say

$$S = \coprod_{r \in R} (rN) \tag{4.3}$$

where $\coprod_\alpha A_\alpha$ is used to denote the union of pair-wise disjoint sets $\{A_\alpha\}$. By Eqs. (4.2) and (4.3) we find that

$$\nu(S) = \sum_{r\in R} \nu(rN) = \sum_{r\in R} \nu(N).$$

The right member from this equation is either 0 or $\infty$, 0 if $\nu(N) = 0$ and $\infty$ if $\nu(N) > 0$. In either case it is not equal $\nu(S) \in (0,1)$. Thus we have reached the desired contradiction.

**Proof.** (Second proof) For $N \subset [0,1)$ and $\alpha \in [0,1)$, let

$$\begin{aligned} N^\alpha &= N + \alpha \operatorname{mod} 1 \\ &= \{a + \alpha \operatorname{mod} 1 \in [0,1) : a \in N\} \\ &= (\alpha + N \cap \{a < 1-\alpha\}) \cup ((\alpha - 1) + N \cap \{a \geq 1-\alpha\}). \end{aligned}$$

If $\mu$ is a measure satisfying the properties of the Question we would have

$$\begin{aligned} \mu(N^\alpha) &= \mu(\alpha + N \cap \{a < 1-\alpha\}) + \mu((\alpha-1) + N \cap \{a \geq 1-\alpha\}) \\ &= \mu(N \cap \{a < 1-\alpha\}) + \mu(N \cap \{a \geq 1-\alpha\}) \\ &= \mu(N \cap \{a < 1-\alpha\} \cup (N \cap \{a \geq 1-\alpha\})) \\ &= \mu(N). \end{aligned} \tag{4.4}$$

We will now construct a bad set $N$ which coupled with Eq. (4.4) will lead to a contradiction.

Set

$$Q_x \equiv \{x + r \in \mathbb{R} : r \in \mathbb{Q}\} = x + \mathbb{Q}.$$

Notice that $Q_x \cap Q_y \neq \emptyset$ implies that $Q_x = Q_y$. Let $\mathcal{O} = \{Q_x : x \in \mathbb{R}\}$ – the orbit space of the $\mathbb{Q}$ action. For all $A \in \mathcal{O}$ choose $f(A) \in [0,1/3) \cap A$.[7] Define $N = f(\mathcal{O})$. Then observe:

1. $f(A) = f(B)$ implies that $A \cap B \neq \emptyset$ which implies that $A = B$ so that $f$ is injective.
2. $\mathcal{O} = \{Q_n : n \in N\}$.

Let $R$ be the countable set,

$$R \equiv \mathbb{Q} \cap [0,1).$$

We now claim that

$$N^r \cap N^s = \emptyset \text{ if } r \neq s \text{ and} \tag{4.5}$$

$$[0,1) = \cup_{r\in R} N^r. \tag{4.6}$$

Indeed, if $x \in N^r \cap N^s \neq \emptyset$ then $x = r + n \operatorname{mod} 1$ and $x = s + n' \operatorname{mod} 1$, then $n - n' \in \mathbb{Q}$, i.e. $Q_n = Q_{n'}$. That is to say, $n = f(Q_n) = f(Q_{n'}) = n'$ and hence that $s = r \operatorname{mod} 1$, but $s, r \in [0,1)$ implies that $s = r$. Furthermore, if $x \in [0,1)$ and $n := f(Q_x)$, then $x - n = r \in \mathbb{Q}$ and $x \in N^{r \operatorname{mod} 1}$.

[7] We have used the Axiom of choice here, i.e. $\prod_{A\in\mathcal{F}}(A \cap [0,1/3]) \neq \emptyset$

Now that we have constructed $N$, we are ready for the contradiction. By Equations (4.4–4.6) we find

$$1 = \mu([0,1)) = \sum_{r\in R} \mu(N^r) = \sum_{r\in R} \mu(N) = \begin{cases} \infty & \text{if} \quad \mu(N) > 0 \\ 0 & \text{if} \quad \mu(N) = 0 \end{cases}.$$

which is certainly inconsistent. Incidentally we have just produced an example of so called "non – measurable" set. ■

Because of this example and our desire to have a measure $\mu$ on $\mathbb{R}$ satisfying the properties in Question 1, we need to modify our definition of a measure. We will give up on trying to measure all subsets $A \subset \mathbb{R}$, i.e. we will only try to define $\mu$ on a smaller collection of "measurable" sets. Such collections will be called $\sigma$ – algebras which we now introduce.

### 4.3. **Algebras and $\sigma$ – algebras.**

**Definition 4.5.** A collection of subsets $\mathcal{A}$ of $X$ is an **Algebra** if

1. $\emptyset, X \in \mathcal{A}$
2. $A \in \mathcal{A}$ implies that $A^c \in \mathcal{A}$
3. $\mathcal{A}$ is closed under finite unions, i.e. if $A_1, \dots, A_n \in \mathcal{A}$ then $A_1 \cap \dots \cap A_n \in \mathcal{A}$.
4. $\mathcal{A}$ is closed under finite intersections.

**Definition 4.6.** A collection of subsets $\mathcal{M}$ of $X$ is a $\sigma$ – algebra ($\sigma$ – field) if $\mathcal{M}$ is an algebra which also closed under countable unions, i.e. if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{M}$.

Notice that since $\mathcal{M}$ is also closed under taking complements, $\mathcal{M}$ is also closed under taking countable intersections.

The reader should compare these definitions with that of a topology, see Definition 3.16. Recall that the elements of a topology are called open sets. Analogously, we will often refer to elements of and algebra $\mathcal{A}$ or a $\sigma$ – algebra $\mathcal{M}$ as **measurable** sets.

**Example 4.7.** Here are a number of examples.

1. $\tau = \mathcal{M} = \mathcal{P}(X)$ in which case all subsets of $X$ are open, closed, and measurable.
2. Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, X, \{2, 3\}\}$ is a topology on $X$ which is not an algebra.
3. $\tau = \mathcal{A} = \{\{1\}, \{2, 3\}, \emptyset, X\}$. is a topology, an algebra, and a $\sigma$ – algebra on $X$. The sets $X$, $\{1\}$, $\{2, 3\}, \phi$ are open and closed. The sets $\{1, 2\}$ and $\{1, 3\}$ are neither open nor **closed** and are not measurable.

**Proposition 4.8.** *Let $\mathcal{E}$ be any collection of subsets of $X$. Then there exists a unique smallest topology $\tau(\mathcal{E})$, algebra $\mathcal{A}(\mathcal{E})$ and $\sigma$-algebra $\sigma(\mathcal{E})$ which contains $\mathcal{E}$.*

**Proof.** Note $\mathcal{P}(X)$ is a topology and an algebra and a $\sigma$-algebra and $\mathcal{E} \subseteq \mathcal{P}(X)$, so that $\mathcal{E}$ is always a subset of a topology, algebra, and $\sigma$ – algebra. One may now easily check that

$$\tau(\mathcal{E}) \equiv \bigcap \{\tau : \tau \text{ is a topology and } \mathcal{E} \subset \tau\}$$

is a topology which is clearly the smallest topology containing $\mathcal{E}$. The analogous construction works for the other cases as well. ■

We may give explicit descriptions of $\tau(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$.

**Proposition 4.9.** *Let $X$ be a set and $\mathcal{E} \subset \mathcal{P}(X)$. For simplicity of notation, assume that $X, \emptyset \in \mathcal{E}$ (otherwise adjoin them to $\mathcal{E}$ if necessary) and let $\mathcal{E}^c \equiv \{A^c : A \in \mathcal{E}\}$ and $\mathcal{E}_c = \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$ Then $\tau(\mathcal{E}) = \tau$ and $\mathcal{A}(\mathcal{E}) = \mathcal{A}$ where*

$$\tau := \{\text{arbitrary unions of finite intersections of elements from } \mathcal{E}\} \tag{4.7}$$

*and*

$$\mathcal{A} := \{\text{finite unions of finite intersections of elements from } \mathcal{E}_c\}. \tag{4.8}$$

**Proof.** From the definition of a topology and an algebra, it is clear that $\mathcal{E} \subset \tau \subset \tau(\mathcal{E})$ and $\mathcal{E} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{E})$. Hence to finish that proof it suffices to show $\tau$ is a topology and $\mathcal{A}$ is an algebra. The proof of these assertions are routine except for possibly showing that $\tau$ is closed under taking finite intersections and $\mathcal{A}$ is closed under complementation.

To check $\mathcal{A}$ is closed under complementation, let $Z \in \mathcal{A}$ be expressed as

$$Z = \bigcup_{i=1}^{N} \bigcap_{j=1}^{K} A_{ij}$$

where $A_{ij} \in \mathcal{E}_c$. Therefore, writing $B_{ij} = A_{ij}^c \in \mathcal{E}_c$, we find that

$$Z^c = \bigcap_{i=1}^{N} \bigcup_{j=1}^{K} B_{ij} = \bigcup_{j_1, \dots, j_N = 1}^{K} (B_{1j_1} \cap B_{2j_2} \cap \cdots \cap B_{Nj_N}) \in \mathcal{A}(\mathcal{E})$$

wherein we have used the fact that $B_{1j_1} \cap B_{2j_2} \cap \cdots \cap B_{Nj_N}$ is a finite intersection of sets from $\mathcal{E}_c$.

To show $\tau$ is closed under finite intersections it suffices to show for $V, W \in \tau$ that $V \cap W \in \tau$. Write

$$V = \cup_{\alpha \in A} V_\alpha \text{ and } W = \cup_{\beta \in B} W_\beta$$

where $V_\alpha$ and $W_\beta$ are sets which are finite intersection of elements from $\mathcal{E}$. Then

$$V \cap W = (\cup_{\alpha \in A} V_\alpha) \cap (\cup_{\beta \in B} W_\beta) = \bigcup_{(\alpha,\beta) \in A \times B} V_\alpha \cap W_\beta \in \tau$$

since for each $(\alpha, \beta) \in A \times B$, $V_\alpha \cap W_\beta$ is still a finite intersection of elements from $\mathcal{E}$. ■

*Remark* 4.10. One might think that in general $\sigma(\mathcal{E})$ may be described as the countable unions of countable intersections of sets in $\mathcal{E}^c$ However this is **false,** since if

$$Z = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{ij}$$

with $A_{ij} \in \mathcal{E}_c$, then

$$Z^c = \bigcup_{j_1=1, j_2=1, \dots j_N=1, \dots}^{\infty} \left( \bigcap_{\ell=1}^{\infty} A_{\ell, j_\ell}^c \right)$$

which is now an **uncountable** union. Thus the above description is not correct. In general it is fairly complicated to explicitly describe $\sigma(\mathcal{E})$, see Proposition 1.23 on page 39 of Folland for details.

**Exercise 4.1.** Let $\tau$ be a topology on a set $X$ and $\mathcal{A} = \mathcal{A}(\tau)$ be the algebra generated by $\tau$. Show $\mathcal{A}$ is the collection of subsets of $X$ which may be written as finite union of sets of the form $F \cap V$ where $F$ is closed and $V$ is open.

The following notion will be useful in the sequel.

**Definition 4.11.** A set $\mathcal{E} \subset \mathcal{P}(X)$ is said to be an **elementary family or elementary class** provided that

- $\emptyset \in \mathcal{E}$
- $\mathcal{E}$ is closed under finite intersections
- if $E \in \mathcal{E}$, then $E^c$ is a finite disjoint union of sets from $\mathcal{E}$. (In particular $X = \emptyset^c$ is a disjoint union of elements from $\mathcal{E}$.)

**Proposition 4.12.** *Suppose $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family, then $\mathcal{A} = \mathcal{A}(\mathcal{E})$ consists of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$.*

**Proof.** This could be proved making use of Proposition 4.12. However it is easier to give a direct proof.

Let $\mathcal{A}$ denote the collection of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$. Clearly $\mathcal{E} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{E})$ so it suffices to show $\mathcal{A}$ is an algebra since $\mathcal{A}(\mathcal{E})$ is the smallest algebra containing $\mathcal{E}$.

By the properties of $\mathcal{E}$, we know that $\emptyset, X \in \mathcal{A}$. Now suppose that $A_i = \coprod_{F\in\Lambda_i} F \in \mathcal{A}$ where, for $i = 1, 2, \ldots, n.$, $\Lambda_i$ is a finite collection of disjoint sets from $\mathcal{E}$. Then

$$\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \left( \coprod_{F\in\Lambda_i} F \right) = \bigcup_{(F_1,,\ldots,F_n)\in\Lambda_1\times\cdots\times\Lambda_n} (F_1 \cap F_2 \cap \cdots \cap F_n)$$

and this is a disjoint (you check) union of elements from $\mathcal{E}$. Therefore $\mathcal{A}$ is closed under finite intersections. Similarly, if $A = \coprod_{F\in\Lambda} F$ with $\Lambda$ being a finite collection of disjoint sets from $\mathcal{E}$, then $A^c = \bigcap_{F\in\Lambda} F^c$. Since by assumption $F^c \in \mathcal{A}$ for $F \in \Lambda \subset \mathcal{E}$ and $\mathcal{A}$ is closed under finite intersections, it follows that $A^c \in \mathcal{A}$. ■

**Exercise 4.2.** Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be elementary families. Show the collection

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is also an elementary family.

**Proposition 4.13.** *If $\mathcal{E} \subseteq \mathcal{P}(X)$ is countable then $\tau(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. In particular $\sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E})$.*

**Proof.** Let $\mathcal{E}_f$ denote the collection of subsets of $X$ which are finite intersection of elements from $\mathcal{E}$ along with $X$ and $\emptyset$. Notice that $\mathcal{E}_f$ is still countable (you prove). A set $Z$ is in $\tau(\mathcal{E})$ iff $Z$ is an arbitrary union of sets from $\mathcal{E}_f$. Therefore $Z = \bigcup\limits_{A\in\mathcal{F}} A$ for some subset $\mathcal{F} \subseteq \mathcal{E}_f$ which is necessarily countable. Since $\mathcal{E}_f \subseteq \sigma(\mathcal{E})$ and $\sigma(\mathcal{E})$ is closed under countable unions it follows that $Z \in \sigma(\mathcal{E})$ and hence that $\tau(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. For the last assertion, since $\mathcal{E} \subset \tau(\mathcal{E}) \subseteq \sigma(\mathcal{E})$ it follows that $\sigma(\mathcal{E}) \subset \sigma(\tau(\mathcal{E})) \subset \sigma(\mathcal{E})$. ■

The analogous notion of elementary class $\mathcal{E}$ for topologies is a basis $\mathcal{B}$ defined below.

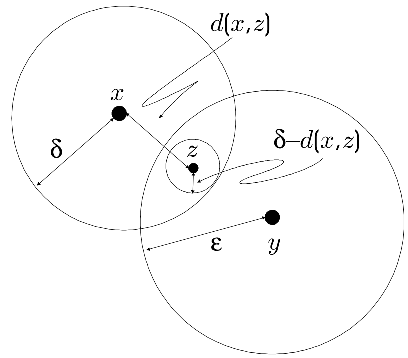


FIGURE 10. Fitting balls in the intersection.

**Definition 4.14.** Let $(X, \tau)$ be a topological space. We say that $\mathcal{S} \subset \tau$ is a **sub-basis** for the topology $\tau$ iff $\tau = \tau(\mathcal{S})$ and $X = \cup \mathcal{S} := \cup_{V \in \mathcal{S}} V$. We say $\mathcal{B} \subset \tau$ is a **basis** for the topology $\tau$ iff $\mathcal{B}$ is a sub-basis with the property that every element $V \in \tau$ may be written as

$$V = \cup \{B \in \mathcal{B} : B \subset V\}.$$

**Exercise 4.3.** Suppose that $\mathcal{S}$ is a sub-basis for a topology $\tau$ on a set $X$. Show $\mathcal{B} := \mathcal{S}_f$ consisting of finite intersections of elements from $\mathcal{S}$ is a basis for $\tau$. (So $\mathcal{S}$ is a basis for a topology iff $\cup \mathcal{S} = X$ and finite intersections of sets from $\mathcal{S}$ may be written as a union of sets from $\mathcal{S}$. Compare with the definition of an elementary class.) Moreover, $\mathcal{S}$ is itself is a basis for $\tau$ iff

$$V_1 \cap V_2 = \cup \{S \in \mathcal{S} : S \subset V_1 \cap V_2\}.$$

for every pair of sets $V_1, V_2 \in \mathcal{S}$.

*Remark* 4.15. Let $(X, d)$ be a metric space, then $\mathcal{E} = \{B_x(\delta) : x \in X \text{ and } \delta > 0\}$ is a basis for $\tau_d$ – the topology associated to the metric $d$. This is the content of Exercise 3.3.

Let us check directly that $\mathcal{E}$ is a basis for a topology. Suppose that $x, y \in X$ and $\epsilon, \delta > 0$. If $z \in B(x, \delta) \cap B(y, \epsilon)$, then

$$B(z, \alpha) \subset B(x, \delta) \cap B(y, \epsilon) \tag{4.9}$$

where $\alpha = \min\{\delta - d(x, z), \epsilon - d(y, z)\}$, see Figure 10. This is a formal consequence of the triangle inequality. For example let us show that $B(z, \alpha) \subset B(x, \delta)$. By the definition of $\alpha$, we have that $\alpha \leq \delta - d(x, z)$ or that $d(x, z) \leq \delta - \alpha$. Hence if $w \in B(z, \alpha)$, then

$$d(x, w) \leq d(x, z) + d(z, w) \leq \delta - \alpha + d(z, w) < \delta - \alpha + \alpha = \delta$$

which shows that $w \in B(x, \delta)$. Similarly we show that $w \in B(y, \epsilon)$ as well. Owing to Exercise 4.3, this shows $\mathcal{E}$ is a basis for a topology. We do not need to use Exercise 4.3 here since in fact Equation (4.9) may be generalized to finite intersection of balls. Namely if $x_i \in X$, $\delta_i > 0$ and $z \in \cap_{i=1}^n B(x_i, \delta_i)$, then

$$B(z, \alpha) \subset \cap_{i=1}^n B(x_i, \delta_i) \tag{4.10}$$

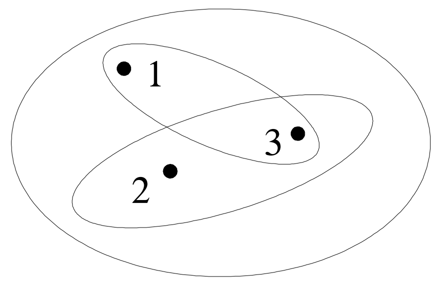


FIGURE 11. A collection of subsets.

where now $\alpha := \min\{\delta_i - d(x_i, z) : i = 1, 2, \ldots, n\}$. By Eq. (4.10) it follows that any finite intersection of open balls may be written as a union of open balls.

**Example 4.16.** Suppose that $\mathcal{E} = \{\emptyset, X, \{1,2\}, \{1,3\}\}$,
Then

$$\tau(\mathcal{E}) = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}\}$$
$$\mathcal{A}(\mathcal{E}) = \sigma(\mathcal{E}) = \mathcal{P}(X).$$

**Definition 4.17.** Let $X$ be a set. We say that a family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is a **partition** of $X$ if $X$ is the disjoint union of the sets in $\mathcal{F}$.

**Example 4.18.** Let $X$ be a set and $\mathcal{E} = \{A_1, \ldots, A_n\}$ where $A_1, \ldots, A_n$ is a partition of $X$. In this case

$$\mathcal{A}(\mathcal{E}) = \sigma(\mathcal{E}) = \tau(\mathcal{E}) = \{\cup_{i\in\Lambda} A_i : \Lambda \subset \{1, 2, \ldots, n\}\}$$

where $\cup_{i\in\Lambda} A_i := \emptyset$ when $\Lambda = \emptyset$. Notice that

$$\#\mathcal{A}(\mathcal{E}) = \#(\mathcal{P}(\{1, 2, \ldots, n\})) = 2^n.$$

**Proposition 4.19.** *Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is a $\sigma$ – algebra and $\mathcal{M}$ is at most a countable set. Then there exists a unique* ***finite*** *partition $\mathcal{F}$ of $X$ such that $\mathcal{F} \subset \mathcal{M}$ and every element $A \in \mathcal{M}$ is of the form*

$$A = \cup_{\alpha\in\mathcal{F}\ni\alpha\subset A}\,\alpha. \tag{4.11}$$

*In particular $\mathcal{M}$ is actually a finite set.*

**Proof.** For each $x \in X$ let

$$A_x = (\cap_{x\in A\in\mathcal{A}} A) \in \mathcal{A}.$$

That is, $A_x$ is the smallest set in $\mathcal{A}$ which contains $x$. Suppose that $C = A_x \cap A_y$ is non-empty. If $x \notin C$ then $x \in A_x \setminus C \in \mathcal{A}$ and hence $A_x \subset A_x \setminus C$ which shows that $A_x \cap C = \emptyset$ which is a contradiction. Hence $x \in C$ and similarly $y \in C$, therefore $A_x \subset C = A_x \cap A_y$ and $A_y \subset C = A_x \cap A_y$ which shows that $A_x = A_y$. Therefore, $\mathcal{F} = \{A_x : x \in X\}$ is a partition of $X$ (which is necessarily countable) and Eq. (4.11) holds for all $A \in \mathcal{M}$. Let $\mathcal{F} = \{P_n\}_{n=1}^N$ where for the moment we allow

$N = \infty$. If $N = \infty$, then $\mathcal{M}$ is one to one correspondence with $\{0,1\}^{\mathbb{N}}$. Indeed to each $a \in \{0,1\}^{\mathbb{N}}$, let $A_a \in \mathcal{M}$ be defined by

$$A_a = \cup\{P_n : a_n = 1\}.$$

This shows that $\mathcal{M}$ is uncountable since $\{0,1\}^{\mathbb{N}}$ is uncountable, think of the base two expansion of numbers in $[0,1]$ for example. Thus any countable $\sigma$ – algebra is necessarily finite. This finishes the proof modulo the uniqueness assertion which is left as an exercise to the reader. ■

As already mentioned the structure of general $\sigma$ – algebras is not so simple.

**Example 4.20.** Let $X = \mathbb{R}$ and

$$\mathcal{E} = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\} = \{(a,\infty) \cap \mathbb{R} : a \in \bar{\mathbb{R}}\} \subseteq \mathcal{P}(\mathbb{R}).$$

Notice that $\mathcal{E}_f = \mathcal{E}$ and that $\mathcal{E}$ is closed under unions, which shows that $\tau(\mathcal{E}) = \mathcal{E}$, i.e. $\mathcal{E}$ is already a topology. Since $(a,\infty)^c = (-\infty, a]$ we find that $\mathcal{E}_c = \{(a,\infty), (-\infty,a], -\infty \le a < \infty\} \cup \{\mathbb{R}, \emptyset\}$. Noting that

$$(a,\infty) \cap (-\infty, b] = (a,b]$$

it is easy to verify that the algebra $\mathcal{A}(\mathcal{E})$ generated by $\mathcal{E}$ may be described as being those sets which are finite disjoint unions of sets from the following list

$$\tilde{\mathcal{E}} := \left\{(a,b] \cap \mathbb{R} : a,b \in \bar{\mathbb{R}}\right\}.$$

(This follows from Proposition 4.12 and the fact that $\tilde{\mathcal{E}}$ is an elementary family of subsets of $\mathbb{R}$.) The $\sigma$ – algebra, $\sigma(\mathcal{E})$, generated by $\mathcal{E}$ is **very complicated**. Here are some sets in $\sigma(\mathcal{E})$ – most of which are not in $\mathcal{A}(\mathcal{E})$.

(a) $(a,b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] \in \sigma(\mathcal{E})$.
(b) All of the standard open subsets of $\mathbb{R}$ are in $\sigma(\mathcal{E})$.
(c) $\{x\} = \bigcap_n \left(x - \frac{1}{n}, x\right] \in \sigma(\mathcal{E})$
(d) $[a,b] = \{a\} \cup (a,b] \in \sigma(\mathcal{E})$
(e) Any countable subset of $\mathbb{R}$ is in $\sigma(\mathcal{E})$.

*Remark* 4.21. In the above example, one may replace $\mathcal{E}$ by $\mathcal{E} = \{(a,\infty) : a \in \mathbb{Q}\} \cup \{\mathbb{R}, \emptyset\}$, in which case $\mathcal{A}(\mathcal{E})$ may be described as being those sets which are finite disjoint unions of sets from the following list

$$\{(a,\infty), (-\infty,a], (a,b] : a,b \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}.$$

This shows that $\mathcal{A}(\mathcal{E})$ is a countable set – a fact we will use later on.

**Notation 4.22.** For a general topological space $(X,\tau)$, the **Borel $\sigma$ – algebra** is the $\sigma$ – algebra, $\mathcal{B}_X = \sigma(\tau)$. We will use $\mathcal{B}_{\mathbb{R}}$ to denote the Borel $\sigma$ - algebra on $\mathbb{R}$.

**Exercise 4.4.** Verify the following identities

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(a,\infty) : a \in \mathbb{R}\} = \sigma(\{(a,\infty) : a \in \mathbb{Q}\} = \sigma(\{[a,\infty) : a \in \mathbb{Q}\}).$$

4.4. **Continuous and Measurable Functions.** Our notion of a "measurable" function will be analogous to that for a continuous function. For motivational purposes, suppose $(X,\mathcal{M},\mu)$ is a measure space and $f: X \to \mathbb{R}_+$. Roughly speaking, in the next section we are going to define $\int\limits_X f d\mu$ by

$$\int\limits_X f d\mu = \lim_{\text{mesh}\to 0} \sum_{0<a_1<a_2<a_3<\dots}^{\infty} a_i \mu(f^{-1}(a_i, a_{i+1}]).$$

For this to make sense we will need to require $f^{-1}((a,b]) \in \mathcal{M}$ for all $a<b$. Because of Lemma 4.28 below, this last condition is equivalent to the condition

$$f^{-1}(\mathcal{B}_{\mathbb{R}}) \subseteq \mathcal{M},$$

where we are using the following notation.

**Notation 4.23.** If $f: X \to Y$ is a function and $\mathcal{E} \subset \mathcal{P}(Y)$ let

$$f^{-1}\mathcal{E} \equiv f^{-1}(\mathcal{E}) \equiv \{f^{-1}(E)|E \in \mathcal{E}\}.$$

If $\mathcal{G} \subset \mathcal{P}(X)$, let

$$f_*\mathcal{G} \equiv \{A \in \mathcal{P}(Y)|f^{-1}(A) \in \mathcal{G}\}.$$

**Exercise 4.5.** Show $f^{-1}\mathcal{E}$ and $f_*\mathcal{G}$ are $\sigma$ – algebras (topologies) provided $\mathcal{E}$ and $\mathcal{G}$ are $\sigma$ – algebras (topologies).

**Definition 4.24.** Let $(X,\mathcal{M})$ and $(Y,\mathcal{F})$ be measurable (topological) spaces. A function $f: X \to Y$ is **measurable (continuous)** if $f^{-1}(\mathcal{F}) \subseteq \mathcal{M}$. We will also say that $f$ is $\mathcal{M}/\mathcal{F}$ – measurable (continuous) or $(\mathcal{M},\mathcal{F})$ – measurable (continuous).

**Example 4.25** (Characteristic Functions)**.** Let $(X,\mathcal{M})$ be a measurable space and $A \subset X$. We define the characteristic function $1_A : X \to \mathbb{R}$ by

$$1_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

If $A \in \mathcal{M}$, then $1_A$ is $(\mathcal{M},\mathcal{P}(\mathbb{R}))$ – measurable because $1_A^{-1}(W)$ is either $\emptyset$, $X$, $A$ or $A^c$ for any $U \subset \mathbb{R}$. Conversely, if $\mathcal{F}$ is any $\sigma$ – algebra on $\mathbb{R}$ containing a set $W \subset \mathbb{R}$ such that $1 \in W$ and $0 \in W^c$, then $A \in \mathcal{M}$ if $1_A$ is $(\mathcal{M},\mathcal{F})$ – measurable. This is because $A = 1_A^{-1}(W) \in \mathcal{M}$.

*Remark* 4.26. Let $f: X \to Y$ be a function. Given a $\sigma$ – algebra (topology) $\mathcal{F} \subset \mathcal{P}(Y)$, the $\sigma$ – algebra (topology) $\mathcal{M} := f^{-1}(\mathcal{F})$ is the smallest $\sigma$ – algebra (topology) on $X$ such that $f$ is $(\mathcal{M},\mathcal{F})$ - measurable (continuous). Similarly, if $\mathcal{M}$ is a $\sigma$ - algebra (topology) on $X$ then $\mathcal{F} = f_*\mathcal{M}$ is the largest $\sigma$ – algebra (topology) on $Y$ such that $f$ is $(\mathcal{M},\mathcal{F})$ - measurable (continuous).

**Lemma 4.27.** *Suppose that $(X,\mathcal{M})$, $(Y,\mathcal{F})$ and $(Z,\mathcal{G})$ are measurable (topological) spaces. If $f:(X,\mathcal{M}) \to (Y,\mathcal{F})$ and $g:(Y,\mathcal{F}) \to (Z,\mathcal{G})$ are measurable (continuous) functions then $g \circ f:(X,\mathcal{M}) \to (Z,\mathcal{G})$ is measurable (continuous) as well.*

**Proof.** This is easy since by assumption $g^{-1}(\mathcal{G}) \subset \mathcal{F}$ and $f^{-1}(\mathcal{F}) \subset \mathcal{M}$ so that

$$(g \circ f)^{-1}(\mathcal{G}) = f^{-1}\left(g^{-1}(\mathcal{G})\right) \subset f^{-1}(\mathcal{F}) \subset \mathcal{M}.$$

■

**Lemma 4.28.** *Suppose that $f : X \to Y$ is a function and $\mathcal{E} \subset \mathcal{P}(Y)$, then*

$$\sigma\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\sigma(\mathcal{E})) \text{ and} \tag{4.12}$$

$$\tau\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\tau(\mathcal{E})). \tag{4.13}$$

*Moreover, if $\mathcal{F} = \sigma(\mathcal{E})$ (or $\mathcal{F} = \tau(\mathcal{E})$) and $\mathcal{M}$ is a $\sigma$ – algebra (topology) on $X$, then $f$ is $(\mathcal{M}, \mathcal{F})$ – measurable (continuous) iff $f^{-1}(\mathcal{E}) \subseteq \mathcal{M}$.*

**Proof.** We will prove Eq. (4.12), the proof of Eq. (4.13) being analogous. If $\mathcal{E} \subset \mathcal{F}$, then $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$ and therefore, (because $f^{-1}(\sigma(\mathcal{E}))$ is a $\sigma$ – algebra)

$$\mathcal{G} := \sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$$

which proves half of Eq. (4.12). For the reverse inclusion notice that

$$f_*\mathcal{G} = \left\{B \subset Y : f^{-1}(B) \in \mathcal{G}\right\}.$$

is a $\sigma$ – algebra which contains $\mathcal{E}$ and thus $\sigma(\mathcal{E}) \subset f_*\mathcal{G}$. Hence if $B \in \sigma(\mathcal{E})$ we know that $f^{-1}(B) \in \mathcal{G}$, i.e.

$$f^{-1}(\sigma(\mathcal{E})) \subset \mathcal{G}.$$

The last assertion of the Lemma is an easy consequence of Eqs. (4.12) and (4.13). ∎

**Proof.**

$$f^{-1}\left(\mathcal{B}_Y\right) = f^{-1}\left(\sigma(\tau)\right) = \sigma\left(f^{-1}(\tau)\right) \subset \mathcal{M}.$$

∎

**Definition 4.29.** A function $f : X \to Y$ between to topological spaces is **Borel measurable** if $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_X$.

**Proposition 4.30.** *Let $X$ and $Y$ be two topological spaces and $f : X \to Y$ be a continuous function. Then $f$ is Borel measurable.*

**Proof.** Using Lemma 4.28 and $\mathcal{B}_Y = \sigma(\tau_Y)$,

$$f^{-1}(\mathcal{B}_Y) = f^{-1}(\sigma(\tau_Y)) = \sigma(f^{-1}(\tau_Y)) \subset \sigma(\tau_X) = \mathcal{B}_X.$$

∎

**Corollary 4.31.** *Suppose that $(X, \mathcal{M})$ is a measurable space. Then $f : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ - measurable iff $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$ iff $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{Q}$ iff $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$, etc. Similarly, if $(X, \mathcal{M})$ is a topological space, then $f : X \to \mathbb{R}$ is $(\mathcal{M}, \tau_{\mathbb{R}})$ - continuous iff $f^{-1}((a, b)) \in \mathcal{M}$ for all $-\infty < a < b < \infty$ iff $f^{-1}((a, \infty)) \in \mathcal{M}$ and $f^{-1}((-\infty, b)) \in \mathcal{M}$ for all $a, b \in \mathbb{Q}$. (We are using $\tau_{\mathbb{R}}$ to denote the standard topology on $\mathbb{R}$ induced by the metric $d(x, y) = |x - y|$.)*

**Proof.** This is an exercise (Exercise 4.7) in using Lemma 4.28. ∎

We will often deal with functions $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let

$$\mathcal{B}_{\bar{\mathbb{R}}} := \sigma\left(\{[a, \infty] : a \in \mathbb{R}\}\right). \tag{4.14}$$

The following Corollary of Lemma 4.28 is a direct analogue of Corollary 4.31.

**Corollary 4.32.** *$f : X \to \bar{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ - measurable iff $f^{-1}((a, \infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$ iff $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$, etc.*

**Proposition 4.33.** *Let $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{\bar{\mathbb{R}}}$ be as above, then*

$$\mathcal{B}_{\bar{\mathbb{R}}} = \{A \subset \bar{\mathbb{R}} : A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}. \tag{4.15}$$

*In particular* $\{\infty\}, \{-\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}$ *and* $\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\bar{\mathbb{R}}}$.

**Proof.** Let us first observe that

$$\{-\infty\} = \cap_{n=1}^{\infty}[-\infty, -n) = \cap_{n=1}^{\infty}[-n, \infty]^c \in \mathcal{B}_{\bar{\mathbb{R}}},$$
$$\{\infty\} = \cap_{n=1}^{\infty}[n, \infty] \in \mathcal{B}_{\bar{\mathbb{R}}} \text{ and } \mathbb{R} = \bar{\mathbb{R}} \setminus \{\pm\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}.$$

Letting $i : \mathbb{R} \to \bar{\mathbb{R}}$ be the inclusion map,

$$\begin{aligned} i^{-1}(\mathcal{B}_{\bar{\mathbb{R}}}) &= \sigma\left(i^{-1}\left(\left\{[a, \infty] : a \in \bar{\mathbb{R}}\right\}\right)\right) = \sigma\left(\left\{i^{-1}([a, \infty]) : a \in \bar{\mathbb{R}}\right\}\right) \\ &= \sigma\left(\left\{[a, \infty] \cap \mathbb{R} : a \in \bar{\mathbb{R}}\right\}\right) = \sigma(\{[a, \infty) : a \in \mathbb{R}\}) = \mathcal{B}_{\mathbb{R}}. \end{aligned}$$

Thus we have shown

$$\mathcal{B}_{\mathbb{R}} = i^{-1}(\mathcal{B}_{\bar{\mathbb{R}}}) = \{A \cap \mathbb{R} : A \in \mathcal{B}_{\bar{\mathbb{R}}}\}.$$

This implies:

1. $A \in \mathcal{B}_{\bar{\mathbb{R}}} \Longrightarrow A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ and
2. if $A \subset \bar{\mathbb{R}}$ is such that $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ there exists $B \in \mathcal{B}_{\bar{\mathbb{R}}}$ such that $A \cap \mathbb{R} = B \cap \mathbb{R}$. Because $A \Delta B \subset \{\pm\infty\}$ and $\{\infty\}, \{-\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}$ we may conclude that $A \in \mathcal{B}_{\bar{\mathbb{R}}}$ as well.

This proves Eq. (4.15). ■

**Proposition 4.34** (Closure under sups, infs and limits)**.** *Suppose that $(X, \mathcal{M})$ is a measurable space and $f_j : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ is a sequence of $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. Then*

$$\sup_j f_j, \quad \inf_j f_j, \quad \limsup_{j\to\infty} f_j \text{ and } \liminf_{j\to\infty} f_j$$

*are all $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. (Note that this result is in generally false when $(X, \mathcal{M})$ is a topological space and measurable is replaced by continuous in the statement.)*

**Proof.** Define $g_+(x) := \sup_j f_j(x)$, then

$$\begin{aligned} \{x : g_+(x) \le a\} &= \{x : f_j(x) \le a \ \forall j\} \\ &= \cap_j \{x : f_j(x) \le a\} \in \mathcal{M} \end{aligned}$$

so that $g_+$ is measurable. Similarly if $g_-(x) = \inf_j f_j(x)$ then

$$\{x : g_-(x) \ge a\} = \cap_j \{x : f_j(x) \ge a\} \in \mathcal{M}.$$

Since

$$\limsup_{j\to\infty} f_j = \inf_n \sup\{f_j : j \ge n\} \text{ and}$$
$$\liminf_{j\to\infty} f_j = \sup_n \inf\{f_j : j \ge n\}$$

we are done by what we have already proved. ■

4.4.1. *More general pointwise limits.*

**Definition 4.35.** Let $(Y,\tau)$ be a topological space. A sequence $\{y_n\}_{n=1}^\infty \subset Y$ **converges** to a point $y \in Y$ if for all $V \in \tau_y$ ($\tau_y$ denotes the open neighborhoods of $y$) $y_n \in V$ for almost all $n$. We will write $y_n \to y$ to indicate the $y_n$ converges to $y$.

With this definition, it is still true that closed sets are closed under limits. Indeed, if $y_n \in C \sqsubset Y$ for all $n$ then $y_n$ can not converge to any element $y \in V := Y \setminus C$ since $V$ is open and $y_n \notin V$ for all $n$. However, limits need not be unique.

**Example 4.36.** Let $Y = \{1,2,3\}$ and $\tau = \{Y, \emptyset, \{1,2\}, \{2,3\}, \{2\}\}$ and $y_n = 2$ for all $n$. Then $y_n \to y$ for every $y \in Y$! Notice that $\sigma(\tau) = \mathcal{P}(Y)$.

**Lemma 4.37.** *Suppose that $(X,\mathcal{M})$ is a measurable space, $(Y,d)$ is a separable metric space and $f_j : X \to Y$ is $(\mathcal{M},\mathcal{B}_Y)$ – measurable for all $j$. Also assume that for each $x \in X$, $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Then $f : X \to Y$ is also $(\mathcal{M},\mathcal{B}_Y)$ – measurable.*

**Proof.** Let $V \in \tau_d$ and $W_m := \{y \in Y : d_{V^c}(y) > 1/m\}$ for $m = 1,2,\dots$. Then $W_m \in \tau_d$,

$$W_m \subset \bar{W}_m \subset \{y \in Y : d_{V^c}(y) \geq 1/m\} \subset V$$

for all $m$ and $W_m \uparrow V$ as $m \to \infty$. The proof will be completed by verifying the identity,

$$f^{-1}(V) = \cup_{m=1}^\infty \cup_{N=1}^\infty \cap_{n\geq N} f_n^{-1}(W_m) \in \mathcal{M}.$$

If $x \in f^{-1}(V)$ then $f(x) \in V$ and hence $f(x) \in W_m$ for some $m$. Since $f_n(x) \to f(x)$, $f_n(x) \in W_m$ for almost all $n$. That is $x \in \cup_{m=1}^\infty \cup_{N=1}^\infty \cap_{n\geq N} f_n^{-1}(W_m)$. Conversely when $x \in \cup_{m=1}^\infty \cup_{N=1}^\infty \cap_{n\geq N} f_n^{-1}(W_m)$ there exists an $m$ such that $f_n(x) \in W_m \subset \bar{W}_m$ for almost all $n$. Since $f_n(x) \to f(x) \in \bar{W}_m \subset V$, it follows that $x \in f^{-1}(V)$. ∎

*Remark* 4.38. In the previous Lemma 4.37 it is possible to let $(Y,\tau)$ be any topological space which has the "regularity" property that if $V \in \tau$ there exists $W_m \in \tau$ such that $W_m \subset \bar{W}_m \subset V$ and $V = \cup_{m=1}^\infty W_m$. Moreover, some extra condition is necessary on the topology $\tau$ in order for Lemma 4.37 to be correct. For example if $(Y,\tau)$ be as in Example 4.36 and $X = \{a,b\}$ with the trivial $\sigma$ – algebra. Let $f_j(a) = f_j(b) = 2$ for all $j$, then $f_j$ is constant and hence measurable. Let $f(a) = 1$ and $f(b) = 2$, then $f_j \to f$ as $j \to \infty$ with $f$ being non-measurable. Notice that the Borel $\sigma$ – algebra on $Y$ is $\mathcal{P}(Y)$.

## 4.5. **Topologies and $\sigma$ – Algebras Generated by Functions.**

**Definition 4.39.** Let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets, $A \subset X$, $i_A : A \to X$ be the inclusion map ($i_A(x) = x$) for all $x \in A$, and

$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{A \cap E : E \in \mathcal{E}\}.$$

When $\mathcal{E} = \tau$ is a topology or $\mathcal{E} = \mathcal{M}$ is a $\sigma$ – algebra we call $\tau_A$ the relative topology and $\mathcal{M}_A$ the relative $\sigma$ – algebra on $A$.

**Proposition 4.40.** *Suppose that $A \subset X$, $\mathcal{M} \subset \mathcal{P}(X)$ is a $\sigma$ – algebra and $\tau \subset \mathcal{P}(X)$ is a topology, then $\mathcal{M}_A \subset \mathcal{P}(A)$ is a $\sigma$ – algebra and $\tau_A \subset \mathcal{P}(A)$ is a topology. (The topology $\tau_A$ is called the relative topology on $A$.) Moreover if $\mathcal{E} \subset \mathcal{P}(X)$ is such that $\mathcal{M} = \sigma(\mathcal{E})$ ($\tau = \tau(\mathcal{E})$) then $\mathcal{M}_A = \sigma(\mathcal{E}_A)$ ($\tau_A = \tau(\mathcal{E}_A)$).*

**Proof.** The first assertion is Exercise 4.5 and the second assertion is a consequence of Lemma 4.28. Indeed,

$$\mathcal{M}_A = i_A^{-1}(\mathcal{M}) = i_A^{-1}(\sigma(\mathcal{E})) = \sigma(i_A^{-1}(\mathcal{E})) = \sigma(\mathcal{E}_A)$$

and similarly

$$\tau_A = i_A^{-1}(\tau) = i_A^{-1}(\tau(\mathcal{E})) = \tau(i_A^{-1}(\mathcal{E})) = \tau(\mathcal{E}_A).$$

■

**Example 4.41.** Suppose that $(X, d)$ is a metric space and $A \subset X$ is a set. Let $\tau = \tau_d$ and $d_A := d|_{A\times A}$ be the metric $d$ restricted to $A$. Then $\tau_A = \tau_{d_A}$, i.e. the relative topology, $\tau_A$, of $\tau_d$ on $A$ is the same as the topology induced by the restriction of the metric $d$ to $A$. Indeed, if $V \in \tau_A$ there exists $W \in \tau$ such that $V \cap A = W$. Therefore for all $x \in A$ there exists $\epsilon > 0$ such that $B_x(\epsilon) \subset W$ and hence $B_x(\epsilon) \cap A \subset V$. Since $B_x(\epsilon) \cap A = B_x^{d_A}(\epsilon)$ is a $d_A$ – ball in $A$, this shows $V$ is $d_A$ – open, i.e. $\tau_A \subset \tau_{d_A}$. Conversely, if $V \in \tau_{d_A}$, then for each $x \in A$ there exists $\epsilon_x > 0$ such that $B_x^{d_A}(\epsilon) = B_x(\epsilon) \cap A \subset V$. Therefore $V = A \cap W$ with $W := \cup_{x\in A} B_x(\epsilon) \in \tau$. This shows $\tau_{d_A} \subset \tau_A$.

**Definition 4.42.** Let $A \subset X$, $f : A \to \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a $\sigma$ – algebra and $\tau \subset \mathcal{P}(X)$ be a topology, then we say that $f|_A$ is measurable (continuous) if $f|_A$ is $\mathcal{M}_A$ – measurable ($\tau_A$ continuous).

**Proposition 4.43.** *Let $A \subset X$, $f : X \to \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a $\sigma$ – algebra and $\tau \subset \mathcal{P}(X)$ be a topology. If $f$ is $\mathcal{M}$ – measurable ($\tau$ continuous) then $f|_A$ is $\mathcal{M}_A$ measurable ($\tau_A$ continuous). Moreover if $A_n \in \mathcal{M}$ ($A_n \in \tau$) such that $X = \cup_{n=1}^\infty A_n$ and $f|A_n$ is $\mathcal{M}_{A_n}$ measurable ($\tau_{A_n}$ continuous) for all $n$, then $f$ is $\mathcal{M}$ – measurable ($\tau$ continuous).*

**Proof.** Notice that $i_A$ is $(\mathcal{M}_A, \mathcal{M})$ – measurable $(\tau_A, \tau)$ – continuous) hence $f|_A = f \circ i_A$ is $\mathcal{M}_A$ measurable ($\tau_A$ – continuous). Let $B \subset \mathbb{C}$ be a Borel set and consider

$$f^{-1}(B) = \cup_{n=1}^\infty \left(f^{-1}(B) \cap A_n\right) = \cup_{n=1}^\infty f|_{A_n}^{-1}(B).$$

If $A \in \mathcal{M}$ ($A \in \tau$), then it is easy to check that

$$\mathcal{M}_A = \{B \in \mathcal{M} : B \subset A\} \subset \mathcal{M} \text{ and}$$
$$\tau_A = \{B \in \tau : B \subset A\} \subset \tau.$$

The second assertion is now an easy consequence of the previous three equations. ■

**Definition 4.44.** Let $X$ and $A$ be sets, and suppose for $\alpha \in A$ we are give a measurable (topological) space $(Y_\alpha, \mathcal{F}_\alpha)$ and a function $f_\alpha : X \to Y_\alpha$. We will write $\sigma(f_\alpha : \alpha \in A)$ ($\tau(f_\alpha : \alpha \in A)$) for the smallest $\sigma$-algebra (topology) on $X$ such that each $f_\alpha$ is measurable (continuous), i.e.

$$\sigma(f_\alpha : \alpha \in A) = \sigma(\cup_\alpha f_\alpha^{-1}(\mathcal{F}_\alpha)) \text{ and}$$
$$\tau(f_\alpha : \alpha \in A) = \tau(\cup_\alpha f_\alpha^{-1}(\mathcal{F}_\alpha)).$$

**Proposition 4.45.** *Assuming the notation in Definition 4.44 and additionally let $(Z, \mathcal{M})$ be a measurable (topological) space and $g : Z \to X$ be a function. Then $g$ is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable ($(\mathcal{M}, \tau(f_\alpha : \alpha \in A))$ – continuous) iff $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$–measurable (continuous) for all $\alpha \in A$.*

**Proof.** (⇒) If $g$ is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable, then the composition $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ – measurable by Lemma 4.27.

(⇐) Let

$$\mathcal{G} = \sigma(f_\alpha : \alpha \in A) = \sigma\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right).$$

If $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ – measurable for all $\alpha$, then

$$g^{-1} f_\alpha^{-1}(\mathcal{F}_\alpha) \subseteq \mathcal{M} \,\forall\, \alpha \in A$$

and therefore

$$g^{-1}\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right) = \cup_{\alpha \in A} g^{-1} f_\alpha^{-1}(\mathcal{F}_\alpha) \subseteq \mathcal{M}.$$

Hence

$$g^{-1}(\mathcal{G}) = g^{-1}\left(\sigma\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right)\right) = \sigma(g^{-1}\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right) \subseteq \mathcal{M}$$

which shows that $g$ is $(\mathcal{M}, \mathcal{G})$ – measurable.

The topological case is proved in the same way. ■

4.6. **Product Spaces.** In this section we consider product topologies and $\sigma$ – algebras. We will start with a finite number of factors first and then later mention what happens for an infinite number of factors.

4.6.1. *Products with a Finite Number of Factors.* Let $\{X_i\}_{i=1}^n$ be a collection of sets, $X := X_1 \times X_2 \times \cdots \times X_n$ and $\pi_i : X \to X_i$ be the projection map $\pi(x_1, x_2, \ldots, x_n) = x_i$ for each $1 \le i \le n$. Let us also suppose that $\tau_i$ is a topology on $X_i$ and $\mathcal{M}_i$ is a $\sigma$ – algebra on $X_i$ for each $i$.

**Notation 4.46.** Let $\mathcal{E}_i \subset \mathcal{P}(X_i)$ be a collection of subsets of $X_i$ for $i = 1, 2, \ldots, n$ we will write, by abuse of notation, $\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ for the collection of subsets of $X_1 \times \cdots \times X_n$ of the form $A_1 \times A_2 \times \cdots \times A_n$ with $A_i \in \mathcal{E}_i$ for all $i$. That is we are identifying $(A_1, A_2, \ldots, A_n)$ with $A_1 \times A_2 \times \cdots \times A_n$.

**Definition 4.47.** The **product topology** on $X$**,** denoted by $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$ is the smallest topology on $X$ so that each map $\pi_i : X \to X_i$ is continuous. Similarly, the **product** $\sigma$ **– algebra** on $X$**,** denoted by $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n$, is the smallest $\sigma$ – algebra on $X$ so that each map $\pi_i : X \to X_i$ is measurable.

*Remark* 4.48. The product topology may also be described as the smallest topology containing sets from $\tau_1 \times \cdots \times \tau_n$, i.e.

$$\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n = \tau(\tau_1 \times \cdots \times \tau_n).$$

Indeed,

$$\begin{aligned}\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n &= \tau(\pi_1, \pi_2, \ldots, \pi_n) \\ &= \tau(\left\{\cap_{i=1}^n \pi_i^{-1}(V_i) : V_i \in \tau_i \text{ for } i = 1, 2, \ldots, n\right\}) \\ &= \tau(\{V_1 \times V_2 \times \cdots \times V_n : V_i \in \tau_i \text{ for } i = 1, 2, \ldots, n\}).\end{aligned}$$

Similarly,

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n = \sigma(\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n).$$

Furthermore if $\mathcal{B}_i \subset \tau_i$ is a basis for the topology $\tau_i$ for each $i$, then $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ is a basis for $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$. To prove this first notice that $\tau_1 \times \cdots \times \tau_n$ is closed under finite intersections and generates $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$. Therefore $\tau_1 \times \cdots \times \tau_n$ is a basis for the product topology. Hence for $W \in \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$ and $x = (x_1, \ldots, x_n) \in W$, there exists $V_1 \times V_2 \times \cdots \times V_n \in \tau_1 \times \cdots \times \tau_n$ such that

$$x \in V_1 \times V_2 \times \cdots \times V_n \subset W.$$

Since $\mathcal{B}_i$ is a basis for $\tau_i$, we may now choose $U_i \in \mathcal{B}_i$ such that $x_i \in U_i \subset V_i$ for each $i$. Thus

$$x \in U_1 \times U_2 \times \cdots \times U_n \subset W$$

and we have shown $W$ may be written as a union of sets from $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$. Since

$$\mathcal{B}_1 \times \cdots \times \mathcal{B}_n \subset \tau_1 \times \cdots \times \tau_n) \subset \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n,$$

this shows $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ is a basis for $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$.

**Lemma 4.49.** *Let $(X_i, d_i)$ for $i = 1, \ldots, n$ be metric spaces, $X := X_1 \times \cdots \times X_n$ and for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $X$ let*

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i). \tag{4.16}$$

*Then the topology, $\tau_d$, associated to the metric $d$ is the product topology on $X$, i.e.*

$$\tau_d = \tau_{d_1} \otimes \tau_{d_2} \otimes \cdots \otimes \tau_{d_n}.$$

**Proof.** Let $\rho(x, y) = \max\{d_i(x_i, y_i) : i = 1, 2, \ldots, n\}$. Then $\rho$ is equivalent to $d$ and hence $\tau_\rho = \tau_d$. Moreover if $\epsilon > 0$ and $x = (x_1, x_2, \ldots, x_n) \in X$, then

$$B_x^\rho(\epsilon) = B_{x_1}^{d_1}(\epsilon) \times \cdots \times B_{x_n}^{d_n}(\epsilon).$$

By Remark 4.15,

$$\mathcal{E} := \{B_x^\rho(\epsilon) : x \in X \text{ and } \epsilon > 0\}$$

is a basis for $\tau_\rho$ and by Remark 4.48 $\mathcal{E}$ is also a basis for $\tau_{d_1} \otimes \tau_{d_2} \otimes \cdots \otimes \tau_{d_n}$. Therefore,

$$\tau_{d_1} \otimes \tau_{d_2} \otimes \cdots \otimes \tau_{d_n} = \tau(\mathcal{E}) = \tau_\rho = \tau_d.$$

■

*Remark* 4.50. Let $(Z, \mathcal{M})$ be a measurable (topological) space, then by Proposition 4.45, a function $f : Z \to X$ is measurable (continuous) iff $\pi_i \circ f : Z \to X_i$ is $(\mathcal{M}, \mathcal{M}_i)$ – measurable $((\tau, \tau_i)$ – continuous) for $i = 1, 2, \ldots, n$. So if we write

$$f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \in X_1 \times X_2 \times \cdots \times X_n,$$

then $f : Z \to X$ is measurable (continuous) iff $f_i : Z \to X_i$ is measurable (continuous) for all $i$.

**Theorem 4.51.** *For $i = 1, 2, \ldots, n$, let $\mathcal{E}_i \subset \mathcal{P}(X_i)$ be a collection of subsets of $X_i$ such that $X_i \in \mathcal{E}_i$ and $\mathcal{M}_i = \sigma(\mathcal{E}_i)$ (or $\tau_i = \tau(\mathcal{E}_i)$) for $i = 1, 2, \ldots, n$, then*

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n = \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \text{ and}$$
$$\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n = \tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n).$$

*Written out more explicitly, these equations state*

$$\sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \text{ and} \tag{4.17}$$

$$\tau(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2) \times \cdots \times \tau(\mathcal{E}_n)) = \tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n). \tag{4.18}$$

*Let us further assume that each $\mathcal{E}_i$ is countable for $i = 1, 2, \ldots, n$, $\tau_i = \tau(\mathcal{E}_i)$ and $\mathcal{M}_i = \sigma(\tau_i)$ is the Borel $\sigma$ – algebra on $i$. Then*

1. *$\mathcal{M}_i = \sigma(\tau_i) = \sigma(\mathcal{E}_i)$ for all $i$ and*
2. *the Borel $\sigma$ – algebra on $X_1 \times X_2 \times \cdots \times X_n$ with the product topology is the product of the Borel $\sigma$ – algebras on the $X_i$ 's, i.e.*

$$\sigma(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n.$$

**Proof.** We will prove Eq. (4.17). The proof of Eq. (4.18) is completely analogous. Let us first do the case of two factors. Since

$$\mathcal{E}_1 \times \mathcal{E}_2 \subset \sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)$$

it follows that

$$\sigma\left(\mathcal{E}_1 \times \mathcal{E}_2\right) \subset \sigma\left(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)\right) = \sigma(\pi_1, \pi_2).$$

To prove the reverse inequality it suffices to show $\pi_i : X_1 \times X_2 \to X_i$ is $\sigma\left(\mathcal{E}_1 \times \mathcal{E}_2\right)$ – $\mathcal{M}_i = \sigma(\mathcal{E}_i)$ measurable for $i = 1, 2$. To prove this suppose that $E \in \mathcal{E}_1$, then

$$\pi_1^{-1}(E) = E \times X_2 \in \mathcal{E}_1 \times \mathcal{E}_2 \subset \sigma\left(\mathcal{E}_1 \times \mathcal{E}_2\right)$$

wherein we have used the fact that $X_2 \in \mathcal{E}_2$. Similarly, for $E \in \mathcal{E}_2$ we have

$$\pi_2^{-1}(E) = X_1 \times E \in \mathcal{E}_1 \times \mathcal{E}_2 \subset \sigma\left(\mathcal{E}_1 \times \mathcal{E}_2\right).$$

This proves the desired measurability, and hence

$$\sigma(\pi_1, \pi_2) \subset \sigma\left(\mathcal{E}_1 \times \mathcal{E}_2\right) \subset \sigma(\pi_1, \pi_2).$$

Let us now assume that each $\mathcal{E}_i$ is countable or $i = 1, 2$. Then it has already been proved in Proposition 4.13 that $\mathcal{M}_i = \sigma(\tau_i) = \sigma(\mathcal{E}_i)$. Moreover, $\mathcal{E}_1 \times \mathcal{E}_2$ is also countable, another application of Proposition 4.13 along with the first two assertions of the theorems gives

$$\begin{aligned}\sigma(\tau_1 \otimes \tau_2) &= \sigma(\tau\left(\tau_1 \times \tau_2\right)) = \sigma(\tau\left(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2)\right)) = \sigma(\tau\left(\mathcal{E}_1 \times \mathcal{E}_2\right)) \\ &= \sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \sigma\left(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)\right) = \sigma\left(\mathcal{M}_1 \times \mathcal{M}_2\right) = \mathcal{M}_1 \otimes \mathcal{M}_2.\end{aligned}$$

The proof for $n$ factors works the same way. Indeed,

$$\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n \subset \sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)$$

implies

$$\sigma\left(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n\right) \subset \sigma\left(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)\right) = \sigma(\pi_1, \ldots, \pi_n)$$

and for $E \in \mathcal{E}_i$,

$$\begin{aligned}\pi_i^{-1}(E) = X_1 \times X_2 \times \cdots \times X_{i-1} \times E \times X_{i+1} \cdots \times X_n &\in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n \\ &\subset \sigma\left(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n\right).\end{aligned}$$

This show $\pi_i$ is $\sigma\left(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n\right)$ – $\mathcal{M}_i = \sigma(\mathcal{E}_i)$ measurable and therefore,

$$\sigma(\pi_1, \ldots, \pi_n) \subset \sigma\left(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n\right) \subset \sigma(\pi_1, \ldots, \pi_n).$$

If the $\mathcal{E}_i$ are countable, then

$$\begin{aligned}\sigma(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n) &= \sigma(\tau(\tau_1 \times \tau_2 \times \cdots \times \tau_n)) \\ &= \sigma(\tau(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2) \times \cdots \times \tau(\mathcal{E}_n))) \\ &= \sigma(\tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n)) \\ &= \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \\ &= \sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)) \\ &= \sigma(\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n) \\ &= \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n.\end{aligned}$$

■

*Remark* 4.52. One can not relax the assumption that $X_i \in \mathcal{E}_i$ in Theorem 4.51. For example, if $X_1 = X_2 = \{1, 2\}$ and $\mathcal{E}_1 = \mathcal{E}_2 = \{\{1\}\}$, then $\sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \{\emptyset, X_1 \times X_2, \{(1, 1)\}\}$ while $\sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)) = \mathcal{P}(X_1 \times X_2)$.

**Proposition 4.53.** *If $(X_i, d_i)$ for $i = 1, \ldots, n$ be metric spaces such that for each $i$ there a countable dense subset $D_i \subseteq X_i$, then*

$$\bigotimes_i \mathcal{B}_{X_i} = \mathcal{B}_{(X_1 \times \cdots \times X_n)}$$

*where $\mathcal{B}_{X_i}$ is the Borel $\sigma$ – algebra on $X_i$ and $\mathcal{B}_{(X_1 \times \cdots \times X_n)}$ is the Borel $\sigma$ – algebra on $X_1 \times \cdots \times X_n$ equipped with the product topology.*

**Proof.** This follows directly from Lemma 4.49 and Theorem 4.51 with

$$\mathcal{E}_i := \left\{B_x^{d_i}(\epsilon) \subset X_i : x \in D_i \text{ and } \epsilon \in \mathbb{Q} \cap (0, \infty)\right\} \text{ for } i = 1, 2, \ldots, n.$$

■

Because all norms on finite dimensional spaces are equivalent, the usual Euclidean norm on $\mathbb{R}^m \times \mathbb{R}^n$ is equivalent to the "product" norm defined by

$$\|(x, y)\|_{\mathbb{R}^m \times \mathbb{R}^n} = \|x\|_{\mathbb{R}^m} + \|y\|_{\mathbb{R}^n}.$$

Hence by Lemma 4.49, the Euclidean topology on $\mathbb{R}^{m+n}$ is the same as the product topology on $\mathbb{R}^{m+n} \cong \mathbb{R}^m \times \mathbb{R}^n$ Here we are identifying $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ by the map

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \to (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{R}^{m+n}.$$

Proposition 4.53 and these comments leads to the following corollaries.

**Corollary 4.54.** *After identifying $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ as above and letting $\mathcal{B}_{\mathbb{R}^n}$ denote the Borel $\sigma$ –algebra on $\mathbb{R}^n$, we have*

$$\mathcal{B}_{\mathbb{R}^{m+n}} = \mathcal{B}_{\mathbb{R}^n} \otimes \mathcal{B}_{\mathbb{R}^m} \text{ and } \mathcal{B}_{\mathbb{R}^n} = \overbrace{\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}}^{n\text{–}times}.$$

**Corollary 4.55.** *If $(X, \mathcal{M})$ is a measurable space, then*

$$f = (f_1, f_2, \ldots, f_n) : X \to \mathbb{R}^n$$

*is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^n})$ – measurable iff $f_i : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable for each $i$. In particular, a function $f : X \to \mathbb{C}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable.*

**Corollary 4.56.** *Let $(X, \mathcal{M})$ be a measurable space and $f, g : X \to \mathbb{C}$ be $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable functions. Then $f \pm g$ and $f \cdot g$ are also $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable.*

**Proof.** Define $F : X \to \mathbb{C}\times\mathbb{C}$, $A_\pm : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ by $F(x) = (f(x), g(x))$, $A_\pm(w, z) = w \pm z$ and $M(w, z) = wz$. Then $A_\pm$ and $M$ are continuous and hence $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$ – measurable. Also $F$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}} \otimes \mathcal{B}_{\mathbb{C}}) = (\mathcal{M}, \mathcal{B}_{\mathbb{C}^2})$ – measurable since $\pi_1 \circ F = f$ and $\pi_2 \circ F = g$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable. Therefore $A_\pm \circ F = f \pm g$ and $M \circ F = f \cdot g$, being the composition of measurable functions, are also measurable. ■

**Lemma 4.57.** *Let $\alpha \in \mathbb{C}$, $(X, \mathcal{M})$ be a measurable space and $f : X \to \mathbb{C}$ be a $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable function. Then*

$$F(x) := \begin{cases} \frac{1}{f(x)} & \text{if} \quad f(x) \neq 0 \\ \alpha & \text{if} \quad f(x) = 0 \end{cases}$$

*is measurable.*

**Proof.** Define $i : \mathbb{C} \to \mathbb{C}$ by

$$i(z) = \begin{cases} \frac{1}{z} & \text{if} \quad z \neq 0 \\ \alpha & \text{if} \quad z = 0. \end{cases}$$

For any open set $V \subset \mathbb{C}$ we have

$$i^{-1}(V) = i^{-1}(V \setminus \{0\}) \cup i^{-1}(V \cap \{0\})$$

Because $i$ is continuous except at $z = 0$, $i^{-1}(V \setminus \{0\})$ is an open set and hence in $\mathcal{B}_{\mathbb{C}}$. Moreover, $i^{-1}(V \cap \{0\}) \in \mathcal{B}_{\mathbb{C}}$ since $i^{-1}(V \cap \{0\})$ is either the empty set or the one point set $\{\alpha\}$. Therefore $i^{-1}(\tau_{\mathbb{C}}) \subseteq \mathcal{B}_{\mathbb{C}}$ and hence $i^{-1}(\mathcal{B}_{\mathbb{C}}) = i^{-1}(\sigma(\tau_{\mathbb{C}})) = \sigma(i^{-1}(\tau_{\mathbb{C}})) \subseteq \mathcal{B}_{\mathbb{C}}$ which shows that $i$ is Borel measurable. Since $F = i \circ f$ is the composition of measurable functions, $F$ is also measurable. ■

4.6.2. *General Product spaces .*

**Definition 4.58.** Suppose$(X_\alpha, \mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of measurable spaces and let $X$ be the product space

$$X = \prod_{\alpha \in A} X_\alpha$$

and $\pi_\alpha : X \to X_\alpha$ be the canonical projection maps. Then the product $\sigma$ – algebra, $\bigotimes_\alpha \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha \equiv \sigma(\pi_\alpha : \alpha \in A) = \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right).$$

Similarly if $(X_\alpha, \mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of topological, the product topology $\bigotimes_\alpha \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha \equiv \tau(\pi_\alpha : \alpha \in A) = \tau\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right).$$

*Remark* 4.59. Let $(Z, \mathcal{M})$ be a measurable (topological) space and

$$\left(X = \prod_{\alpha\in A} X_\alpha, \bigotimes_{\alpha\in A} \mathcal{M}_\alpha\right)$$

be as in Definition 4.58. By Proposition 4.45, a function $f : Z \to X$ is measurable (continuous) iff $\pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{M}_\alpha)$ – measurable (continuous) for all $\alpha \in A$.

**Proposition 4.60.** *Suppose that* $(X_\alpha, \mathcal{M}_\alpha)_{\alpha\in A}$ *is a collection of measurable (topological) spaces and* $\mathcal{E}_\alpha \subseteq \mathcal{M}_\alpha$ *generates* $\mathcal{M}_\alpha$ *for each* $\alpha \in A$*, then*

$$\otimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \quad \left(\tau\left(\cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right)\right) \tag{4.19}$$

*Moreover, suppose that* $A$ *is either finite or countably infinite,* $X_\alpha \in \mathcal{E}_\alpha$ *for each* $\alpha \in A$*, and* $\mathcal{M}_\alpha = \sigma(\mathcal{E}_\alpha)$ *for each* $\alpha \in A$*. Then the product* $\sigma$ *– algebra satisfies*

$$\bigotimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right). \tag{4.20}$$

*Similarly if* $A$ *is finite and* $\mathcal{M}_\alpha = \tau(\mathcal{E}_\alpha)$*, then the product topology satisfies*

$$\bigotimes_{\alpha\in A}\mathcal{M}_\alpha = \tau\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right). \tag{4.21}$$

**Proof.** We will prove Eq. (4.19) in the measure theoretic case since a similar proof works in the topological category. Since $\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \cup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)$, it follows that

$$\mathcal{F} := \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right) = \bigotimes_\alpha \mathcal{M}_\alpha.$$

Conversely,

$$\mathcal{F} \supset \sigma(\pi_\alpha^{-1}(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\sigma(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\mathcal{M}_\alpha)$$

holds for all $\alpha$ implies that

$$\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha) \subset \mathcal{F}$$

and hence that $\bigotimes_\alpha \mathcal{M}_\alpha \subseteq \mathcal{F}$.

We now prove Eq. (4.20). Since we are assuming that $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, we see that

$$\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}$$

and therefore by Eq. (4.19)

$$\bigotimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right).$$

This last statement is true independent as to whether $A$ is countable or not. For the reverse inclusion it suffices to notice that since $A$ is countable,

$$\prod_{\alpha\in A} E_\alpha = \cap_{\alpha\in A}\pi_\alpha^{-1}(E_\alpha) \in \bigotimes_{\alpha\in A}\mathcal{M}_\alpha$$

and hence

$$\sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right) \subset \bigotimes_{\alpha\in A}\mathcal{M}_\alpha.$$

Here is a generalization of Theorem 4.51 to the case of countable number of factors.
■

**Proposition 4.61.** *Let $\{X_\alpha\}_{\alpha\in A}$ be a sequence of sets where $A$ is at most countable. Suppose for each $\alpha \in A$ we are given a countable set $\mathcal{E}_\alpha \subset \mathcal{P}(X_\alpha)$. Let $\tau_\alpha = \tau(\mathcal{E}_\alpha)$ be the topology on $X_\alpha$ generated by $\mathcal{E}_\alpha$ and $X$ be the product space $\prod_{\alpha\in A} X_\alpha$ with equipped with the product topology $\tau := \otimes_{\alpha\in A}\tau(\mathcal{E}_\alpha)$. Then the Borel $\sigma$ – algebra $\mathcal{B}_X = \sigma(\tau)$ is the same as the product $\sigma$ – algebra:*

$$\mathcal{B}_X = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha},$$

*where $\mathcal{B}_{X_\alpha} = \sigma(\tau(\mathcal{E}_\alpha)) = \sigma(\mathcal{E}_\alpha)$ for all $\alpha \in A$.*

**Proof.** By Proposition 4.60, the topology $\tau$ may be described as the smallest topology containing $\mathcal{E} = \cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)$. Now $\mathcal{E}$ is the countable union of countable sets so is still countable. Therefore by Proposition 4.13 and Proposition 4.60 we have

$$\mathcal{B}_X = \sigma(\tau) = \sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E}) = \otimes_{\alpha\in A}\sigma(\mathcal{E}_\alpha) = \otimes_{\alpha\in A}\sigma(\tau_\alpha) = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha}.$$

■

**Lemma 4.62.** *Suppose that $(Y, \mathcal{F})$ is a measurable space and $F : X \to Y$ is a map. Then to every $(\sigma(F), \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable function, $H$ from $X \to \bar{\mathbb{R}}$, there is a $(\mathcal{F}, \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable function $h : Y \to \bar{\mathbb{R}}$ such that $H = h \circ F$.*

**Proof.** First suppose that $H = 1_A$ where $A \in \sigma(F) = F^{-1}(\mathcal{B}_{\bar{\mathbb{R}}})$. Let $J \in \mathcal{B}_{\bar{\mathbb{R}}}$ such that $A = F^{-1}(J)$ then $1_A = 1_{F^{-1}(J)} = 1_J \circ F$ and hence the Lemma is valid in this case with $h = 1_J$. More generally if $H = \sum a_i 1_{A_i}$ is a simple function, then there exists $J_i \in \mathcal{B}_{\bar{\mathbb{R}}}$ such that $1_{A_i} = 1_{J_i} \circ F$ and hence $H = h \circ F$ with $h := \sum a_i 1_{J_i}$ – a simple function on $\bar{\mathbb{R}}$.

For general $(\sigma(F), \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable function, $H$, from $X \to \bar{\mathbb{R}}$, choose simple functions $H_n$ converging to $H$. Let $h_n$ be simple functions on $\bar{\mathbb{R}}$ such that $H_n = h_n \circ F$. Then it follows that

$$H = \lim_{n\to\infty} H_n = \limsup_{n\to\infty} H_n = \limsup_{n\to\infty} h_n \circ F = h \circ F$$

where $h := \limsup_{n\to\infty} h_n$ – a measurable function from $Y$ to $\bar{\mathbb{R}}$. ■

The following is an immediate corollary of Proposition 4.45 and Lemma 4.62.

**Corollary 4.63.** *Let $X$ and $A$ be sets, and suppose for $\alpha \in A$ we are give a measurable space $(Y_\alpha, \mathcal{F}_\alpha)$ and a function $f_\alpha : X \to Y_\alpha$. Let $Y := \prod_{\alpha\in A} Y_\alpha$, $\mathcal{F} := \otimes_{\alpha\in A}\mathcal{F}_\alpha$ be the product $\sigma$ – algebra on $Y$ and $\mathcal{M} := \sigma(f_\alpha : \alpha \in A)$ be the smallest $\sigma$-algebra on $X$ such that each $f_\alpha$ is measurable. Then the function $F : X \to Y$ defined by $[F(x)]_\alpha := f_\alpha(x)$ for each $\alpha \in A$ is $(\mathcal{M}, \mathcal{F})$ – measurable and a function $H : X \to \bar{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable iff there exists a $(\mathcal{F}, \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable function $h$ from $Y$ to $\bar{\mathbb{R}}$ such that $H = h \circ F$.*

## 4.7. **Exercises.**

**Exercise 4.6** (Structure of countable $\sigma$ – algebras.)**.** Removed, since this problem is covered in Proposition 4.19.

**Exercise 4.7.** Prove Corollary 4.31. **Hint:** See Exercise 4.4.

**Exercise 4.8.** Folland, Problem 1.5 on p.24. If $\mathcal{M}$ is the $\sigma$ – algebra generated by $\mathcal{E} \subset \mathcal{P}(X)$, then $\mathcal{M}$ is the union of the $\sigma$ – algebras generated by countable subsets $\mathcal{F} \subset \mathcal{E}$.

**Exercise 4.9.** Let $(X, \mathcal{M})$ be a measure space and $f_n : X \to \mathbb{F}$ be a sequence of measurable functions on $X$. Show that $\{x : \lim_{n\to\infty} f_n(x) \text{ exists}\} \in \mathcal{M}$.

**Exercise 4.10.** Show that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is $(\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})$ – measurable.

**Exercise 4.11.** Folland problem 2.6 on p. 48.

**Exercise 4.12.** Suppose that $X$ is a set, $\{(Y_\alpha, \tau_\alpha) : \alpha \in A\}$ is a family of topological spaces and $f_\alpha : X \to Y_\alpha$ is a given function for all $\alpha \in A$. Assuming that $\mathcal{S}_\alpha \subset \tau_\alpha$ is a sub-basis for the topology $\tau_\alpha$ for each $\alpha \in A$, show $\mathcal{S} := \cup_{\alpha\in A} f_\alpha^{-1}(\mathcal{S}_\alpha)$ is a sub-basis for the topology $\tau := \tau(f_\alpha : \alpha \in A)$.

**Notation 4.64.** Let $X$ be a set and $\mathbf{p} := \{p_n\}_{n=0}^\infty$ be a family of semi-metrics on $X$, i.e. $p_n : X \times X \to [0, \infty)$ are functions satisfying the assumptions of metric except for the assertion that $p_n(x, y) = 0$ implies $x = y$. Further assume that $p_n(x, y) \le p_{n+1}(x, y)$ for all $n$ and if $p_n(x, y) = 0$ for all $n \in \mathbb{N}$ then $x = y$. Given $n \in \mathbb{N}$ and $x \in X$ let

$$B_n(x, \epsilon) := \{y \in X : p_n(x, y) < \epsilon\}.$$

We will write $\tau(\mathbf{p})$ form the smallest topology on $X$ such that $p_n(x, \cdot) : X \to [0, \infty)$ is continuous for all $n \in \mathbb{N}$ and $x \in X$, i.e. $\tau(\mathbf{p}) := \tau(p_n(x\cdot) : n \in \mathbb{N} \text{ and } x \in X)$.

**Exercise 4.13.** Using Notation 4.64, show that collection of balls,

$$\mathcal{B} := \{B_n(x, \epsilon) : n \in \mathbb{N},\ x \in X \text{ and } \epsilon > 0\},$$

forms a basis for the topology $\tau(\mathbf{p})$. **Hint:** Use Exercise 4.12 to show $\mathcal{B}$ is a sub-basis for the topology $\tau(\mathbf{p})$ and then use Exercise 4.3 to show $\mathcal{B}$ is in fact a basis for the topology $\tau(\mathbf{p})$.

**Exercise 4.14.** Using the notation in 4.64, let

$$d(x, y) = \sum_{n=0}^\infty 2^{-n} \frac{p_n(x, y)}{1 + p_n(x, y)}.$$

Show $d$ is a metric on $X$ and $\tau_d = \tau(\mathbf{p})$. Conclude that a sequence $\{x_k\}_{k=1}^\infty \subset X$ converges to $x \in X$ iff

$$\lim_{k\to\infty} p_n(x_k, x) = 0 \text{ for all } n \in \mathbb{N}.$$

**Exercise 4.15.** Let $\{(X_n, d_n)\}_{n=1}^\infty$ be a sequence of metric spaces, $X := \prod_{n=1}^\infty X_n$, and for $x = (x(n))_{n=1}^\infty$ and $y = (y(n))_{n=1}^\infty$ in $X$ let

$$d(x, y) = \sum_{n=1}^\infty 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}.$$

(See Exercise 3.25.) Moreover, let $\pi_i : X \to X_i$ be the projection maps, show

$$\tau_d = \otimes_{n=1}^\infty \tau_{d_i} := \tau(\{\pi_i : i \in \mathbb{N}\}).$$

That is show the $d$ – metric topology is the same as the product topology on $X$.