1. 240C Midterm / Practice Qual, Due (in class) Friday May 17, 2002

Notation: Throughout this test, $\mathcal{B}$ denotes the Borel σ – algebra on $\mathbb{R}$, $\mathcal{B}_{[0,1]}$ is the Borel σ – algebra on $[0,1]$, m denotes Lebesgue measure on $\mathcal{B}$ and $C([0,1])$ is the Banach space of continuous functions on $[0,1]$ equipped with the supremum norm. Also $L^p([0,1]) = L^p([0,1], \mathcal{B}_{[0,1]}, m)$ and $L^p(\mathbb{R}) = L^p(\mathbb{R}, \mathcal{B}, m)$ equipped with the usual L^p – norms.

Instructions: Clearly explain and justify your answers. You may cite theorems from textbooks or that were proved in class as long as they are not what the problem explicitly asks you to prove. Make sure to state the results that you are using and be sure to verify their hypotheses.

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(1) Let X and Y be topological spaces, $A \subset X$ and $B \subset Y$ be subsets. Show $\overline{A \times B} = \bar{A} \times \bar{B}$. (If you get stuck in the general case, prove the result when X and Y are metric spaces to get partial credit.)

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(2) Compute the following limits (allowing for $\pm\infty$ as possible values) and justify the calculations:

$$\text{a)} \qquad \lim_{n\to\infty} \int_0^1 e^{nx(x-1)}\, dx,$$

$$\text{b)} \qquad \lim_{n\to\infty} \int_0^n \sum_{k=0}^{n} x^k e^{-2x}\, dx, \text{ and}$$

$$\text{c)} \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+(x+n)^2}\, dx.$$

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(3) Suppose that $G : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function and for $f \in L^2([0,1])$ and $x \in [0,1]$ let

$$Tf(x) = \int_0^1 G(x,y) f(y) dm(y). \tag{1.1}$$

(a) Show $Tf \in C([0,1])$ and that $T : L^2([0,1]) \to C([0,1])$ is a bounded operator.

(b) Show that T takes bounded subsets of $L^2([0,1])$ to precompact subsets of $C([0,1])$.

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(4) Let $f \in L^2([0,1])$, $g(x) := Tf(x)$ be as in Eq. (1.1) with $G(x,y) := \min(x,y)$. Show $g \in C^1((0,1))$ and

$$g'(x) = \int_x^1 f(y)dm(y).$$

Hint: Either (a) compute the derivative using the definition of the derivative or (b) prove the result first for "nice" f and then pass to the limit.

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(5) Let H be a separable Hilbert space and $T : H \to H$ be a linear (but not necessarily bounded map) such that for all $x \in H$,

$$M := \sup\{\|TPx\| : P \text{ is orthogonal projection with } \dim(PH) < \infty\} < \infty$$

where $PH = \{Px : x \in H\}$ is the range of P. Show T is bounded.

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(6) Suppose that $f \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{C}$ and $f * f(x) = \lambda f(x)$ for m – a.e. x. Show that $f = 0$ a.e.

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(7) Suppose μ is a complex measure on $\mathcal{B}_{[0,1]}$ such that

$$\int_{[0,1]} e^{nx} d\mu(x) = 0 \text{ for all } n \in \mathbb{N}.$$

Show $\mu \equiv 0$.