4. Algebras, σ – Algebras and Measurability

4.1. Introduction: What are measures and why "measurable" sets.

Definition 4.1 (Preliminary)**.** Suppose that X is a set and $\mathcal{P}(X)$ denotes the collection of all subsets of X. A measure μ on X is a function $\mu : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. If $\{A_i\}_{i=1}^N$ is a finite ($N < \infty$) or countable ($N = \infty$) collection of subsets of X which are pair-wise disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$\mu(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mu(A_i).$$

Example 4.2. Suppose that X is any set and $x \in X$ is a point. For $A \subset X$, let

$$\delta_x(A) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu = \delta_x$ is a measure on X called the at x.

Example 4.3. Suppose that μ is a measure on X and $\lambda > 0$, then $\lambda\mu$ is also a measure on X. Moreover, if $\{\mu_\alpha : \alpha \in J\}$ are all measures on X, then $\mu = \sum_{\alpha \in J} \mu_\alpha$, i.e.

$$\mu(A) = \sum_{\alpha \in J} \mu_\alpha(A) \text{ for all } A \subset X$$

is a measure on X. (See Section 2 for the meaning of this sum.) We must show that μ is countably additive. Suppose that $\{A_i\}_{i=1}^\infty$ is a collection of pair-wise disjoint subsets of X, then

$$\begin{aligned} \mu(\cup_{i=1}^\infty A_i) &= \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \sum_{\alpha \in J} \mu_\alpha(A_i) \\ &= \sum_{\alpha \in J} \sum_{i=1}^\infty \mu_\alpha(A_i) = \sum_{\alpha \in J} \mu_\alpha(\cup_{i=1}^\infty A_i) \\ &= \mu(\cup_{i=1}^\infty A_i) \end{aligned}$$

where in the third equality we used Theorem 2.21 below and in the fourth we used that fact that μ_α is a measure.

Example 4.4. Suppose that X is a set $\lambda : X \to [0, \infty]$ is a function. Then

$$\mu := \sum_{x \in X} \lambda(x) \delta_x$$

is a measure, explicitly

$$\mu(A) = \sum_{x \in A} \lambda(x)$$

for all $A \subset X$.

4.2. **The problem with Lebesgue "measure".**

Question 1. Does there exist a measure $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that
1. $\mu([a, b)) = (b - a)$ for all $a < b$ and
2. $\mu(A + x) = \mu(A)$ for all $x \in \mathbb{R}$?

The unfortunate answer is no which we now demonstrate. In fact the answer is no even if we replace (1) by the condition that $0 < \mu((0, 1]) < \infty$.

Let us identify $[0, 1)$ with the unit circle $S := \{z \in \mathbb{C} : |z| = 1\}$ by the map $\phi(t) = e^{i2\pi t} \in S$ for $t \in [0, 1)$. Using this identification we may use μ to define a function ν on $\mathcal{P}(S)$ by $\nu(\phi(A)) = \mu(A)$ for all $A \subset [0, 1)$. This new function is a measure on S with the property that $0 < \nu((0, 1]) < \infty$. For $z \in S$ and $N \subset S$ let

$$zN := \{zn \in S : n \in N\}, \tag{4.1}$$

that is to say $e^{i\theta} N$ is N rotated counter clockwise by angle θ. We now claim that ν is invariant under these rotations, i.e.

$$\nu(zN) = \nu(N) \tag{4.2}$$

for all $z \in S$ and $N \subset S$. To verify this, write $N = \phi(A)$ and $z = \phi(t)$ for some $t \in [0, 1)$ and $A \subset [0, 1)$. Then

$$\phi(t)\phi(A) = \phi(t + A \bmod 1)$$

where For $N \subset [0, 1)$ and $\alpha \in [0, 1)$, let

$$\begin{aligned} t + A \bmod 1 &= \{a + t \bmod 1 \in [0, 1) : a \in N\} \\ &= (a + A \cap \{a < 1 - t\}) \cup ((t - 1) + A \cap \{a \geq 1 - t\}). \end{aligned}$$

Thus

$$\begin{aligned} \nu(\phi(t)\phi(A)) &= \mu(t + A \bmod 1) \\ &= \mu\left((a + A \cap \{a < 1 - t\}) \cup ((t - 1) + A \cap \{a \geq 1 - t\})\right) \\ &= \mu\left((a + A \cap \{a < 1 - t\})\right) + \mu\left(((t - 1) + A \cap \{a \geq 1 - t\})\right) \\ &= \mu\left(A \cap \{a < 1 - t\}\right) + \mu\left(A \cap \{a \geq 1 - t\}\right) \\ &= \mu\left((A \cap \{a < 1 - t\}) \cup (A \cap \{a \geq 1 - t\})\right) \\ &= \mu(A) = \nu(\phi(A)). \end{aligned}$$

Therefore it suffices to prove that no finite measure ν on S such that Eq. (4.2) holds. To do this we will "construct" a non-measurable set $N = \phi(A)$ for some $A \subset [0, 1)$.

To do this let R be the countable set

$$R := \{z = e^{i2\pi t} : t \in [0, 1) \cap \mathbb{Q}\}.$$

As above R acts on S by rotations and divides S up into equivalence classes, where $z, w \in S$ are equivalent if $z = rw$ for some $r \in R$. Choose (using the axiom of choice) one representative point n from each of these equivalence classes and let $N \subset S$ be the set of these representative points. Then every point $z \in S$ may be uniquely written as $z = nr$ with $n \in N$ and $r \in R$. That is to say

$$S = \coprod_{r \in R} (rN) \tag{4.3}$$

where $\coprod_\alpha A_\alpha$ is used to denote the union of pair-wise disjoint sets $\{A_\alpha\}$. By Eqs. (4.2) and (4.3) we find that

$$\nu(S) = \sum_{r\in R} \nu(rN) = \sum_{r\in R} \nu(N).$$

The right member from this equation is either 0 or ∞, 0 if $\nu(N) = 0$ and ∞ if $\nu(N) > 0$. In either case it is not equal $\nu(S) \in (0,1)$. Thus we have reached the desired contradiction.

Proof. (Second proof) For $N \subset [0,1)$ and $\alpha \in [0,1)$, let

$$\begin{aligned} N^\alpha &= N + \alpha \operatorname{mod} 1 \\ &= \{a + \alpha \operatorname{mod} 1 \in [0,1) : a \in N\} \\ &= (\alpha + N \cap \{a < 1-\alpha\}) \cup ((\alpha - 1) + N \cap \{a \geq 1-\alpha\}). \end{aligned}$$

If μ is a measure satisfying the properties of the Question we would have

$$\begin{aligned} \mu(N^\alpha) &= \mu(\alpha + N \cap \{a < 1-\alpha\}) + \mu((\alpha-1) + N \cap \{a \geq 1-\alpha\}) \\ &= \mu(N \cap \{a < 1-\alpha\}) + \mu(N \cap \{a \geq 1-\alpha\}) \\ &= \mu(N \cap \{a < 1-\alpha\} \cup (N \cap \{a \geq 1-\alpha\})) \\ &= \mu(N). \end{aligned} \tag{4.4}$$

We will now construct a bad set N which coupled with Eq. (4.4) will lead to a contradiction.

Set

$$Q_x \equiv \{x + r \in \mathbb{R} : r \in \mathbb{Q}\} = x + \mathbb{Q}.$$

Notice that $Q_x \cap Q_y \neq \emptyset$ implies that $Q_x = Q_y$. Let $\mathcal{O} = \{Q_x : x \in \mathbb{R}\}$ – the orbit space of the $\mathbb{Q}$ action. For all $A \in \mathcal{O}$ choose $f(A) \in [0,1/3) \cap A$.[7] Define $N = f(\mathcal{O})$. Then observe:

1. $f(A) = f(B)$ implies that $A \cap B \neq \emptyset$ which implies that $A = B$ so that f is injective.
2. $\mathcal{O} = \{Q_n : n \in N\}$.

Let R be the countable set,

$$R \equiv \mathbb{Q} \cap [0,1).$$

We now claim that

$$N^r \cap N^s = \emptyset \text{ if } r \neq s \text{ and} \tag{4.5}$$

$$[0,1) = \cup_{r\in R} N^r. \tag{4.6}$$

Indeed, if $x \in N^r \cap N^s \neq \emptyset$ then $x = r + n \operatorname{mod} 1$ and $x = s + n' \operatorname{mod} 1$, then $n - n\prime \in \mathbb{Q}$, i.e. $Q_n = Q_{n'}$. That is to say, $n = f(Q_n) = f(Q_{n'}) = n'$ and hence that $s = r \operatorname{mod} 1$, but $s, r \in [0,1)$ implies that $s = r$. Furthermore, if $x \in [0,1)$ and $n := f(Q_x)$, then $x - n = r \in \mathbb{Q}$ and $x \in N^{r \operatorname{mod} 1}$.

[7] We have used the Axiom of choice here, i.e. $\prod_{A\in\mathcal{F}}(A \cap [0,1/3]) \neq \emptyset$

Now that we have constructed N, we are ready for the contradiction. By Equations (4.4–4.6) we find

$$1 = \mu([0,1)) = \sum_{r\in R}\mu(N^r) = \sum_{r\in R}\mu(N)$$
$$= \begin{cases} \infty & \text{if } \mu(N) > 0 \\ 0 & \text{if } \mu(N) = 0 \end{cases}.$$

which is certainly inconsistent. Incidentally we have just produced an example of so called "non – measurable" set. ■

Because of this example and our desire to have a measure μ on $\mathbb{R}$ satisfying the properties in Question 1, we need to modify our definition of a measure. We will give up on trying to measure all subsets $A \subset \mathbb{R}$, i.e. we will only try to define μ on a smaller collection of "measurable" sets. Such collections will be called σ – algebras which we now introduce.

4.3. Algebras and σ – algebras.

Definition 4.5. A collection of subsets $\mathcal{A}$ of X is an **Algebra** if

1. $\emptyset, X \in \mathcal{A}$
2. $A \in \mathcal{A}$ implies that $A^c \in \mathcal{A}$
3. $\mathcal{A}$ is closed under finite unions, i.e. if $A_1, \dots, A_n \in \mathcal{A}$ then $A_1 \cap \cdots \cap A_n \in \mathcal{A}$.
4. $\mathcal{A}$ is closed under finite intersections.

Definition 4.6. A collection of subsets $\mathcal{M}$ of X is a σ – algebra (σ – field) if $\mathcal{M}$ is an algebra which also closed under countable unions, i.e. if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{M}$.

Notice that since $\mathcal{M}$ is also closed under taking complements, $\mathcal{M}$ is also closed under taking countable intersections.

The reader should compare these definitions with that of a topology, see Definition 3.16. Recall that the elements of a topology are called open sets. Analogously, we will often refer to elements of and algebra $\mathcal{A}$ or a σ – algebra $\mathcal{M}$ as **measurable** sets.

Example 4.7. Here are a number of examples.

1. $\tau = \mathcal{M} = \mathcal{P}(X)$ in which case all subsets of X are open, closed, and measurable.
2. Let $X = \{1,2,3\}$, then $\tau = \{\emptyset, X, \{2,3\}\}$ is a topology on X which is not an algebra.
3. $\tau = \mathcal{A} = \{\{1\}, \{2,3\}, \emptyset, X\}$. is a topology, an algebra, and a σ – algebra on X. The sets X, $\{1\}$, $\{2,3\}, \phi$ are open and closed. The sets $\{1,2\}$ and $\{1,3\}$ are neither open nor **closed** and are not measurable.

Proposition 4.8. *Let $\mathcal{E}$ be any collection of subsets of X. Then there exists a unique smallest topology $\tau(\mathcal{E})$, algebra $\mathcal{A}(\mathcal{E})$ and σ-algebra $\mathcal{M}(\mathcal{E})$ which contains $\mathcal{E}$. I will also tend to write $\sigma(\mathcal{E})$ for $\mathcal{M}(\mathcal{E})$, i.e. $\sigma(\mathcal{E}) = \mathcal{M}(\mathcal{E})$. The notation $\mathcal{M}(\mathcal{E})$ is used in Folland, but $\sigma(\mathcal{E})$ is the more standard notation.*

Proof. Note $\mathcal{P}(X)$ is a topology and an algebra and a σ-algebra and $\mathcal{E} \subseteq \mathcal{P}(X)$, so that $\mathcal{E}$ is always a subset of a topology, algebra, and σ – algebra. One may now easily check that

$$\tau(\mathcal{E}) \equiv \bigcap\{\tau : \tau \text{ is a topology and } \mathcal{E} \subset \tau\}$$

is a topology which is clearly the smallest topology containing $\mathcal{E}$. The analogous construction works for the other cases as well. ■

We may give explicit descriptions of $\tau(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$.

Proposition 4.9. *Let X and $\mathcal{E} \subset \mathcal{P}(X)$. For simplicity of notation, assume that $X, \emptyset \in \mathcal{E}$ (otherwise adjoin them to $\mathcal{E}$ if necessary) and let $\mathcal{E}^c \equiv \{A^c : A \in \mathcal{E}\}$ and $\mathcal{E}_c = \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$ Then*

$$\tau(\mathcal{E}) = \{\text{arbitrary unions of finite intersections of elements from } \mathcal{E}\} \tag{4.7}$$

and

$$\mathcal{A}(\mathcal{E}) = \{\text{finite unions of finite intersections of elements from } \mathcal{E}_c\}. \tag{4.8}$$

Proof. From the definition of a topology and an algebra, it is clear that $\tau(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$ contain those sets in the right side of Eqs. (4.7) and (4.8) respectively. Hence it suffices to show that the right members of Eqs. (4.7) and (4.8form a topology and an algebra respectively. The proof of these assertions are routine except for possibly showing that

$$\mathcal{A} := \{\text{finite unions of finite intersections of element from } \mathcal{E}_c\}$$

is closed under complementation. To check this, we notice that the typical element $Z \in \mathcal{A}$ is of the form

$$Z = \bigcup_{i=1}^{N} \bigcap_{j=1}^{K} A_{ij}$$

where $A_{ij} \in \mathcal{E}_c$. Therefore, writing $B_{ij} = A_{ij}^c \in \mathcal{E}_c$, we find that

$$Z^c = \bigcap_{i=1}^{N} \bigcup_{j=1}^{K} B_{ij} = \bigcup_{j_1, \dots, j_N = 1}^{K} (B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}) \in \mathcal{A}(\mathcal{E})$$

wherein we have used the fact that $B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}$ is a finite intersection of sets from $\mathcal{E}_c$. ■

Remark 4.10. Let (X, d) be a metric space, then the associated topology τ_d on X may be described as the topology generated by $\mathcal{E} = \{B_x(\delta) : x \in X \text{ and } \delta > 0\}$ – the collection of open balls in X.

In order to see directly that the open sets are those which may be written as the union of open balls we must show that the finite intersection of open balls can be expressed as a union of open balls. Suppose $B(x, \delta)$ and $B(y, \epsilon)$ are two open balls in X and $z \in B(x, \delta) \cap B(y, \epsilon)$, then

$$B(z, \alpha) \subset B(x, \delta) \cap B(y, \epsilon) \tag{4.9}$$

where $\alpha = \min\{\delta - d(x, z), \epsilon - d(y, z)\}$, see Figure 10. This is a formal consequence of the triangle inequality. For example let us show that $B(z, \alpha) \subset B(x, \delta)$. By the definition of α, we have that $\alpha \leq \delta - d(x, z)$ or that $d(x, z) \leq \delta - \alpha$. Hence if $w \in B(z, \alpha)$, then

$$d(x, w) \leq d(x, z) + d(z, w) \leq \delta - \alpha + d(z, w) < \delta - \alpha + \alpha = \delta$$

which shows that $w \in B(x, \delta)$. Similarly we show that $w \in B(y, \epsilon)$ as well.

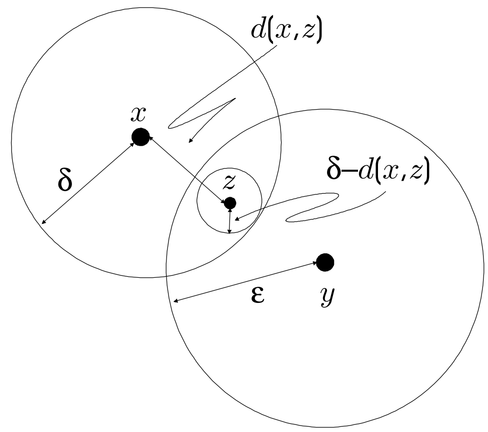

FIGURE 10. Fitting balls in the intersection.

Proof. Equation (4.9) may be generalized to finite intersection of balls, namely if $x_i \in X$, $\delta_i > 0$ and $z \in \cap_{i=1}^n B(x_i, \delta_i)$, then

$$B(z, \alpha) \subset \cap_{i=1}^n B(x_i, \delta_i) \tag{4.10}$$

where now $\alpha := \min\{\delta_i - d(x_i, z) : i = 1, 2, \ldots, n\}$. By Eq. (4.10) it follows that any finite intersection of open balls may be written as a union of open balls. ■

Remark 4.11. One might think that in general $\mathcal{M}(\mathcal{E})$ may be described as the countable unions of countable intersections of sets in $\mathcal{E}^c$ However this is **false,** since if

$$Z = \bigcup_{i=1}^\infty \bigcap_{j=1}^\infty A_{ij}$$

with $A_{ij} \in \mathcal{E}_c$, then

$$Z^c = \bigcup_{j_1=1, j_2=1, \ldots j_N=1, \ldots}^\infty \left(\bigcap_{\ell=1}^\infty B_{1 j_\ell} \right)$$

which is now an **uncountable** union. Thus the above description is not correct. In general it is fairly complicated to explicitly describe $\mathcal{M}(\mathcal{E})$, see Proposition 1.23 on page 39 of Folland for details.

Exercise 4.1. Let τ be a topology on a set X and $\mathcal{A} = \mathcal{A}(\tau)$ be the algebra generated by τ. Show $\mathcal{A}$ is the collection of subsets of X which may be written as finite union of sets of the form $F \cap V$ where F is closed and V is open.

The following notion will be useful in the sequel.

Definition 4.12. A set $\mathcal{E} \subset \mathcal{P}(X)$ is said to be an **elementary family or elementary class** provided that

- $\emptyset \in \mathcal{E}$
- $\mathcal{E}$ is closed under finite intersections
- if $E \in \mathcal{E}$, then E^c is a finite disjoint union of sets from $\mathcal{E}$.

Proposition 4.13. *Suppose $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family, then $\mathcal{A} = \mathcal{A}(\mathcal{E})$ consists of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$.*

Proof. (First Proof.) By Proposition 4.9

$$\mathcal{A}(\mathcal{E}) = \{\text{finite unions of finite intersections of elements from } \mathcal{E}_c\},$$

where $\mathcal{E}^c \equiv \{A^c : A \in \mathcal{E}\}$ and $\mathcal{E}_c = \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$. Using the definition of an elementary family we see that $\mathcal{A}(\mathcal{E})$ may be described more simply as

$$\mathcal{A}(\mathcal{E}) = \{\text{finite unions of elements from } \mathcal{E}\}.$$

Let $A = \cup_{i=1}^n E_i \in \mathcal{A}(\mathcal{E})$ with $E_i \in \mathcal{E}$. To finish the proof we need to show that A may be written disjoint union of elements from $\mathcal{E}$. We prove this by induction on n. For $n = 1$ and $A = E_1$ there is nothing to prove. If $n = 2$ and $A = E_1 \cup E_2$, let $E_2^c = \coprod_{i=1}^k F_i$ with $F_i \in \mathcal{E}$. Then

$$E_2 \backslash E_1 = E_2 \cap E_1^c = \coprod_{i=1}^k E_2 \cap F_i$$

so that

$$A = E_1 \cup \left(\coprod_{i=1}^k E_2 \cap F_i \right)$$

is the desired decomposition. Now for the induction step, suppose that

$$A = \cup_{i=1}^n E_i = B \cup E_n = (B \backslash E_n) \cup E_n$$

where $B = \coprod_{j=1}^N E_j'$ where $\left\{E_j'\right\} \subset \mathcal{E}$ are pairwise disjoint. Write $E_n^c = \coprod_{i=1}^k F_i$ with $F_i \in \mathcal{E}$, then

$$B \backslash E_n = B \cap E_n^c = \coprod_{i=1}^k B \cap F_i = \coprod_{i=1}^k \coprod_{j=1}^N E_j' \cap F_i$$

and hence

$$A = \left(\coprod_{i=1}^k \coprod_{j=1}^N E_j' \cap F_i \right) \coprod E_n$$

is the desired decomposition.

(**Second more direct proof.**) Let $\mathcal{A}$ denote the collection of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$. Clearly $\mathcal{A} \subset \mathcal{A}(\mathcal{E})$ so it suffices to show that $\mathcal{A}$ is an algebra. By the properties of $\mathcal{E}$, we know that $\emptyset, X \in \mathcal{A}$. Further, if $A = \coprod_{i=1}^n E_i$ with $E_i \in \mathcal{A}$, then

$$A^c = \cap_{i=1}^n E_i^c.$$

Since $\mathcal{E}$ is an elementary class, for each i there exists a collection of disjoint sets $\{F_{ij}\}_j \subset \mathcal{E}$ such that $E_i^c = \coprod_j F_{ij}$. Therefore,

$$A^c = \cap_{i=1}^n (\cup_j F_{ij}) = \bigcup_{j_1, j_2, \ldots, j_n} (F_{1j_1} \cap F_{2j_2} \cap \cdots \cap F_{n,j_n})$$

and this is a disjoint union. Hence $A^c \in \mathcal{A}$, i.e. $\mathcal{A}$ is closed under complementation.

Now suppose that $A_i = \coprod_j F_{ij} \in \mathcal{A}$ for $i = 1, 2, \ldots, n$, then

$$\cap_i A_i = \bigcup_{j_1, j_2, \ldots, j_n} (F_{1j_1} \cap F_{2j_2} \cap \cdots \cap F_{n,j_n})$$

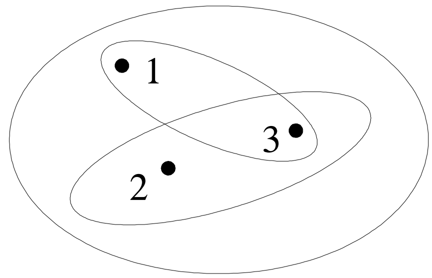

FIGURE 11. A collection of subsets.

which is again a disjoint unison of sets from $\mathcal{E}$ so that $\mathcal{A}$ is closed under finite intersections. ■

Exercise 4.2. Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be elementary families. Show the collection

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is also an elementary family.

Proposition 4.14. *If $\mathcal{E} \subseteq \mathcal{P}(X)$ is countable then $\tau(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E}) = \sigma(\mathcal{E})$. In particular $\sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E})$.*

Proof. Let $\mathcal{E}_f$ denote the collection of subsets of X which are finite intersection of elements from $\mathcal{E}$ along with X and $\emptyset$. Notice that $\mathcal{E}_f$ is still countable (you prove). A set Z is in $\tau(\mathcal{E})$ iff Z is an arbitrary union of sets from $\mathcal{E}_f$. Therefore $Z = \bigcup_{A \in \mathcal{F}} A$ for some subset $\mathcal{F} \subseteq \mathcal{E}_f$ which is necessarily countable. Since $\mathcal{E}_f \subseteq \mathcal{M}(\mathcal{E})$ and $\mathcal{M}(\mathcal{E})$ is closed under countable unions it follows that $Z \in \mathcal{M}(\mathcal{E})$ and hence that $\tau(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E})$. ■

Example 4.15. Suppose that $\mathcal{E} = \{\emptyset, X, \{1,2\}, \{1,3\}\}$,
Then

$$\tau(\mathcal{E}) = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}\}$$
$$\mathcal{A}(\mathcal{E}) = \mathcal{M}(\mathcal{E}) = \mathcal{P}(X).$$

Example 4.16. Let X be a set and $\mathcal{E} = \{A_1, \dots, A_n\} \cup \{X, \emptyset\}$ where $A_1, \dots, A_n$ is a partition of X, i.e. $X = \cup_{j=1}^n A_j$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. In this case

$$\mathcal{A}(\mathcal{E}) = \mathcal{M}(\mathcal{E}) = \tau(\mathcal{E}) = \{\cup_{i \in \Lambda} A_i : \Lambda \subset \{1, 2, \dots, n\}\}$$

where $\cup_{i \in \Lambda} A_i := \emptyset$ when $\Lambda = \emptyset$. Notice that

$$\#\mathcal{A}(\mathcal{E}) = \#(\mathcal{P}(\{1, 2, \dots, n\})) = 2^n.$$

Proposition 4.17. *Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is a σ – algebra and $\mathcal{M}$ is at most a countable set. Then there exists a unique **finite** partition $\mathcal{F}$ of X such that $\mathcal{F} \subset \mathcal{M}$ and every element $A \in \mathcal{M}$ is of the form*

$$A = \cup_{\alpha \in \mathcal{F} \ni \alpha \subset A} \alpha. \tag{4.11}$$

In particular $\mathcal{M}$ is actually a finite set.

Proof. For each $x \in X$ let

$$A_x = (\cap_{x \in A \in \mathcal{A}} A) \in \mathcal{A}.$$

That is, A_x is the smallest set in $\mathcal{A}$ which contains x. Suppose that $C = A_x \cap A_y$ is non-empty. If $x \notin C$ then $x \in A_x \setminus C \in \mathcal{A}$ and hence $A_x \subset A_x \setminus C$ which shows that $A_x \cap C = \emptyset$ which is a contradiction. Hence $x \in C$ and similarly $y \in C$, therefore $A_x \subset C = A_x \cap A_y$ and $A_y \subset C = A_x \cap A_y$ which shows that $A_x = A_y$. Therefore, $\mathcal{F} = \{A_x : x \in X\}$ is a partition of X (which is necessarily countable) and Eq. (4.11) holds for all $A \in \mathcal{M}$. Let $\mathcal{F} = \{P_n\}_{n=1}^N$ where for the moment we allow $N = \infty$. If $N = \infty$, then $\mathcal{M}$ is one to one correspondence with $\{0,1\}^{\mathbb{N}}$. Indeed to each $a \in \{0,1\}^{\mathbb{N}}$, let $A_a \in \mathcal{M}$ be defined by

$$A_a = \cup\{P_n : a_n = 1\}.$$

This shows that $\mathcal{M}$ is uncountable since $\{0,1\}^{\mathbb{N}}$ is uncountable, think of the base two expansion of numbers in $[0,1]$ for example. Thus any countable σ – algebra is necessarily finite. This finishes the proof modulo the uniqueness assertion which is left as an exercise to the reader. ■

Unfortunately, as already mentioned the structure of general σ – algebras is not so simple.

Example 4.18. Let $X = \mathbb{R}$ and $\mathcal{E} = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\} \subseteq \mathcal{P}(\mathbb{R})$. Notice that $\mathcal{E}_f = \mathcal{E}$ and that $\mathcal{E}$ is closed under unions, which shows that $\tau(\mathcal{E}) = \mathcal{E}$, i.e. $\mathcal{E}$ is already a topology. Since $(a,\infty)^c = (-\infty, a]$ we find that $\mathcal{E}_c = \{(a,\infty), (-\infty,a], -\infty \le a < \infty\} \cup \{\mathbb{R}, \emptyset\}$. Noting that

$$(a,\infty) \cap (-\infty, b] = (a,b]$$

it is easy to verify that the algebra $\mathcal{A}(\mathcal{E})$ generated by $\mathcal{E}$ may be described as being those sets which are finite disjoint unions of sets from the following list

$$\{(a,\infty), (-\infty,a], (a,b] : a,b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

The σ – algebra, $\sigma(\mathcal{E})$, generated by $\mathcal{E}$ is **very complicated**. Here are some sets in $\sigma(\mathcal{E})$.

(a) $(a,b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] \in \sigma(\mathcal{E})$.
(b) All of the standard open subsets of $\mathbb{R}$ are in $\sigma(\mathcal{E})$.
(c) $\{x\} = \bigcap_n \left(x - \frac{1}{n}, x\right] \in \sigma(\mathcal{E})$
(d) $[a,b] = \{a\} \cup (a,b] \in \sigma(\mathcal{E})$
(e) Any countable subset of $\mathbb{R}$ is in $\sigma(\mathcal{E})$.

Remark 4.19. In the above example, one may replace $\mathcal{E}$ by $\mathcal{E} = \{(a,\infty) : a \in \mathbb{Q}\} \cup \{\mathbb{R}, \emptyset\}$, in which case $\mathcal{A}(\mathcal{E})$ may be described as being those sets which are finite disjoint unions of sets from the following list

$$\{(a,\infty), (-\infty,a], (a,b] : a,b \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}.$$

This shows that $\mathcal{A}(\mathcal{E})$ is a countable set – a fact we will use later on.

Notation 4.20. For a general topological space (X,τ), the **Borel σ – algebra** is the σ – algebra, $\mathcal{B}_X = \sigma(\tau)$. We will use $\mathcal{B}_{\mathbb{R}}$ to denote the Borel σ - algebra on $\mathbb{R}$.

Exercise 4.3. Verify the following identities

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(a,\infty) : a \in \mathbb{R}\} = \sigma(\{(a,\infty) : a \in \mathbb{Q}\} = \sigma(\{[a,\infty) : a \in \mathbb{Q}\}).$$

4.4. **Continuous and Measurable Functions.** Our notion of a "measurable" function will be analogous to that for a continuous function. For motivational purposes, suppose $(X,\mathcal{M},\mu)$ is a measure space and $f:X\to\mathbb{R}_+$. Roughly speaking, in the next section we are going to define $\int\limits_X f d\mu$ by

$$\int\limits_X f d\mu = \lim_{\text{mesh}\to 0}\sum_{0<a_1<a_2<a_3<\dots}^{\infty} a_i\mu(f^{-1}(a_i,a_{i+1}]).$$

For this to make sense we will need to require $f^{-1}((a,b])\in\mathcal{M}$ for all $a<b$. Because of Lemma 4.25 below, this last condition is equivalent to the condition

$$f^{-1}(\mathcal{B}_{\mathbb{R}})\subseteq\mathcal{M},$$

where we are using the following notation.

Notation 4.21. If $f:X\to Y$ is a function and $\mathcal{E}\subset\mathcal{P}(Y)$ let

$$f^{-1}\mathcal{E}\equiv f^{-1}\left(\mathcal{E}\right)\equiv\{f^{-1}(E)|E\in\mathcal{E}\}.$$

If $\mathcal{G}\subset\mathcal{P}(X)$, let

$$f_*\mathcal{G}\equiv\{A\in\mathcal{P}(Y)|f^{-1}(A)\in\mathcal{G}\}.$$

Exercise 4.4. Show $f^{-1}\mathcal{E}$ and $f_*\mathcal{G}$ are σ – algebras (topologies) provided $\mathcal{E}$ and $\mathcal{G}$ are σ – algebras (topologies).

Definition 4.22. Let $(X,\mathcal{M})$ and $(Y,\mathcal{F})$ be measurable (topological) spaces. A function $f:X\to Y$ is **measurable (continuous)** if $f^{-1}(\mathcal{F})\subseteq\mathcal{M}$. We will also say that f is $\mathcal{M}/\mathcal{F}$ – measurable (continuous) or $(\mathcal{M},\mathcal{F})$ – measurable (continuous).

Remark 4.23. Let $f:X\to Y$ be a function. Given a σ – algebra (topology) $\mathcal{F}\subset\mathcal{P}(Y)$, the σ – algebra (topology) $\mathcal{M}:=f^{-1}(\mathcal{F})$ is the smallest σ – algebra (topology) on X such that f is $(\mathcal{M},\mathcal{F})$ - measurable (continuous). Similarly, if $\mathcal{M}$ is a σ - algebra (topology) on X then $\mathcal{F}=f_*\mathcal{M}$ is the largest σ – algebra (topology) on Y such that f is $(\mathcal{M},\mathcal{F})$ - measurable (continuous).

Lemma 4.24. *Suppose that $(X,\mathcal{M})$, $(Y,\mathcal{F})$ and $(Z,\mathcal{G})$ are measurable (topological) spaces. If $f:(X,\mathcal{M})\to(Y,\mathcal{F})$ and $g:(Y,\mathcal{F})\to(Z,\mathcal{G})$ are measurable (continuous) functions then $g\circ f:(X,\mathcal{M})\to(Z,\mathcal{G})$ is measurable (continuous) as well.*

Proof. This is easy since by assumption $g^{-1}(\mathcal{G})\subset\mathcal{F}$ and $f^{-1}\left(\mathcal{F}\right)\subset\mathcal{M}$ so that

$$(g\circ f)^{-1}\left(\mathcal{G}\right)=f^{-1}\left(g^{-1}\left(\mathcal{G}\right)\right)\subset f^{-1}\left(\mathcal{F}\right)\subset\mathcal{M}.$$

■

Lemma 4.25. *Suppose that $f:X\to Y$ is a function and $\mathcal{E}\subset\mathcal{P}(Y)$, then*

$$\sigma\left(f^{-1}(\mathcal{E})\right)=f^{-1}(\sigma(\mathcal{E}))\text{ and}\tag{4.12}$$

$$\tau\left(f^{-1}(\mathcal{E})\right)=f^{-1}(\tau(\mathcal{E})).\tag{4.13}$$

Moreover, if $\mathcal{F}=\sigma(\mathcal{E})$ (or $\mathcal{F}=\tau(\mathcal{E})$) and $\mathcal{M}$ is a σ – algebra (topology) on X, then f is $(\mathcal{M},\mathcal{F})$ – measurable (continuous) iff $f^{-1}(\mathcal{E})\subseteq\mathcal{M}$.

Proof. We will prove Eq. (4.12), the proof of Eq. (4.13) being analogous. If $\mathcal{E}\subset\mathcal{F}$, then $f^{-1}(\mathcal{E})\subset f^{-1}(\sigma(\mathcal{E}))$ and therefore, (because $f^{-1}(\sigma(\mathcal{E}))$ is a σ – algebra)

$$\mathcal{G}:=\sigma(f^{-1}(\mathcal{E}))\subset f^{-1}(\sigma(\mathcal{E}))$$

which proves half of Eq. (4.12). For the reverse inclusion notice that

$$f_*\mathcal{G} = \left\{B \subset Y : f^{-1}(B) \in \mathcal{G}\right\}.$$

is a $\sigma-$algebra which contains $\mathcal{E}$ and thus $\sigma(\mathcal{E}) \subset f_*\mathcal{G}$. Hence if $B \in \sigma(\mathcal{E})$ we know that $f^{-1}(B) \in \mathcal{G}$, i.e.

$$f^{-1}(\sigma(\mathcal{E})) \subset \mathcal{G}.$$

The last assertion of the Lemma is an easy consequence of Eqs. (4.12) and (4.13). ■

Corollary 4.26. *Suppose that $(X, \mathcal{M})$ is a measurable space. Then $f : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ - measurable iff $f^{-1}((a,\infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$ iff $f^{-1}((a,\infty)) \in \mathcal{M}$ for all $a \in \mathbb{Q}$ iff $f^{-1}((-\infty,a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$, etc. Similarly, if $(X, \mathcal{M})$ is a topological space, then $f : X \to \mathbb{R}$ is $(\mathcal{M}, \tau_{\mathbb{R}})$ - continuous iff $f^{-1}((a,b)) \in \mathcal{M}$ for all $-\infty < a < b < \infty$ iff $f^{-1}((a,\infty)) \in \mathcal{M}$ and $f^{-1}((-\infty,b)) \in \mathcal{M}$ for all $a, b \in \mathbb{Q}$. (We are using $\tau_{\mathbb{R}}$ to denote the standard topology on $\mathbb{R}$ induced by the metric $d(x,y) = |x-y|$.)*

Proof. An exercise in using Lemma 4.25. ■

We will often deal with functions $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let

$$\mathcal{B}_{\bar{\mathbb{R}}} := \sigma\left(\left\{[a,\infty] : a \in \bar{\mathbb{R}}\right\}\right). \tag{4.14}$$

The following Corollary of Lemma 4.25 is a direct analogue of Corollary 4.26.

Corollary 4.27. *$f : X \to \bar{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ - measurable iff $f^{-1}((a,\infty]) \in \mathcal{M}$ for all $a \in \bar{\mathbb{R}}$ iff $f^{-1}((-\infty,a]) \in \mathcal{M}$ for all $a \in \bar{\mathbb{R}}$, etc.*

Proposition 4.28. *A subset $A \subset \bar{\mathbb{R}}$ is in $\mathcal{B}_{\bar{\mathbb{R}}}$ iff $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$. In particular $\{\pm\infty\}$, $\{\infty\}$ and $\{-\infty\}$ are in $\mathcal{B}_{\bar{\mathbb{R}}}$.*

Proof. Let $i : \mathbb{R} \to \bar{\mathbb{R}}$ be the inclusion map. Since $i^{-1}(([a,\infty])) = [a,\infty) \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ for all $a \in \bar{\mathbb{R}}$, i is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\bar{\mathbb{R}}})$– measurable. In particular if $A \in \mathcal{B}_{\bar{\mathbb{R}}}$, then $i^{-1}(A) = A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$.

For the converse, we begin with the observations:

$$\begin{aligned}
\{-\infty\} &= \cap_{n=1}^{\infty}[-\infty,-n) = \cap_{n=1}^{\infty}[-n,\infty]^c \in \mathcal{B}_{\bar{\mathbb{R}}}, \\
\{\infty\} &= [\infty,\infty] \in \mathcal{B}_{\bar{\mathbb{R}}} \text{ and} \\
\mathbb{R} &= \bar{\mathbb{R}} \setminus \{\pm\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}.
\end{aligned}$$

Using these facts you may easily show that

$$\mathcal{M} = \{A \subset \mathbb{R} : A \in \mathcal{B}_{\bar{\mathbb{R}}}\}$$

is a σ – algebra on $\mathbb{R}$ which contains (a,∞) for all $a \in \mathbb{R}$. Hence $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}$, i.e. $\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\bar{\mathbb{R}}}$. Using these observations, if $A \subset \bar{\mathbb{R}}$ and $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$, then

$$A = (A \cap \mathbb{R}) \cup (A \cap \{\pm\infty\}) \in \mathcal{B}_{\bar{\mathbb{R}}}.$$

■

Proposition 4.29 (Closure under sups, infs and limits)**.** *Suppose that $(X, \mathcal{M})$ is a measurable space and $f_j : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ is a sequence of $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. Then*

$$\sup_j f_j, \quad \inf_j f_j, \quad \lim_{j\to\infty}\sup\ f_j \text{ and } \lim_{j\to\infty}\inf\ f_j$$

are all $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. (Note that this result is in generally false when $(X,\mathcal{M})$ is a topological space and measurable is replaced by continuous in the statement.)

Proof. Define $g_+(x) := \sup_j f_j(x)$, then

$$\{x : g_+(x) \le a\} = \{x : f_j(x) \le a \ \forall j\} = \cap_j \{x : f_j(x) \le a\} \in \mathcal{M}$$

so that g_+ is measurable. Similarly if $g_-(x) = \inf_j f_j(x)$ then

$$\{x : g_-(x) \ge a\} = \cap_j \{x; f_j(x) \ge a\} \in \mathcal{M}.$$

Since

$$\limsup \ f_j = \inf_n \sup \{f_j : j \ge n\} \text{ and}$$
$$\underline{\lim} \ f_j = \sup_n \inf \{f_j : j \ge n\}$$

we are done by what we have already proved. ∎

Lemma 4.30. *Suppose that $(X,\mathcal{M})$ is a measurable space, (Y,τ) be a topological space and $f_j : X \to Y$ is $(\mathcal{M},\mathcal{B}_Y)$ – measurable for all j. Also assume that for each $x \in X$, $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Then $f : X \to Y$ is also $(\mathcal{M},\mathcal{B}_Y)$ – measurable.*

Proof. Suppose that $V \subset Y$ is an open set, then

$$f^{-1}(V) = \{x : f(x) \in V\} = \{x : f_n(x) \in V \text{ for almost all } n\} = \cup_{N=1}^\infty \cap_{n=N}^\infty f_n^{-1}(V) \in \mathcal{M}$$

since $f_n^{-1}(V) \in \mathcal{M}$ because each f_n is measurable. Therefore $f^{-1}(\tau) \subset \mathcal{M}$ and thus

$$f^{-1}(\mathcal{B}_Y) = f^{-1}(\sigma(\tau)) = \sigma\left(f^{-1}(\tau)\right) \subset \mathcal{M}.$$

∎

Definition 4.31. A function $f : X \to Y$ between to topological spaces is **Borel measurable** if $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_X$.

Proposition 4.32. *Let X and Y be two topological spaces and $f : X \to Y$ be a continuous function. Then f is Borel measurable.*

Proof. Using Lemma 4.25 and $\mathcal{B}_Y = \sigma(\tau_Y)$,

$$f^{-1}(\mathcal{B}_Y) = f^{-1}(\sigma(\tau_Y)) = \sigma(f^{-1}(\tau_Y)) \subset \sigma(\tau_X) = \mathcal{B}_X.$$

∎

4.5. Relative Topologies and σ – Algebras.

Definition 4.33. Let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets, $A \subset X$, $i_A : A \to X$ be the inclusion map $(i_A(x) = x)$ for all $x \in A$, and

$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{A \cap E : E \in \mathcal{E}\}.$$

Proposition 4.34. *Suppose that $A \subset X$, $\mathcal{M} \subset \mathcal{P}(X)$ is a σ – algebra and $\tau \subset \mathcal{P}(X)$ is a topology, then $\mathcal{M}_A \subset \mathcal{P}(A)$ is a σ – algebra and $\tau_A \subset \mathcal{P}(A)$ is a topology. (The topology τ_A is called the relative topology on A.) Moreover if $\mathcal{E} \subset \mathcal{P}(X)$ is such that $\mathcal{M} = \sigma(\mathcal{E})$ $(\tau = \tau(\mathcal{E}))$ then $\mathcal{M}_A = \sigma(\mathcal{E}_A)$ $(\tau_A = \tau(\mathcal{E}_A))$.*

Proof. The first assertion is easy to check as remarked after Notation 4.21. The second assertion is a consequence of Lemma 4.25. Indeed,

$$\mathcal{M}_A = i_A^{-1}(\mathcal{M}) = i_A^{-1}(\sigma(\mathcal{E})) = \sigma(i_A^{-1}(\mathcal{E})) = \sigma(\mathcal{E}_A)$$

and similarly

$$\tau_A = i_A^{-1}(\tau) = i_A^{-1}(\tau(\mathcal{E})) = \tau(i_A^{-1}(\mathcal{E})) = \tau(\mathcal{E}_A).$$

■

Definition 4.35. Let $A \subset X$, $f : A \to \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a σ – algebra and $\tau \subset \mathcal{P}(X)$ be a topology, then we say that $f|_A$ is measurable (continuous) if $f|_A$ is $\mathcal{M}_A$ – measurable (τ_A continuous).

Proposition 4.36. *Let $A \subset X$, $f : X \to \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a σ – algebra and $\tau \subset \mathcal{P}(X)$ be a topology. If f is $\mathcal{M}$ – measurable (τ continuous) then $f|_A$ is $\mathcal{M}_A$ measurable (τ_A continuous). Moreover if $A_n \in \mathcal{M}$ ($A_n \in \tau$) such that $X = \cup_{n=1}^{\infty} A_n$ and $f|A_n$ is $\mathcal{M}_{A_n}$ measurable (τ_{A_n} continuous) for all n, then f is $\mathcal{M}$ – measurable (τ continuous).*

Proof. Notice that i_A is $(\mathcal{M}_A, \mathcal{M})$ – measurable (τ_A, τ) – continuous) hence $f|_A = f \circ i_A$ is $\mathcal{M}_A$ measurable (τ_A – continuous). Let $B \subset \mathbb{C}$ be a Borel set and consider

$$f^{-1}(B) = \cup_{n=1}^{\infty} \left(f^{-1}(B) \cap A_n \right) = \cup_{n=1}^{\infty} f|_{A_n}^{-1}(B).$$

If $A \in \mathcal{M}$ ($A \in \tau$), then it is easy to check that

$$\mathcal{M}_A = \{B \in \mathcal{M} : B \subset A\} \subset \mathcal{M} \text{ and}$$
$$\tau_A = \{B \in \tau : B \subset A\} \subset \tau.$$

The second assertion is now an easy consequence of the previous three equations. ■

Definition 4.37. Let X and A be sets, and suppose for $\alpha \in A$ we are give a measurable (topological) space $(Y_\alpha, \mathcal{F}_\alpha)$ and a function $f_\alpha : X \to Y_\alpha$. We will write $\sigma(f_\alpha : \alpha \in A)$ $(\tau(f_\alpha : \alpha \in A))$ for the smallest σ-algebra (topology) on X such that each f_α is measurable (continuous), i.e.

$$\sigma(f_\alpha : \alpha \in A) = \sigma(\cup_\alpha f_\alpha^{-1}(\mathcal{F}_\alpha)) \text{ and}$$
$$\tau(f_\alpha : \alpha \in A) = \tau(\cup_\alpha f_\alpha^{-1}(\mathcal{F}_\alpha)).$$

Proposition 4.38. *Assuming the notation in Definition 4.37 and additionally let $(Z, \mathcal{M})$ be a measurable (topological) space and $g : Z \to X$ be a function. Then g is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable ($(\mathcal{M}, \tau(f_\alpha : \alpha \in A))$ – continuous) iff $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$–measurable (continuous) for all $\alpha \in A$.*

Proof. ($\Rightarrow$) If g is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable, then the composition $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ – measurable by Lemma 4.24.
($\Leftarrow$) Let

$$\mathcal{G} = \sigma(f_\alpha : \alpha \in A) = \sigma\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right).$$

If $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ – measurable for all α, then

$$g^{-1} f_\alpha^{-1}(\mathcal{F}_\alpha) \subseteq \mathcal{M} \, \forall \, \alpha \in A$$

and therefore

$$g^{-1}\left(\cup_{\alpha\in A}f_\alpha^{-1}(\mathcal{F}_\alpha)\right)=\cup_{\alpha\in A}g^{-1}f_\alpha^{-1}(\mathcal{F}_\alpha)\subseteq\mathcal{M}.$$

Hence

$$g^{-1}(\mathcal{G})=g^{-1}\left(\sigma\left(\cup_{\alpha\in A}f_\alpha^{-1}(\mathcal{F}_\alpha)\right)\right)=\sigma(g^{-1}\left(\cup_{\alpha\in A}f_\alpha^{-1}(\mathcal{F}_\alpha)\right)\subseteq\mathcal{M}$$

which shows that g is $(\mathcal{M},\mathcal{G})$ – measurable.

The topological case is proved in the same way. ■

4.6. **Product Spaces.** In this section we consider product topologies and σ – algebras. We will start with a finite number of factors first and then later mention what happens for an infinite number of factors.

4.6.1. *Products with a Finite Number of Factors.* Let $\{X_i\}_{i=1}^n$ be a collection of sets, $X:=X_1\times X_2\times\cdots\times X_n$ and $\pi_i:X\to X_i$ be the projection map $\pi(x_1,x_2,\dots,x_n)=x_i$ for each $1\le i\le n$. Let us also suppose that τ_i is a topology on X_i and $\mathcal{M}_i$ is a σ – algebra on X_i for each i.

Definition 4.39. The **product topology** on X, denoted by $\tau_1\otimes\tau_2\otimes\cdots\otimes\tau_n$ is the smallest topology on X so that each map $\pi_i:X\to X_i$ is continuous. Similarly, the **product σ – algebra** on X, denoted by $\mathcal{M}_1\otimes\mathcal{M}_2\otimes\cdots\otimes\mathcal{M}_n$, is the smallest σ – algebra on X so that each map $\pi_i:X\to X_i$ is measurable.

Remark 4.40. The product topology may also be described as the smallest topology containing sets of the form, $V_1\times V_2\times\cdots\times V_n$ with $V_i\in\tau_i$ for $i=1,2,\dots,n$. Indeed,

$$\begin{aligned}\tau_1\otimes\tau_2\otimes\cdots\otimes\tau_n&=\tau(\pi_1,\pi_2,\dots,\pi_n)\\&=\tau(\left\{\cap_{i=1}^n\pi_i^{-1}(V_i):V_i\in\tau_i\text{ for }i=1,2,\dots,n\right\})\\&=\tau(\{V_1\times V_2\times\cdots\times V_n:V_i\in\tau_i\text{ for }i=1,2,\dots,n\}.\end{aligned}$$

Similarly, $\mathcal{M}_1\otimes\mathcal{M}_2\otimes\cdots\otimes\mathcal{M}_n$, is the smallest σ – algebra on X containing sets of the form, $A_1\times A_2\times\cdots\times A_n$ with $A_i\in\mathcal{M}_i$ for $i=1,2,\dots,n$.

Remark 4.41. If (X_i,d_i) for $i=1,\dots,n$ be metric spaces, $X:=X_1\times\cdots\times X_n$ and for $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$ in X let

$$d(x,y)=\sum_{i=1}^n d_i(x_i,y_i). \tag{4.15}$$

Then the topology, τ_d, associated to the metric d is the product topology on X, i.e.

$$\tau_d=\tau_{d_1}\otimes\tau_{d_2}\otimes\cdots\otimes\tau_{d_n}.$$

This is a consequence of Remark 4.40 and the following statement and : if $\epsilon>0$ and $x\in X$ then

$$B(x_1,\epsilon/n)\times\cdots\times B(x_n,\epsilon/n)\subset B((x_1,x_2,\dots,x_n),\epsilon)\subset B(x_1,\epsilon)\times\cdots\times B(x_n,\epsilon).$$

Remark 4.42. Let $(Z,\mathcal{M})$ be a measurable (topological) space, then by Proposition 4.38, a function $f:Z\to X$ is measurable (continuous) iff $\pi_i\circ f:Z\to X_i$ is $(\mathcal{M},\mathcal{M}_i)$ – measurable ((τ,τ_i) – continuous) for $i=1,2,\dots,n$. So if we write

$$f(z)=(f_1(z),f_2(z),\dots,f_n(z))\in X_1\times X_2\times\cdots\times X_n,$$

then $f:Z\to X$ is measurable (continuous) iff $f_i:Z\to X_i$ is measurable (continuous) for all i.

Theorem 4.43. *For $i = 1, 2, \ldots, n$, let $\mathcal{E}_i \subset \mathcal{P}(X_i)$ be a collection of subsets of X_i such that $X_i \in \mathcal{E}_i$ and $\mathcal{M}_i = \sigma(\mathcal{E}_i)$ (or $\tau_i = \tau(\mathcal{E}_i)$) for $i = 1, 2, \ldots, n$, then*

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n = \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \text{ or}$$
$$\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n = \tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n).$$

In short we have

$$\sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) = \sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)) \text{ and} \tag{4.16}$$
$$\tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) = \tau(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2) \times \cdots \times \tau(\mathcal{E}_n)). \tag{4.17}$$

Let us further assume that each $\mathcal{E}_i$ is countable for $i = 1, 2, \ldots, n$, $\tau_i = \tau(\mathcal{E}_i)$ and $\mathcal{M}_i = \sigma(\tau_i)$ is the Borel σ – algebra on i. Then

1. $\mathcal{M}_i = \sigma(\tau_i) = \sigma(\mathcal{E}_i)$ *for all i and*
2. $$\sigma(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n.$$

   That is the Borel σ – algebra on $X_1 \times X_2 \times \cdots \times X_n$ with the product topology is the product of the Borel σ – algebras on X_i.

Proof. We will prove Eq. (4.16). The proof of Eq. (4.17) is completely analogous. Since

$$\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n \subset \sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)$$

it follows that

$$\sigma\left(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n\right) \subset \sigma\left(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)\right).$$

To prove the reverse inequality we have

$$\begin{aligned}\sigma(\mathcal{E}_1) \times X_2 \times \cdots \times X_n &= \pi_1^{-1}(\sigma(\mathcal{E}_1)) = \sigma(\pi_1^{-1}(\mathcal{E}_1)) \\ &= \sigma(\mathcal{E}_1 \times X_2 \times \cdots \times X_n) \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n)\end{aligned}$$

wherein the last containment we have used the fact that $X_i \in \mathcal{E}_i$ for each i. Similarly one shows for each i that

$$X_1 \times X_2 \times \cdots \times X_{i-1} \times \sigma(\mathcal{E}_i) \times X_{i+1} \cdots \times X_n \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n)$$

for each i and therefore. Since $\sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n)$ is closed under finite intersections, we learn that

$$A_1 \times A_2 \times \cdots \times A_n = \cap_{i=1}^n \left(X_1 \times X_2 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \cdots \times X_n\right) \in \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n)$$

when $A_i \in \sigma(\mathcal{E}_i)$. This shows that

$$\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n) \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n)$$

and therefore that

$$\sigma\left(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)\right) \subset \sigma\left(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n\right).$$

Let us now assume that each $\mathcal{E}_i$ is countable. Then it has already been proved in Proposition 4.14 that $\mathcal{M}_i = \sigma(\tau_i) = \sigma(\mathcal{E}_i)$. Moreover, $\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ is also countable, another application of Proposition 4.14 along with the first two assertions of the theorems gives

$$\begin{aligned}\sigma(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n) &= \sigma(\tau\left(\tau_1 \times \tau_2 \times \cdots \times \tau_n\right)) \\ &= \sigma(\tau\left(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2) \times \cdots \times \tau(\mathcal{E}_n)\right)) \\ &= \sigma(\tau\left(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n\right)) \\ &= \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \\ &= \sigma\left(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)\right) \\ &= \sigma\left(\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n\right) \\ &= \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n.\end{aligned}$$

■

Remark 4.44. One can not relax the assumption that $X_i \in \mathcal{E}_i$ in Theorem 4.43. For example, if $X_1 = X_2 = \{1,2\}$ and $\mathcal{E}_1 = \mathcal{E}_2 = \{\{1\}\}$, then $\sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \{\emptyset, X_1 \times X_2, \{(1,1)\}\}$ while $\sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)) = \mathcal{P}(X_1 \times X_2)$.

Proposition 4.45. *If (X_i, d_i) for $i = 1, \ldots, n$ be metric spaces such that for each i there a countable dense subset $D_i \subseteq X_i$, then*

$$\bigotimes_i \mathcal{B}_{X_i} = \mathcal{B}_{(X_1 \times \cdots \times X_n)}$$

where $\mathcal{B}_{X_i}$ is the Borel σ – algebra on X_i and $\mathcal{B}_{(X_1 \times \cdots \times X_n)}$ is the Borel σ – algebra on $X_1 \times \cdots \times X_n$ equipped with the product topology.

Proof. This follows directly from Theorem 4.43 with

$$\mathcal{E}_i := \left\{B_x^{d_i}(\epsilon) \subset X_i : x \in D_i \text{ and } \epsilon \in \mathbb{Q} \cap (0,\infty)\right\} \text{ for } i = 1, 2, \ldots, n.$$

■

Because all norms on finite dimensional spaces are equivalent, the usual Euclidean norm on $\mathbb{R}^m \times \mathbb{R}^n$ is equivalent to the norm "product" norm defined by

$$\|(x,y)\|_{\mathbb{R}^m \times \mathbb{R}^n} = \|x\|_{\mathbb{R}^m} + \|y\|_{\mathbb{R}^n}.$$

Hence by Remark 4.41, the Euclidean topology on $\mathbb{R}^{m+n}$ is the same as the product topology on $\mathbb{R}^{m+n} \cong \mathbb{R}^m \times \mathbb{R}^n$ Here we are identifying $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ by the map

$$(x,y) \in \mathbb{R}^m \times \mathbb{R}^n \to (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{R}^{m+n}.$$

Proposition 4.45 and these comments leads to the following corollaries.

Corollary 4.46. *After identifying $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ as above, $\mathcal{B}_{\mathbb{R}^{m+n}} = \mathcal{B}_{\mathbb{R}^n} \otimes \mathcal{B}_{\mathbb{R}^m}$, where $\mathcal{B}_{\mathbb{R}^n}$ is the Borel σ –algebra on $\mathbb{R}^n$. Similarly, $\mathcal{B}_{\mathbb{R}^n} = \overbrace{\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}}^{n-times}$ where we identify $\mathbb{R}^n$ with $\overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n-times}$ in the usual way.*

Corollary 4.47. *If $(X, \mathcal{M})$ is a measurable space, then*

$$f = (f_1, f_2, \ldots, f_n) : X \to \mathbb{R}^n$$

is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^n})$ – measurable iff $f_i : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable for each i. In particular, a function $f : X \to \mathbb{C}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable.

Corollary 4.48. *Let $(X,\mathcal{M})$ be a measurable space and $f,g : X \to \mathbb{C}$ be $(\mathcal{M},\mathcal{B}_{\mathbb{C}})$ – measurable functions. Then $f \pm g$ and $f \cdot g$ are also $(\mathcal{M},\mathcal{B}_{\mathbb{C}})$ – measurable.*

Proof. Define $F : X \to \mathbb{C}\times\mathbb{C}$, $A_\pm : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ by $F(x) = (f(x), g(x))$, $A_\pm(w,z) = w \pm z$ and $M(w,z) = wz$. Then $A_\pm$ and M are continuous and hence $(\mathcal{B}_{\mathbb{C}^2},\mathcal{B}_{\mathbb{C}})$ – measurable. Also F is $(\mathcal{M},\mathcal{B}_{\mathbb{C}}\otimes\mathcal{B}_{\mathbb{C}}) = (\mathcal{M},\mathcal{B}_{\mathbb{C}^2})$ – measurable since $\pi_1 \circ F = f$ and $\pi_2 \circ F = g$ are $(\mathcal{M},\mathcal{B}_{\mathbb{C}})$ – measurable measurable. Therefore $A_\pm \circ F = f \pm g$ and $M \circ F = f \cdot g$ being the composition of measurable functions are also measurable. ■

Lemma 4.49. *Let $\alpha \in \mathbb{C}$, $(X,\mathcal{M})$ be a measurable space and $f : X \to \mathbb{C}$ be a $(\mathcal{M},\mathcal{B}_{\mathbb{C}})$ – measurable function. Then*

$$F(x) := \begin{cases} \frac{1}{f(x)} & \text{if} \quad f(x) \neq 0 \\ \alpha & \text{if} \quad f(x) = 0 \end{cases}$$

is measurable.

Proof. Define $i : \mathbb{C} \to \mathbb{C}$ by

$$i(z) = \begin{cases} \frac{1}{z} & \text{if} \quad z \neq 0 \\ \alpha & \text{if} \quad z = 0. \end{cases}$$

For any open set $V \subset \mathbb{C}$ we have

$$i^{-1}(V) = i^{-1}(V \setminus \{0\}) \cup i^{-1}(V \cap \{0\})$$

Because i is continuous except at $z = 0$, $i^{-1}(V \setminus \{0\})$ is an open set and hence in $\mathcal{B}_{\mathbb{C}}$. Moreover, $i^{-1}(V \cap \{0\}) \in \mathcal{B}_{\mathbb{C}}$ since $i^{-1}(V \cap \{0\})$ is either the empty set or the one point set $\{\alpha\}$. Therefore $i^{-1}(\tau_{\mathbb{C}}) \subseteq \mathcal{B}_{\mathbb{C}}$ and hence $i^{-1}(\mathcal{B}_{\mathbb{C}}) = i^{-1}(\sigma(\tau_{\mathbb{C}})) = \sigma(i^{-1}(\tau_{\mathbb{C}})) \subseteq \mathcal{B}_{\mathbb{C}}$ which shows that i is Borel measurable. Since $F = i \circ f$ is the composition of measurable functions, F is also measurable. ■

4.7. **Appendix: General Product spaces .**

Definition 4.50. Suppose$(X_\alpha \mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of measurable spaces and let X be the product space

$$X = \prod_{\alpha\in A} X_\alpha$$

and $\pi_\alpha : X \to X_\alpha$ be the canonical projection maps. Then the product σ – algebra, $\bigotimes_\alpha \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha \equiv \sigma(\pi_\alpha : \alpha \in A) = \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right).$$

Similarly if $(X_\alpha \mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of topological, the product topology $\bigotimes_\alpha \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha \equiv \tau(\pi_\alpha : \alpha \in A) = \tau\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right).$$

Remark 4.51. Let $(Z,\mathcal{M})$ be a measurable (topological) space and

$$\left(X = \prod_{\alpha\in A} X_\alpha, \bigotimes_{\alpha\in A} \mathcal{M}_\alpha\right)$$

be as in Definition 4.50. By Proposition 4.38, a function $f : Z \to X$ is measurable (continuous) iff $\pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{M}_\alpha)$ – measurable (continuous) for all $\alpha \in A$.

Proposition 4.52. *Suppose that $(X_\alpha \mathcal{M}_\alpha)_{\alpha \in A}$ is a collection of measurable (topological) spaces and $\mathcal{E}_\alpha \subseteq \mathcal{M}_\alpha$ generates $\mathcal{M}_\alpha$ for each $\alpha \in A$, then*

$$\otimes_{\alpha \in A} \mathcal{M}_\alpha = \sigma\left(\cup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \quad \left(\tau\left(\cup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right)\right) \tag{4.18}$$

Moreover, suppose that A is either finite or countably infinite, $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, and $\mathcal{M}_\alpha = \sigma(\mathcal{E}_\alpha)$ is a σ – algebra for all $\alpha \in A$. Then the product σ – algebra satisfies

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \sigma\left(\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right). \tag{4.19}$$

Similarly if A is finite and $\mathcal{M}_\alpha = \tau(\mathcal{E}_\alpha)$, then the product topology satisfies

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \tau\left(\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right). \tag{4.20}$$

Proof. We will prove Eq. (4.18) in the measure theoretic case since a similar proof works in the topological category. Since $\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \cup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)$, it follows that

$$\mathcal{F} := \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right) = \bigotimes_\alpha \mathcal{M}_\alpha.$$

Conversely,

$$\mathcal{F} \supset \sigma(\pi_\alpha^{-1}(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\sigma(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\mathcal{M}_\alpha)$$

holds for all α implies that

$$\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha) \subset \mathcal{F}$$

and hence that $\bigotimes_\alpha \mathcal{M}_\alpha \subseteq \mathcal{F}$.

We now prove Eq. (4.19). Since we are assuming that $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, we see that

$$\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}$$

and therefore by Eq. (4.18)

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right).$$

This last statement is true independent as to whether A is countable or not. For the reverse inclusion it suffices to notice that since A is countable,

$$\prod_{\alpha \in A} E_\alpha = \cap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha) \in \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$$

and hence

$$\sigma\left(\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right) \subset \bigotimes_{\alpha \in A} \mathcal{M}_\alpha.$$

Here is a generalization of Theorem 4.43 to the case of countable number of factors.
■

Proposition 4.53. *Let $\{X_\alpha\}_{\alpha\in A}$ be a sequence of sets where A is at most countable. Suppose for each $\alpha \in A$ we are given a countable set $\mathcal{E}_\alpha \subset \mathcal{P}(X_\alpha)$. Let $\tau_\alpha = \tau(\mathcal{E}_\alpha)$ be the topology on X_α generated by $\mathcal{E}_\alpha$ and X be the product space $\prod_{\alpha\in A} X_\alpha$ with equipped with the product topology $\tau := \otimes_{\alpha\in A}\tau(\mathcal{E}_\alpha)$. Then the Borel σ – algebra $\mathcal{B}_X = \sigma(\tau)$ is the same as the product σ – algebra:*

$$\mathcal{B}_X = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha},$$

where $\mathcal{B}_{X_\alpha} = \sigma(\tau(\mathcal{E}_\alpha)) = \sigma(\mathcal{E}_\alpha)$ for all $\alpha \in A$.

Proof. By Proposition 4.52, the topology τ may be described as the smallest topology containing $\mathcal{E} = \cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)$. Now $\mathcal{E}$ is the countable union of countable sets so is still countable. Therefore by Proposition 4.14 and Proposition 4.52 we have

$$\mathcal{B}_X = \sigma(\tau) = \sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E}) = \otimes_{\alpha\in A}\sigma(\mathcal{E}_\alpha) = \otimes_{\alpha\in A}\sigma(\tau_\alpha) = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha}.$$

■

4.8. **Exercises.**

Definition 4.54. Let X be a set. We say that a family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is a **partition** of X if X is the disjoint union of the sets in $\mathcal{F}$.

Exercise 4.5 (Structure of countable σ – algebras.)**.** Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is a σ – algebra of sets and that $\mathcal{M}$ is countable (i.e. countable or finite). Show that there is a unique partition $\mathcal{F} \subset \mathcal{M} \subset \mathcal{P}(X)$ such that for each $A \in \mathcal{M}$,

$$A = \bigcup_{\alpha\in\mathcal{F}\,:\,\alpha\subset A} \alpha.$$

Use this result to show that $\mathcal{M}$ must in fact be finite.

Hints: 1. For each $x \in X$ let

$$A_x = \bigcap\{A : A \in \mathcal{M} \,:\, x \in A\}$$

i.e. A_x is the smallest element in $\mathcal{M}$ which contains x. Show that if $A_x \cap A_y \neq \emptyset$ then $A_x = A_y$. Then show that $\mathcal{F} = \{A_x \in \mathcal{M} : x \in X\} \subset \mathcal{M}$ is the desired partition.

2. If $\mathcal{F} = \{P_n\}_{n=1}^N$, (allowing $N = \infty$ for the moment), show that $\mathcal{M}$ is in one to one correspondence with the set of sequences $\{a_n\}_{n=1}^N$ with $a_n \in \{0, 1\}$ for all n. The latter set is uncountable if $N = \infty$.

Exercise 4.6. Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \mathbb{R}$ be a function. Show f is $\mathcal{M} - \mathcal{B}_\mathbb{R}$ measurable iff any one of the following conditions holds:

1. $f^{-1}((a,\infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
2. $f^{-1}((a,\infty)) \in \mathcal{M}$ for all $a \in \mathbb{Q}$.
3. $f^{-1}([a,\infty)) \in \mathcal{M}$ for all $a \in \mathbb{Q}$.

Hint: See Exercise 4.3.

Exercise 4.7. Folland, Problem 1.5 on p.24. If $\mathcal{M}$ is the σ – algebra generated by $\mathcal{E} \subset \mathcal{P}(X)$, then $\mathcal{M}$ is the union of the σ – algebras generated by countable subsets $\mathcal{F} \subset \mathcal{E}$.

Exercise 4.8. Let $(X, \mathcal{M})$ be a measure space and $f_n : X \to \mathbb{F}$ be a sequence of measurable functions on X. Show that $\{x : \lim_{n\to\infty} f_n(x) \text{ exists}\} \in \mathcal{M}$.

Exercise 4.9. Show that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is $(\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})$ – measurable.

Exercise 4.10. Folland problem 2.6 on p. 48.

4.9. **Solutions.**