Appendix A. Multinomial Theorems and Calculus Results

Given a multi-index $\alpha \in \mathbb{Z}_+^n$, let $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$,

$$x^\alpha := \prod_{j=1}^n x_j^{\alpha_j} \text{ and } \partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha := \prod_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}.$$

We also write

$$\partial_v f(x) := \frac{d}{dt} f(x+tv)|_{t=0}.$$

A.1. **Multinomial Theorems and Product Rules.** For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$, $m \in \mathbb{N}$ and $(i_1, \ldots, i_m) \in \{1, 2, \ldots, n\}^m$ let $\hat{\alpha}_j(i_1, \ldots, i_m) = \#\{k : i_k = j\}$. Then

$$\left(\sum_{i=1}^n a_i\right)^m = \sum_{i_1,\ldots,i_m=1}^n a_{i_1} \ldots a_{i_m} = \sum_{|\alpha|=m} C(\alpha) a^\alpha$$

where

$$C(\alpha) = \#\{(i_1, \ldots, i_m) : \hat{\alpha}_j(i_1, \ldots, i_m) = \alpha_j \text{ for } j = 1, 2, \ldots, n\}$$

I claim that $C(\alpha) = \frac{m!}{\alpha!}$. Indeed, one possibility for such a sequence $(a_1, \ldots, a_{i_m})$ for a given α is gotten by choosing

$$(\overbrace{a_1, \ldots, a_1}^{\alpha_1}, \overbrace{a_2, \ldots, a_2}^{\alpha_2}, \ldots, \overbrace{a_n, \ldots, a_n}^{\alpha_n}).$$

Now there are $m!$ permutations of this list. However, only those permutations leading to a distinct list are to be counted. So for each of these $m!$ permutations we must divide by the number of permutation which just rearrange the groups of a_i's among themselves for each i. There are $\alpha! := \alpha_1! \cdots \alpha_n!$ such permutations. Therefore, $C(\alpha) = m!/\alpha!$ as advertised. So we have proved

$$\left(\sum_{i=1}^n a_i\right)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} a^\alpha. \tag{A.1}$$

Now suppose that $a, b \in \mathbb{R}^n$ and α is a multi-index, we have

$$(a+b)^\alpha = \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} a^\beta b^{\alpha-\beta} = \sum_{\beta+\delta=\alpha} \frac{\alpha!}{\beta!\delta!} a^\beta b^\delta \tag{A.2}$$

Indeed, by the standard Binomial formula,

$$(a_i + b_i)^{\alpha_i} = \sum_{\beta_i \le \alpha_i} \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!} a^{\beta_i} b^{\alpha_i - \beta_i}$$

from which Eq. (A.2) follows. Eq. (A.2) generalizes in the obvious way to

$$(a_1 + \cdots + a_k)^\alpha = \sum_{\beta_1+\cdots+\beta_k=\alpha} \frac{\alpha!}{\beta_1! \cdots \beta_k!} a_1^{\beta_1} \ldots a_k^{\beta_k} \tag{A.3}$$

where $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$.

Now let us consider the product rule for derivatives. Let us begin with the one variable case (write $d^n f$ for $f^{(n)} = \frac{d^n}{dx^n} f$) where we will show by induction that

$$d^n(fg) = \sum_{k=0}^n \binom{n}{k} d^k f \cdot d^{n-k} g. \tag{A.4}$$

Indeed assuming Eq. (A.4) we find

$$\begin{aligned} d^{n+1}(fg) &= \sum_{k=0}^{n} \binom{n}{k} d^{k+1} f \cdot d^{n-k} g + \sum_{k=0}^{n} \binom{n}{k} d^{k} f \cdot d^{n-k+1} g \\ &= \sum_{k=1}^{n+1} \binom{n}{k-1} d^{k} f \cdot d^{n-k+1} g + \sum_{k=0}^{n} \binom{n}{k} d^{k} f \cdot d^{n-k+1} g \\ &= \sum_{k=1}^{n+1} \left[\binom{n}{k-1} + \binom{n}{k} \right] d^{k} f \cdot d^{n-k+1} g + d^{n+1} f \cdot g + f \cdot d^{n+1} g. \end{aligned}$$

Since

$$\begin{aligned} \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} \\ &= \frac{n!}{(k-1)!\,(n-k)!} \left[\frac{1}{(n-k+1)} + \frac{1}{k} \right] \\ &= \frac{n!}{(k-1)!\,(n-k)!} \frac{n+1}{(n-k+1)\,k} = \binom{n+1}{k} \end{aligned}$$

the result follows.

Now consider the multi-variable case

$$\begin{aligned} \partial^{\alpha}(fg) &= \left(\prod_{i=1}^{n} \partial_i^{\alpha_i} \right) (fg) = \prod_{i=1}^{n} \left[\sum_{k_i=0}^{\alpha_i} \binom{\alpha_i}{k_i} \partial_i^{k_i} f \cdot \partial_i^{\alpha_i - k_i} g \right] \\ &= \sum_{k_1=0}^{\alpha_1} \cdots \sum_{k_n=0}^{\alpha_n} \prod_{i=1}^{n} \binom{\alpha_i}{k_i} \partial^{k} f \cdot \partial^{\alpha-k} g = \sum_{k \le \alpha} \binom{\alpha}{k} \partial^{k} f \cdot \partial^{\alpha-k} g \end{aligned}$$

where $k = (k_1, k_2, \ldots, k_n)$ and

$$\binom{\alpha}{k} := \prod_{i=1}^{n} \binom{\alpha_i}{k_i} = \frac{\alpha!}{k!(\alpha-k)!}.$$

So we have proved

$$\partial^{\alpha}(fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^{\beta} f \cdot \partial^{\alpha-\beta} g. \tag{A.5}$$

A.2. **Taylor's Theorem.**

Theorem A.1. *Suppose $X \subset \mathbb{R}^n$ is an open set, $x : [0,1] \to X$ is a C^1 – path, and $f \in C^N(X, \mathbb{C})$. Let $v_s := x(1) - x(s)$ and $v = v_1 = x(1) - x(0)$, then*

$$f(x(1)) = \sum_{m=0}^{N-1} \frac{1}{m!} \left(\partial_v^m f \right)(x(0)) + R_N \tag{A.6}$$

where

$$R_N = \frac{1}{(N-1)!} \int_0^1 \left(\partial_{\dot{x}(s)} \partial_{v_s}^{N-1} f \right)(x(s)) ds = \frac{1}{N!} \int_0^1 \left(-\frac{d}{ds} \partial_{v_s}^{N} f \right)(x(s)) ds. \tag{A.7}$$

and $0! := 1$.

Proof. By the fundamental theorem of calculus and the chain rule,

$$f(x(t)) = f(x(0)) + \int_0^t \frac{d}{ds} f(x(s))ds = f(x(0)) + \int_0^t \left(\partial_{\dot{x}(s)} f\right)(x(s))ds \tag{A.8}$$

and in particular,

$$f(x(1)) = f(x(0)) + \int_0^1 \left(\partial_{\dot{x}(s)} f\right)(x(s))ds.$$

This proves Eq. (A.6) when $N = 1$. We will now complete the proof using induciton on N.

Applying Eq. (A.8) with f replaced by $\frac{1}{(N-1)!}\left(\partial_{\dot{x}(s)}\partial_{v_s}^{N-1} f\right)$ gives

$$\begin{aligned}\frac{1}{(N-1)!}\left(\partial_{\dot{x}(s)}\partial_{v_s}^{N-1} f\right)(x(s)) &= \frac{1}{(N-1)!}\left(\partial_{\dot{x}(s)}\partial_{v_s}^{N-1} f\right)(x(0)) \\ &\quad + \frac{1}{(N-1)!}\int_0^s \left(\partial_{\dot{x}(s)}\partial_{v_s}^{N-1}\partial_{\dot{x}(t)} f\right)(x(t))dt \\ &= -\frac{1}{N!}\left(\frac{d}{ds}\partial_{v_s}^N f\right)(x(0)) - \frac{1}{N!}\int_0^s \left(\frac{d}{ds}\partial_{v_s}^N \partial_{\dot{x}(t)} f\right)(x(t))dt\end{aligned}$$

wherein we have used the fact that mixed partial derivatives commute to show $\frac{d}{ds}\partial_{v_s}^N f = N\partial_{\dot{x}(s)}\partial_{v_s}^{N-1} f$. Integrating this equation on $s \in [0,1]$ shows, using the fundamental theorem of calculus,

$$\begin{aligned}R_N &= \frac{1}{N!}\left(\partial_v^N f\right)(x(0)) - \frac{1}{N!}\int_{0\le t\le s\le 1}\left(\frac{d}{ds}\partial_{v_s}^N \partial_{\dot{x}(t)} f\right)(x(t))dsdt \\ &= \frac{1}{N!}\left(\partial_v^N f\right)(x(0)) + \frac{1}{(N+1)!}\int_{0\le t\le 1}\left(\partial_{w_t}^N \partial_{\dot{x}(t)} f\right)(x(t))dt \\ &= \frac{1}{N!}\left(\partial_v^N f\right)(x(0)) + R_{N+1}\end{aligned}$$

which completes the inductive proof. ■

Remark A.2. Using Eq. (A.1) with a_i replaced by $v_i\partial_i$ (although $\{v_i\partial_i\}_{i=1}^n$ are not complex numbers they are commuting symbols), we find

$$\partial_v^m f = \left(\sum_{i=1}^n v_i\partial_i\right)^m f = \sum_{|\alpha|=m}\frac{m!}{\alpha!}v^\alpha\partial^\alpha.$$

Using this fact we may write Eqs. (A.6) and (A.7) as

$$f(x(1)) = \sum_{|\alpha|\le N-1}\frac{1}{\alpha!}v^\alpha\partial^\alpha f(x(0)) + R_N$$

and

$$R_N = \sum_{|\alpha|=N}\frac{1}{\alpha!}\int_0^1\left(-\frac{d}{ds}v_s^\alpha\partial^\alpha f\right)(x(s))ds.$$

Corollary A.3. *Suppose $X \subset \mathbb{R}^n$ is an open set which contains $x(s) = (1-s)x_0 + sx_1$ for $0 \le s \le 1$ and $f \in C^N(X, \mathbb{C})$. Then*

(A.9)
$$f(x_1) = \sum_{m=0}^{N-1} \frac{1}{m!} \left(\partial_v^m f\right)(x_0) + \frac{1}{N!} \int_0^1 \left(\partial_v^N f\right)(x(s)) d\nu_N(s)$$

(A.10)
$$= \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial^\alpha f(x(0))(x_1 - x_0)^\alpha + \sum_{\alpha:|\alpha|=N} \frac{1}{\alpha!} \left[\int_0^1 \partial^\alpha f(x(s)) d\nu_N(s)\right] (x_1 - x_0)^\alpha$$

where $v := x_1 - x_0$ and $d\nu_N$ is the probability measure on $[0,1]$ given by

$$d\nu_N(s) := N(1-s)^{N-1} ds. \tag{A.11}$$

If we let $x = x_0$ and $y = x_1 - x_0$ (so $x + y = x_1$) Eq. (A.10) may be written as

$$f(x+y) = \sum_{|\alpha|<N} \frac{\partial_x^\alpha f(x)}{\alpha!} y^\alpha + \sum_{\alpha:|\alpha|=N} \frac{1}{\alpha!} \left(\int_0^1 \partial_x^\alpha f(x+sy) d\nu_N(s)\right) y^\alpha. \tag{A.12}$$

Proof. This is a special case of Theorem A.1. Notice that

$$v_s = x(1) - x(s) = (1-s)(x_1 - x_0) = (1-s)v$$

and hence

$$R_N = \frac{1}{N!} \int_0^1 \left(-\frac{d}{ds}(1-s)^N \partial_v^N f\right)(x(s)) ds = \frac{1}{N!} \int_0^1 \left(\partial_v^N f\right)(x(s)) N(1-s)^{N-1} ds.$$

■

Example A.4. Let $X = (-1,1) \subset \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(x) = (1-x)^\beta$. The reader should verify

$$f^{(m)}(x) = (-1)^m \beta(\beta-1)\dots(\beta-m+1)(1-x)^{\beta-m}$$

and therefore by Taylor's theorem (Eq. (F.21) with $x = 0$ and $y = x$)

$$(1-x)^\beta = 1 + \sum_{m=1}^{N-1} \frac{1}{m!} (-1)^m \beta(\beta-1)\dots(\beta-m+1) x^m + R_N(x) \tag{A.13}$$

where

$$\begin{aligned} R_N(x) &= \frac{x^N}{N!} \int_0^1 (-1)^N \beta(\beta-1)\dots(\beta-N+1)(1-sx)^{\beta-N} d\nu_N(s) \\ &= \frac{x^N}{N!} (-1)^N \beta(\beta-1)\dots(\beta-N+1) \int_0^1 \frac{N(1-s)^{N-1}}{(1-sx)^{N-\beta}} ds. \end{aligned}$$

Now for $x \in (-1,1)$ and $N > \beta$,

$$0 \le \int_0^1 \frac{N(1-s)^{N-1}}{(1-sx)^{N-\beta}} ds \le \int_0^1 \frac{N(1-s)^{N-1}}{(1-s)^{N-\beta}} ds = \int_0^1 N(1-s)^{\beta-1} ds = \frac{N}{\beta}$$

and therefore,

$$|R_N(x)| \le \frac{|x|^N}{(N-1)!} |(\beta-1)\dots(\beta-N+1)| =: \rho_N.$$

Since

$$\limsup_{N\to\infty}\frac{\rho_{N+1}}{\rho_N}=|x|\cdot\limsup_{N\to\infty}\frac{N-\beta}{N}=|x|<1$$

and so by the Ratio test, $|R_N(x)|\le\rho_N\to 0$ (exponentially fast) as $N\to\infty$. Therefore by passing to the limit in Eq. (A.13) we have proved

$$(1-x)^\beta=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\beta(\beta-1)\dots(\beta-m+1)x^m \tag{A.14}$$

which is valid for $|x|<1$ and $\beta\in\mathbb{R}$. An important special cases is $\beta=-1$ in which case, Eq. (A.14) becomes $\frac{1}{1-x}=\sum_{m=0}^{\infty}x^m$, the standard geometric series formula. Another another useful special case is $\beta=1/2$ in which case Eq. (A.14) becomes

$$\begin{aligned}\sqrt{1-x}&=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-m+1)x^m\\&=1-\sum_{m=1}^{\infty}\frac{(2m-3)!!}{2^m m!}x^m\text{ for all }|x|<1.\end{aligned}\tag{A.15}$$