Conclude from this that

$$I(f) = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} f(c_i)(F(t_{i+1}) - F(t_i)).$$

As usual we will write this integral as $\int_{-M}^{M} f dF$ and as $\int_{-M}^{M} f(t)dt$ if $F(t) = t$.

Exercise 11.5. Folland problem 1.28.

Exercise 11.6. Suppose that $F \in C^1(\mathbb{R})$ is an increasing function and μ_F is the unique Borel measure on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all $a \le b$. Show that $d\mu_F = \rho dm$ for some function $\rho \ge 0$. Find ρ explicitly in terms of F.

Exercise 11.7. Suppose that $F(x) = e1_{x\ge 3} + \pi 1_{x\ge 7}$ and μ_F is the is the unique Borel measure on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all $a \le b$. Give an explicit description of the measure μ_F.

Exercise 11.8. Let $E \in \mathcal{B}_{\mathbb{R}}$ with $m(E) > 0$. Then for any $\alpha \in (0,1)$ there exists an open interval $J \subset \mathbb{R}$ such that $m(E \cap J) \ge \alpha m(J)$. **Hints:** 1. Reduce to the case where $m(E) \in (0,\infty)$. 2) Approximate E from the outside by an open set $V \subset \mathbb{R}$. 3. Make use of Exercise 3.43, which states that V may be written as a disjoint union of open intervals.

11.10.1. *The Laws of Large Number Exercises.* For the rest of the problems of this section, let ν be a probability measure on $\mathcal{B}_{\mathbb{R}}$ such that $\int_{\mathbb{R}} |x|\, d\nu(x) < \infty$, $\mu_n := \nu$ for $n \in \mathbb{N}$ and μ denote the infinite product measure as constructed in Corollary 11.40. So μ is the unique measure on $(X := \mathbb{R}^{\mathbb{N}}, \mathcal{B} := \mathcal{B}_{\mathbb{R}^{\mathbb{N}}})$ such that

$$\int_X f(x_1, x_2, \dots, x_N) d\mu(x) = \int_{\mathbb{R}^N} f(x_1, x_2, \dots, x_N) d\nu(x_1) \dots d\nu(x_N) \tag{11.43}$$

for all $N \in \mathbb{N}$ and bounded measurable functions $f : \mathbb{R}^N \to \mathbb{R}$. We will also use the following notation:

$$\begin{aligned} S_n(x) &:= \frac{1}{n}\sum_{k=1}^{n} x_k \text{ for } x \in X, \\ m &:= \int_{\mathbb{R}} x d\nu(x) \text{ the average of } \nu, \\ \sigma^2 &:= \int_{\mathbb{R}} (x-m)^2 d\nu(x) \text{ the variance of } \nu \text{ and} \\ \gamma &:= \int_{\mathbb{R}} (x-m)^4 d\nu(x). \end{aligned}$$

The variance may also be written as $\sigma^2 = \int_{\mathbb{R}} x^2 d\nu(x) - m^2$.

Exercise 11.9 (Weak Law of Large Numbers)**.** Suppose further that $\sigma^2 < \infty$, show $\int_X S_n d\mu = m$,

$$\|S_n - m\|_2^2 = \int_X (S_n - m)^2\, d\mu = \frac{\sigma^2}{n}$$

and $\mu(|S_n - m| > \epsilon) \le \frac{\sigma^2}{n\epsilon^2}$ for all $\epsilon > 0$ and $n \in \mathbb{N}$.

Exercise 11.10 (A simple form of the Strong Law of Large Numbers)**.** Suppose now that $\gamma := \int_{\mathbb{R}} (x-m)^4 d\nu(x) < \infty$. Show for all $\epsilon > 0$ and $n \in \mathbb{N}$ that

$$\begin{aligned}\|S_n - m\|_4^4 = \int_X (S_n - m)^4 \, d\mu &= \frac{1}{n^4}\left(n\gamma + 3n(n-1)\sigma^4\right) \\ &= \frac{1}{n^2}\left[n^{-1}\gamma + 3\left(1-n^{-1}\right)\sigma^4\right] \text{ and} \\ \mu(|S_n - m| > \epsilon) &\leq \frac{n^{-1}\gamma + 3\left(1-n^{-1}\right)\sigma^4}{\epsilon^4 n^2}.\end{aligned}$$

Conclude from the last estimate and the first Borel Cantelli Lemma 5.22 that $\lim_{n\to\infty} S_n(x) = m$ for μ – a.e. $x \in X$.

Exercise 11.11. Suppose $\gamma := \int_{\mathbb{R}} (x-m)^4 d\nu(x) < \infty$ and $m = \int_{\mathbb{R}} (x-m) d\nu(x) \neq 0$. For $\lambda > 0$ let $T_\lambda : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be defined by $T_\lambda(x) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n, \ldots)$, $\mu_\lambda = \mu \circ T_\lambda^{-1}$ and

$$X_\lambda := \left\{ x \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_j = \lambda \right\}.$$

Show

$$\mu_\lambda(X_{\lambda'}) = \delta_{\lambda,\lambda'} = \begin{cases} 1 & \text{if} \quad \lambda = \lambda' \\ 0 & \text{if} \quad \lambda \neq \lambda' \end{cases}$$

and use this to show if $\lambda \neq 1$, then $d\mu_\lambda \neq \rho d\mu$ for any measurable function $\rho : \mathbb{R}^{\mathbb{N}} \to [0, \infty]$.