11. Construction of Measures

Now that we have developed integration theory relative to a measure on a $\sigma -$ algebra, it is time to show how to construct the measures that we have been using. This is a bit technical because there tends to be no "explicit" description of the general element of the typical $\sigma -$ algebras. On the other hand, we do know how to explicitly describe algebras which are generated by some class of sets $\mathcal{E} \subset \mathcal{P}(X)$. Therefore, we might try to define measures on $\sigma(\mathcal{E})$ by there restrictions to $\mathcal{A}(\mathcal{E})$. Theorem 6.5 shows this is a plausible method.

So the strategy of this section is as follows: 1) construct finitely additive measure on an algebra, 2) construct "integrals" associated to such finitely additive measures, 3) extend these integrals (Daniell's method) when possible to a larger class of functions, 4) construct a measure from the extended integral (Daniell – Stone construction theorem).

11.1. **Finitely Additive Measures and Associated Integrals.**

Definition 11.1. Suppose that $\mathcal{E} \subset \mathcal{P}(X)$ is a collection of subsets of a set X and $\mu : \mathcal{E} \to [0, \infty]$ is a function. Then

1. μ is **additive on** $\mathcal{E}$ if $\mu(E) = \sum_{i=1}^n \mu(E_i)$ whenever $E = \coprod_{i=1}^n E_i \in \mathcal{E}$ with $E_i \in \mathcal{E}$ for $i = 1, 2, \ldots, n < \infty$.
2. μ is $\sigma -$ **additive (or countable additive) on** $\mathcal{E}$ if Item 1. holds even when $n = \infty$.
3. μ is **subadditive on** $\mathcal{E}$ if $\mu(E) \le \sum_{i=1}^n \mu(E_i)$ whenever $E = \coprod_{i=1}^n E_i \in \mathcal{E}$ with $E_i \in \mathcal{E}$ and $n \in \mathbb{N} \cup \{\infty\}$.
4. μ is $\sigma -$ **finite on** $\mathcal{E}$ if there exist $E_n \in \mathcal{E}$ such that $X = \cup_n E_n$ and $\mu(E_n) < \infty$.

The reader should check if $\mathcal{E} = \mathcal{A}$ is an algebra and μ is additive on $\mathcal{A}$, then μ is $\sigma -$ **finite on** $\mathcal{A}$ iff there exists $X_n \in \mathcal{A}$ such that $X_n \uparrow X$ and $\mu(X_n) < \infty$ for all n.

Proposition 11.2. *Suppose $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family (see Definition 4.11) and $\mathcal{A} = \mathcal{A}(\mathcal{E})$ is the algebra generated by $\mathcal{E}$. Then every additive function $\mu : \mathcal{E} \to [0, \infty]$ extends uniquely to an additive measure (which we still denote by μ) on $\mathcal{A}$.*

Proof. Since by Proposition 4.12, every element $A \in \mathcal{A}$ is of the form $A = \coprod_i E_i$ with $E_i \in \mathcal{E}$, it is clear that if μ extends to a measure the extension is unique and must be given by

$$\mu(A) = \sum_i \mu(E_i). \tag{11.1}$$

To prove the existence of the extension, the main point is to show that defining $\mu(A)$ by Eq. (11.1) is well defined, i.e. if we also have $A = \coprod_j F_j$ with $F_j \in \mathcal{E}$, then we must show

$$\sum_i \mu(E_i) = \sum_j \mu(F_j). \tag{11.2}$$

But $E_i = \coprod_j (E_i \cap F_j)$ and the property that μ is additive on $\mathcal{E}$ implies $\mu(E_i) = \sum_j \mu(E_i \cap F_j)$ and hence

$$\sum_i \mu(E_i) = \sum_i \sum_j \mu(E_i \cap F_j) = \sum_{i,j} \mu(E_i \cap F_j).$$

By symmetry or an analogous argument,

$$\sum_j \mu(F_j) = \sum_{i,j} \mu(E_i \cap F_j)$$

which combined with the previous equation shows that Eq. (11.2) holds. It is now easy to verify that μ extended to $\mathcal{A}$ as in Eq. (11.1) is an additive measure on $\mathcal{A}$. ■

Proposition 11.3. *Let $X = \mathbb{R}$ and $\mathcal{E}$ be the elementary class*

$$\mathcal{E} = \{(a, b] \cap \mathbb{R} : -\infty \le a \le b \le \infty\},$$

and $\mathcal{A} = \mathcal{A}(\mathcal{E})$ be the algebra of disjoint union of elements from $\mathcal{E}$. Suppose that $\mu : \mathcal{A} \to [0, \infty]$ is an additive measure such that $\mu((a, b]) < \infty$ for all $-\infty < a < b < \infty$. Then there is a unique increasing function $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ such that $F(0) = 0$, $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$ and

$$\mu((a, b] \cap \mathbb{R}) = F(b) - F(a) \ \forall \ a \le b \text{ in } \bar{\mathbb{R}}. \tag{11.3}$$

Conversely, given an increasing function $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ such that $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$ there is a unique measure $\mu = \mu_F$ on $\mathcal{A}$ such that the relation in Eq. (11.3) holds.

So the finitely additive measures μ on $\mathcal{A}(\mathcal{E})$ which are finite on bounded sets are in one to one correspondence with increasing functions $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ such that $F(0) = 0$, $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$.

Proof. If F is going to exist, then

$$\mu((0, b] \cap \mathbb{R}) = F(b) - F(0) = F(b) \text{ if } b \in [0, \infty],$$
$$\mu((a, 0]) = F(0) - F(a) = -F(a) \text{ if } a \in [-\infty, 0]$$

from which we learn

$$F(x) = \begin{cases} -\mu((x, 0]) & \text{if} \quad x \le 0 \\ \mu((0, x] \cap \mathbb{R}) & \text{if} \quad x \ge 0. \end{cases}$$

Moreover, one easily checks using the additivity of μ that Eq. (11.3) holds for this F.

Conversely, suppose $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ is an increasing function such that $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$. Define μ on $\mathcal{E}$ using the formula in Eq. (11.3). I claim that μ is additive on $\mathcal{E}$ and hence has a unique extension to $\mathcal{A}$ which will finish the argument. Suppose that

$$(a, b] = \coprod_{i=1}^{n} (a_i, b_i].$$

By reordering $(a_i, b_i]$ if necessary, we may assume that

$$a = a_1 > b_1 = a_2 < b_2 = a_3 < \cdots < a_n < b_n = b.$$

Therefore,

$$\mu((a,b]) = F(b) - F(a) = \sum_{i=1}^{n} [F(b_i) - F(a_i)] = \sum_{i=1}^{n} \mu((a_i, b_i])$$

as desired. ■

11.1.1. *Integrals associated to finitely additive measures.*

Definition 11.4. Let μ be a finitely additive measure on an algebra $\mathcal{A} \subset \mathcal{P}(X)$, $\mathbb{S} = \mathbb{S}_f(\mathcal{A}, \mu)$ be the collection of simple functions defined in Notation 9.1 and for $f \in \mathbb{S}$ defined the **integral** $I(f) = I_\mu(f)$ by

$$I_\mu(f) = \sum_{y \in \mathbb{R}} y \mu(f = y). \tag{11.4}$$

The same proof used for Proposition 5.14 shows $I_\mu : \mathbb{S} \to \mathbb{R}$ is linear and positive, i.e. $I(f) \geq 0$ if $f \geq 0$. Taking absolute values of Eq. (11.4) gives

$$|I(f)| \leq \sum_{y \in \mathbb{R}} |y| \, \mu(f = y) \leq \|f\|_\infty \, \mu(f \neq 0) \tag{11.5}$$

where $\|f\|_\infty = \sup_{x \in X} |f(x)|$. For $A \in \mathcal{A}$, let $\mathbb{S}_A := \{f \in \mathbb{S} : \{f \neq 0\} \subset A\}$. The estimate in Eq. (11.5) implies

$$|I(f)| \leq \mu(A) \, \|f\|_\infty \text{ for all } f \in \mathbb{S}_A. \tag{11.6}$$

The B.L.T. theorem then implies that I has a unique extension I_A to $\bar{\mathbb{S}}_A \subset B(X)$ for any $A \in \mathcal{A}$ such that $\mu(A) < \infty$. The extension I_A is still positive. Indeed, let $f \in \bar{\mathbb{S}}_A$ with $f \geq 0$ and let $f_n \in \mathbb{S}_A$ be a sequence such that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. Then $f_n \vee 0 \in \mathbb{S}_A$ and

$$\|f - f_n \vee 0\|_\infty \leq \|f - f_n\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, $I_A(f) = \lim_{n \to \infty} I_A(f_n \vee 0) \geq 0$.

Suppose that $A, B \in \mathcal{A}$ are sets such that $\mu(A) + \mu(B) < \infty$, then $\mathbb{S}_A \cup \mathbb{S}_B \subset \mathbb{S}_{A \cup B}$ and so $\bar{\mathbb{S}}_A \cup \bar{\mathbb{S}}_B \subset \bar{\mathbb{S}}_{A \cup B}$. Therefore $I_A(f) = I_{A \cup B}(f) = I_B(f)$ for all $f \in \bar{\mathbb{S}}_A \cap \bar{\mathbb{S}}_B$. The next proposition summarizes these remarks.

Proposition 11.5. *Let $(\mathcal{A}, \mu, I = I_\mu)$ be as in Definition 11.4, then we may extend I to*

$$\tilde{\mathbb{S}} := \cup \{\bar{\mathbb{S}}_A : A \in \mathcal{A} \text{ with } \mu(A) < \infty\}$$

by defining $I(f) = I_A(f)$ when $f \in \bar{\mathbb{S}}_A$ with $\mu(A) < \infty$. Moreover this extension is still positive.

Notation 11.6. Suppose $X = \mathbb{R}$, $\mathcal{A} = \mathcal{A}(\mathcal{E})$, F and μ are as in Proposition 11.3. For $f \in \tilde{\mathbb{S}}$, we will write $I(f)$ as $\int_{-\infty}^{\infty} f dF$ or $\int_{-\infty}^{\infty} f(x) dF(x)$ and refer to $\int_{-\infty}^{\infty} f dF$ as the **Riemann Stieljtes integral** of f relative to F.

Lemma 11.7. *Using the notation above, the map $f \in \tilde{\mathbb{S}} \to \int_{-\infty}^{\infty} f dF$ is linear, positive and satisfies the estimate*

$$\left| \int_{-\infty}^{\infty} f dF \right| \leq (F(b) - F(a)) \, \|f\|_\infty \tag{11.7}$$

if $\operatorname{supp}(f) \subset (a, b)$. *Moreover $C_c(\mathbb{R}, \mathbb{R}) \subset \tilde{\mathbb{S}}$.*

Proof. The only new point of the lemma is to prove $C_c(\mathbb{R},\mathbb{R}) \subset \tilde{\mathbb{S}}$, the remaining assertions follow directly from Proposition 11.5. The fact that $C_c(\mathbb{R},\mathbb{R}) \subset \tilde{\mathbb{S}}$ has essentially already been done in Example 5.24. In more detail, let $f \in C_c(\mathbb{R},\mathbb{R})$ and choose $a < b$ such that $\operatorname{supp}(f) \subset (a,b)$. Then define $f_k \in \mathbb{S}$ as in Example 5.24, i.e.

$$f_k(x) = \sum_{l=0}^{n_k-1} \min\left\{f(x) : a_l^k \le x \le a_{l+1}^k\right\} 1_{(a_l^k, a_{l+1}^k]}(x)$$

where $\pi_k = \{a = a_0^k < a_1^k < \cdots < a_{n_k}^k = b\}$, for $k = 1,2,3,\dots$, is a sequence of refining partitions such that $\operatorname{mesh}(\pi_k) \to 0$ as $k \to \infty$. Since $\operatorname{supp}(f)$ is compact and f is continuous, f is uniformly continuous on $\mathbb{R}$. Therefore $\|f - f_k\|_\infty \to 0$ as $k \to \infty$, showing $f \in \tilde{\mathbb{S}}$. Incidently, for $f \in C_c(\mathbb{R},\mathbb{R})$, it follows that

$$\int_{-\infty}^{\infty} f dF = \lim_{k\to\infty} \sum_{l=0}^{n_k-1} \min\left\{f(x) : a_l^k \le x \le a_{l+1}^k\right\} \left[F(a_{l+1}^k) - F(a_l^k)\right]. \tag{11.8}$$

■

The most important special case of a Riemann Stieljtes integral is when $F(x) = x$ in which case $\int_{-\infty}^{\infty} f(x)dF(x) = \int_{-\infty}^{\infty} f(x)dx$ is the ordinary Riemann integral. The following Exercise is an abstraction of Lemma 11.7.

Exercise 11.1. Continue the notation of Definition 11.4 and Proposition 11.5. Further assume that X is a metric space, there exists open sets $X_n \subset_o X$ such that $X_n \uparrow X$ and for each $n \in \mathbb{N}$ and $\delta > 0$ there exists a finite collection of sets $\{A_i\}_{i=1}^k \subset \mathcal{A}$ such that $diam(A_i) < \delta$, $\mu(A_i) < \infty$ and $X_n \subset \cup_{i=1}^k A_i$. Then $C_c(X,\mathbb{R}) \subset \tilde{\mathbb{S}}$ and so I is well defined on $C_c(X,\mathbb{R})$.

Proposition 11.8. *Suppose that (X,τ) is locally compact Hausdorff space and I is a positive linear functional on $C_c(X,\mathbb{R})$. Then for each compact subset $K \subset X$ there is a constant $C_K < \infty$ such that $|I(f)| \le C_K \|f\|_\infty$ for all $f \in C_c(X,\mathbb{R})$ with* $\operatorname{supp}(f) \subset K$. *Moreover, if $f_n \in C_c(X,[0,\infty))$ and $f_n \downarrow 0$ (**pointwise**) as $n \to \infty$, then $I(f_n) \downarrow 0$ as $n \to \infty$.*

Proof. Let $f \in C_c(X,\mathbb{R})$ with $\operatorname{supp}(f) \subset K$. By Lemma 8.15 there exists $\psi_K \prec X$ such that $\psi_K = 1$ on K. Since $\|f\|_\infty \psi_K \pm f \ge 0$,

$$0 \le I(\|f\|_\infty \psi_K \pm f) = \|f\|_\infty I(\psi_K) \pm I(f)$$

from which it follows that $|I(f)| \le I(\psi_K)\|f\|_\infty$. So the first assertion holds with $C_K = I(\psi_K) < \infty$.

Now suppose that $f_n \in C_c(X,[0,\infty))$ and $f_n \downarrow 0$ as $n \to \infty$. Let $K = \operatorname{supp}(f_1)$ and notice that $\operatorname{supp}(f_n) \subset K$ for all n. By Dini's Theorem (see Exercise 3.11), $\|f_n\|_\infty \downarrow 0$ as $n \to \infty$ and hence

$$0 \le I(f_n) \le C_K \|f_n\|_\infty \downarrow 0 \text{ as } n \to \infty.$$

■

This result applies to the Riemann Stieljtes integral in Lemma 11.7 restricted to $C_c(\mathbb{R},\mathbb{R})$. However it is not generally true in this case that $I(f_n) \downarrow 0$ for all $f_n \in \mathbb{S}$ such that $f_n \downarrow 0$. Proposition 11.10 below addresses this question.

Definition 11.9. A countably additive function μ on an algebra $\mathcal{A} \subset 2^X$ is called a **premeasure**.

As for measures (see Remark 5.2 and Proposition 5.3), one easily shows if μ is a premeasure on $\mathcal{A}$, $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$ and if $A_n \uparrow A \in \mathcal{A}$ then $\mu(A_n) \uparrow \mu(A)$ as $n \to \infty$ or if $\mu(A_1) < \infty$ and $A_n \downarrow \emptyset$ then $\mu(A_n) \downarrow 0$ as $n \to \infty$ Now suppose that μ in Proposition 11.3 were a premeasure on $\mathcal{A}(\mathcal{E})$. Letting $A_n = (a, b_n]$ with $b_n \downarrow b$ as $n \to \infty$ we learn,

$$F(b_n) - F(a) = \mu((a, b_n]) \downarrow \mu((a, b]) = F(b) - F(a)$$

from which it follows that $\lim_{y \downarrow b} F(y) = F(b)$, i.e. F is right continuous. We will see below that in fact μ is a premeasure on $\mathcal{A}(\mathcal{E})$ iff F is right continuous.

Proposition 11.10. *Let $(\mathcal{A}, \mu, \mathbb{S} = \mathbb{S}_f(\mathcal{A}, \mu), I = I_\mu)$ be as in Definition 11.4. If μ is a premeasure on $\mathcal{A}$, then*

$$\forall\ f_n \in \mathbb{S} \text{ with } f_n \downarrow 0 \Longrightarrow I(f_n) \downarrow 0 \text{ as } n \to \infty. \tag{11.9}$$

Proof. Let $\epsilon > 0$ be given. Then

$$f_n = f_n 1_{f_n > \epsilon f_1} + f_n 1_{f_n \le \epsilon f_1} \le f_1 1_{f_n > \epsilon f_1} + \epsilon f_1,$$

$$I(f_n) \le I\left(f_1 1_{f_n > \epsilon f_1}\right) + \epsilon I(f_1) = \sum_{a>0} a \mu\left(f_1 = a,\ f_n > \epsilon a\right) + \epsilon I(f_1),$$

and hence

$$\limsup_{n\to\infty} I(f_n) \le \sum_{a>0} a \limsup_{n\to\infty} \mu\left(f_1 = a,\ f_n > \epsilon a\right) + \epsilon I(f_1). \tag{11.10}$$

Because, for $a > 0$,

$$\mathcal{A} \ni \{f_1 = a,\ f_n > \epsilon a\} := \{f_1 = a\} \cap \{f_n > \epsilon a\} \downarrow \emptyset \text{ as } n \to \infty$$

and $\mu(f_1 = a) < \infty$, $\limsup_{n\to\infty} \mu\left(f_1 = a,\ f_n > \epsilon a\right) = 0$. Combining this with Eq. (11.10) and making use of the fact that $\epsilon > 0$ is arbitrary we learn $\limsup_{n\to\infty} I(f_n) = 0$. ■

11.2. The Daniell-Stone Construction Theorem.

Definition 11.11. A vector subspace $\mathbb{S}$ of real valued functions on a set X is a **lattice** if it is closed under the lattice operations; $f \vee g = \max(f, g)$ and $f \wedge g = \min(f, g)$.

Remark 11.12. Notice that a lattice $\mathbb{S}$ is closed under the absolute value operation since $|f| = f \vee 0 - f \wedge 0$. Furthermore if $\mathbb{S}$ is a vector space of real valued functions, to show that $\mathbb{S}$ is a lattice it suffices to show $f^+ = f \vee 0 \in \mathbb{S}$ for all $f \in \mathbb{S}$. This is because

$$|f| = f^+ + (-f)^+,$$
$$f \vee g = \frac{1}{2}\left(f + g + |f - g|\right) \text{ and}$$
$$f \wedge g = \frac{1}{2}\left(f + g - |f - g|\right).$$

Notation 11.13. Given a collection of extended real valued functions $\mathcal{C}$ on X, let $\mathcal{C}^+ := \{f \in \mathcal{C} : f \ge 0\}$ – denote the subset of positive functions $f \in \mathcal{C}$.

Definition 11.14. A linear functional I on $\mathbb{S}$ is said to be **positive** (i.e. non-negative) if $I(f) \ge 0$ for all $f \in \mathbb{S}^+$. (This is equivalent to the statement the $I(f) \le I(g)$ if $f, g \in \mathbb{S}$ and $f \le g$.)

Definition 11.15 (Property (D))**.** A non-negative linear functional I on $\mathbb{S}$ is said to be continuous under monotone limits if $I(f_n) \downarrow 0$ for all $\{f_n\}_{n=1}^{\infty} \subset \mathbb{S}^+$ satisfying (**pointwise**) $f_n \downarrow 0$. A positive linear functional on $\mathbb{S}$ satisfying property (D) is called a **Daniell integral** on $\mathbb{S}$. We will also write $\mathbb{S}$ as $D(I)$ – the domain of I.

Example 11.16. Let (X, τ) be a locally compact Hausdorff space and I be a positive linear functional on $\mathbb{S} := C_c(X, \mathbb{R})$. It is easily checked that $\mathbb{S}$ is a lattice and Proposition 11.8 shows I is automatically a Daniell integral. In particular if $X = \mathbb{R}$ and F is an increasing function on $\mathbb{R}$, then the corresponding Riemann Stieljtes integral restricted to $\mathbb{S} := C_c(\mathbb{R}, \mathbb{R})$ $(f \in C_c(\mathbb{R}, \mathbb{R}) \to \int_{\mathbb{R}} f dF)$ is a Daniell integral.

Example 11.17. Let $(\mathcal{A}, \mu, \mathbb{S} = \mathbb{S}_f(\mathcal{A}, \mu), I = I_\mu)$ be as in Definition 11.4. It is easily checked that $\mathbb{S}$ is a lattice. Proposition 11.10 guarantees that I is a Daniell integral on $\mathbb{S}$ when μ is a premeasure on $\mathcal{A}$.

Lemma 11.18. *Let I be a non-negative linear functional on a lattice $\mathbb{S}$. Then property (D) is equivalent to either of the following two properties:*

- **D_1:** *If $\phi, \phi_n \in \mathbb{S}$ satisfy; $\phi_n \leq \phi_{n+1}$ for all n and $\phi \leq \lim_{n\to\infty} \phi_n$, then $I(\phi) \leq \lim_{n\to\infty} I(\phi_n)$.*
- **D_2:** *If $u_j \in \mathbb{S}^+$ and $\phi \in \mathbb{S}$ is such that $\phi \leq \sum_{j=1}^{\infty} u_j$ then $I(\phi) \leq \sum_{j=1}^{\infty} I(u_j)$.*

Proof. (D) $\Longrightarrow$ (D_1) Let $\phi, \phi_n \in \mathbb{S}$ be as in D_1. Then $\phi \wedge \phi_n \uparrow \phi$ and $\phi - (\phi \wedge \phi_n) \downarrow 0$ which implies

$$I(\phi) - I(\phi \wedge \phi_n) = I(\phi - (\phi \wedge \phi_n)) \downarrow 0.$$

Hence

$$I(\phi) = \lim_{n\to\infty} I(\phi \wedge \phi_n) \leq \lim_{n\to\infty} I(\phi_n).$$

(D_1) $\Longrightarrow$ (D_2) Apply (D_1) with $\phi_n = \sum_{j=1}^{n} u_j$.

(D_2) $\Longrightarrow$ (D) Suppose $\phi_n \in \mathbb{S}$ with $\phi_n \downarrow 0$ and let $u_n = \phi_n - \phi_{n+1}$. Then $\sum_{n=1}^{N} u_n = \phi_1 - \phi_{N+1} \uparrow \phi_1$ and hence

$$I(\phi_1) \leq \sum_{n=1}^{\infty} I(u_n) = \lim_{N\to\infty} \sum_{n=1}^{N} I(u_n) = \lim_{N\to\infty} I(\phi_1 - \phi_{N+1}) = I(\phi_1) - \lim_{N\to\infty} I(\phi_{N+1})$$

from which it follows that $\lim_{N\to\infty} I(\phi_{N+1}) \leq 0$. Since $I(\phi_{N+1}) \geq 0$ for all N we conclude that $\lim_{N\to\infty} I(\phi_{N+1}) = 0$. ■

In the remainder of this section, $\mathbb{S}$ will denote a lattice of bounded real valued functions on a set X and $I : \mathbb{S} \to \mathbb{R}$ will be a Daniell integral on $\mathbb{S}$.

Lemma 11.19. *Suppose that $\{f_n\}, \{g_n\} \subset \mathbb{S}$.*

1. *If $f_n \uparrow f$ and $g_n \uparrow g$ with $f, g : X \to (-\infty, \infty]$ such that $f \leq g$, then*

$$\lim_{n\to\infty} I(f_n) \leq \lim_{n\to\infty} I(g_n). \tag{11.11}$$

2. *If $f_n \downarrow f$ and $g_n \downarrow g$ with $f, g : X \to [-\infty, \infty)$ such that $f \leq g$, then Eq. (11.11) still holds.*

In particular, in either case if $f = g$, then $\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(g_n)$.

Proof.

1. Fix $n \in \mathbb{N}$, then $g_k \wedge f_n \uparrow f_n$ as $k \to \infty$ and $g_k \wedge f_n \le g_k$ and hence
$$I(f_n) = \lim_{k\to\infty} I(g_k \wedge f_n) \le \lim_{k\to\infty} I(g_k).$$
Passing to the limit $n \to \infty$ in this equation proves Eq. (11.11).
2. Since $-f_n \uparrow (-f)$ and $-g_n \uparrow (-g)$ and $-g \le (-f)$, what we just proved shows
$$-\lim_{n\to\infty} I(g_n) = \lim_{n\to\infty} I(-g_n) \le \lim_{n\to\infty} I(-f_n) = -\lim_{n\to\infty} I(f_n)$$
which is equivalent to Eq. (11.11).

■

Definition 11.20. Let
$$\mathbb{S}_\uparrow = \{f : X \to (-\infty, \infty] : \exists\, f_n \in \mathbb{S} \text{ such that } f_n \uparrow f\}$$
and for $f \in \mathbb{S}_\uparrow$ let $I(f) = \lim_{n\to\infty} I(f_n) \in (-\infty, \infty]$.

Lemma 11.19 shows this extension of I to $\mathbb{S}_\uparrow$ is well defined and positive, i.e. $I(f) \le I(g)$ if $f \le g$.

Definition 11.21. Let $\mathbb{S}_\downarrow = \{f : X \to [-\infty, \infty) : \exists\, f_n \in \mathbb{S} \text{ such that } f_n \downarrow f\}$ and define $I(f) = \lim_{n\to\infty} I(f_n)$ on $\mathbb{S}_\downarrow$.

Exercise 11.2. Show $\mathbb{S}_\downarrow = -\mathbb{S}_\uparrow$ and for $f \in \mathbb{S}_\downarrow \cup \mathbb{S}_\uparrow$ that $I(-f) = -I(f) \in \bar{\mathbb{R}}$.

We are now in a position to state the main construction theorem. The theorem we state here is not as general as possible but it will suffice for our present purposes. See Section 12 for a more general version and the full proof.

Theorem 11.22 (Daniell-Stone)**.** *Let $\mathbb{S}$ be a lattice of bounded functions on a set X such that $1 \wedge \phi \in \mathbb{S}$ and let I be a Daniel integral on $\mathbb{S}$. Further assume there exists $\chi \in \mathbb{S}_\uparrow$ such that $I(\chi) < \infty$ and $\chi(x) > 0$ for all $x \in X$. Then there exists a unique measure μ on $\mathcal{M} := \sigma(\mathbb{S})$ such that*

$$I(f) = \int_X f d\mu \text{ for all } f \in \mathbb{S}. \tag{11.12}$$

Moreover, for all $g \in L^1(X, \mathcal{M}, \mu)$,

$$\sup\{I(f) : \mathbb{S}_\downarrow \ni f \le g\} = \int_X g d\mu = \inf\{I(h) : g \le h \in \mathbb{S}_\uparrow\}. \tag{11.13}$$

Proof. Only a sketch of the proof will be given here. Full details may be found in Section 12 below.

Existence. For $g : X \to \bar{\mathbb{R}}$, define
$$\bar{I}(g) := \inf\{I(h) : g \le h \in \mathbb{S}_\uparrow\},$$
$$\underline{I}(g) := \sup\{I(f) : \mathbb{S}_\downarrow \ni f \le g\}$$
and set
$$L^1(I) := \{g : X \to \bar{\mathbb{R}} : \bar{I}(g) = \underline{I}(g) \in \mathbb{R}\}.$$
For $g \in L^1(I)$, let $\hat{I}(g) = \bar{I}(g) = \underline{I}(g)$. Then, as shown in Proposition 12.10, $L^1(I)$ is a "extended" vector space and $\hat{I} : L^1(I) \to \mathbb{R}$ is linear as defined in Definition 12.1 below. By Proposition 12.6, if $f \in \mathbb{S}_\uparrow$ with $I(f) < \infty$ then $f \in L^1(I)$. Moreover, $\hat{I}$ obeys the monotone convergence theorem, Fatou's lemma, and the

dominated convergence theorem, see Theorem 12.11, Lemma 12.12 and Theorem 12.15 respectively.

Let

$$\mathcal{R} := \left\{A \subset X : 1_A \wedge f \in L^1(I) \text{ for all } f \in \mathbb{S}\right\}$$

and for $A \in \mathcal{R}$ set $\mu(A) := \bar{I}(1_A)$. It can then be shown: 1) $\mathcal{R}$ is a σ algebra (Lemma 12.23) containing $\sigma(\mathbb{S})$ (Lemma 12.24), μ is a measure on $\mathcal{R}$ (Lemma 12.25), and that Eq. (11.12) holds. In fact it is shown in Theorem 12.28 and Proposition 12.29 below that $L^1(X, \mathcal{M}, \mu) \subset L^1(I)$ and

$$\hat{I}(g) = \int_X g d\mu \text{ for all } g \in L^1(X, \mathcal{M}, \mu).$$

The assertion in Eq. (11.13) is a consequence of the definition of $L^1(I)$ and $\hat{I}$ and this last equation.

Uniqueness. Suppose that ν is another measure on $\sigma(\mathbb{S})$ such that

$$I(f) = \int_X f d\nu \text{ for all } f \in \mathbb{S}.$$

By the monotone convergence theorem and the definition of I on $\mathbb{S}_\uparrow$,

$$I(f) = \int_X f d\nu \text{ for all } f \in \mathbb{S}_\uparrow.$$

Therefore if $A \in \sigma(\mathbb{S}) \subset \mathcal{R}$,

$$\begin{aligned} \mu(A) &= \bar{I}(1_A) = \inf\{I(h) : 1_A \leq h \in \mathbb{S}_\uparrow\} \\ &= \inf\{\int_X h d\nu : 1_A \leq h \in \mathbb{S}_\uparrow\} \geq \int_X 1_A d\nu = \nu(A) \end{aligned}$$

which shows$\nu \leq \mu$. If $A \in \sigma(\mathbb{S}) \subset \mathcal{R}$ with $\mu(A) < \infty$, then, by Remark 12.22 below, $1_A \in L^1(I)$ and therefore

$$\begin{aligned} \mu(A) &= \bar{I}(1_A) = \hat{I}(1_A) = \underline{I}(1_A) = \sup\{I(f) : \mathbb{S}_\downarrow \ni f \leq 1_A\} \\ &= \sup\{\int_X f d\nu : \mathbb{S}_\downarrow \ni f \leq 1_A\} \leq \nu(A). \end{aligned}$$

Hence $\mu(A) \leq \nu(A)$ for all $A \in \sigma(\mathbb{S})$ and $\nu(A) = \mu(A)$ when $\mu(A) < \infty$.

To prove $\nu(A) = \mu(A)$ for all $A \in \sigma(\mathbb{S})$, let $X_n := \{\chi \geq 1/n\} \in \sigma(\mathbb{S})$. Since $1_{X_n} \leq n\chi$,

$$\mu(X_n) = \int_X 1_{X_n} d\mu \leq \int_X n\chi d\mu = nI(\chi) < \infty.$$

Since $\chi > 0$ on X, $X_n \uparrow X$ and therefore by continuity of ν and μ,

$$\nu(A) = \lim_{n\to\infty} \nu(A \cap X_n) = \lim_{n\to\infty} \mu(A \cap X_n) = \mu(A)$$

for all $A \in \sigma(\mathbb{S})$. ■

The rest of this chapter is devoted to applications of the Daniell – Stone construction theorem.

Remark 11.23. To check the hypothesis in Theorem 11.22 that there exists $\chi \in \mathbb{S}_\uparrow$ such that $I(\chi) < \infty$ and $\chi(x) > 0$ for all $x \in X$, it suffices to find $\phi_n \in \mathbb{S}^+$ such that $\sum_{n=1}^\infty \phi_n > 0$ on X. To see this let $M_n := \max(\|\phi_n\|_u, I(\phi_n), 1)$ and define $\chi := \sum_{n=1}^\infty \frac{1}{M_n 2^n}\phi_n$, then $\chi \in \mathbb{S}_\uparrow$, $0 < \chi \leq 1$ and $I(\chi) \leq 1 < \infty$.

11.3. **Extensions of premeasures to measures I.** In this section let X be a set, $\mathcal{A}$ be a subalgebra of 2^X and $\mu_0 : \mathcal{A} \to [0,\infty]$ be a premeasure on $\mathcal{A}$.

Definition 11.24. Let $\mathcal{E}$ be a collection of subsets of X, let $\mathcal{E}_\sigma$ denote the collection of subsets of X which are finite or countable unions of sets from $\mathcal{E}$. Similarly let $\mathcal{E}_\delta$ denote the collection of subsets of X which are finite or countable intersections of sets from $\mathcal{E}$. We also write $\mathcal{E}_{\sigma\delta} = (\mathcal{E}_\sigma)_\delta$ and $\mathcal{E}_{\delta\sigma} = (\mathcal{E}_\delta)_\sigma$, etc.

Remark 11.25. Let μ_0 be a premeasure on an algebra $\mathcal{A}$. Any $A = \cup_{n=1}^\infty A'_n \in \mathcal{A}_\sigma$ with $A'_n \in \mathcal{A}$ may be written as $A = \coprod_{n=1}^\infty A_n$, with $A_n \in \mathcal{A}$ by setting $A_n := A'_n \setminus (A'_1 \cup \dots \cup A'_{n-1})$. If we also have $A = \coprod_{n=1}^\infty B_n$ with $B_n \in \mathcal{A}$, then $A_n = \coprod_{k=1}^\infty (A_n \cap B_k)$ and therefore because μ_0 is a premeasure,

$$\mu_0(A_n) = \sum_{k=1}^\infty \mu_0(A_n \cap B_k).$$

Summing this equation on n shows,

$$\sum_{n=1}^\infty \mu_0(A_n) = \sum_{n=1}^\infty \sum_{k=1}^\infty \mu_0(A_n \cap B_k)$$

By symmetry (i.e. the same argument with the A's and B's interchanged) and Fubini's theorem for sums,

$$\sum_{k=1}^\infty \mu_0(B_k) = \sum_{k=1}^\infty \sum_{n=1}^\infty \mu_0(A_n \cap B_k) = \sum_{n=1}^\infty \sum_{k=1}^\infty \mu_0(A_n \cap B_k)$$

and hence $\sum_{n=1}^\infty \mu_0(A_n) = \sum_{k=1}^\infty \mu_0(B_k)$. Therefore we may extend μ_0 to $\mathcal{A}_\sigma$ by setting

$$\mu_0(A) := \sum_{n=1}^\infty \mu_0(A_n)$$

if $A = \coprod_{n=1}^\infty A_n$, with $A_n \in \mathcal{A}$. In future we will tacitly assume this extension has been made.

Theorem 11.26. *Let X be a set, $\mathcal{A}$ be a subalgebra of 2^X and μ_0 be a premeasure on $\mathcal{A}$ which is σ – finite on $\mathcal{A}$, i.e. there exists $X_n \in \mathcal{A}$ such that $\mu_0(X_n) < \infty$ and $X_n \uparrow X$ as $n \to \infty$. Then μ_0 has a unique extension to a measure, μ, on $\mathcal{M} := \sigma(\mathcal{A})$. Moreover, if $A \in \mathcal{M}$ and $\epsilon > 0$ is given, there exists $B \in \mathcal{A}_\sigma$ such that $A \subset B$ and $\mu(B \setminus A) < \epsilon$. In particular,*

$$\mu(A) = \inf\{\mu_0(B) : A \subset B \in \mathcal{A}_\sigma\} \tag{11.14}$$

$$= \inf\{\sum_{n=1}^\infty \mu_0(A_n) : A \subset \coprod_{n=1}^\infty A_n \text{ with } A_n \in \mathcal{A}\}. \tag{11.15}$$

Proof. Let $(\mathcal{A}, \mu_0, I = I_{\mu_0})$ be as in Definition 11.4. As mentioned in Example 11.17, I is a Daniell integral on the lattice $\mathbb{S} = \mathbb{S}_f(\mathcal{A}, \mu_0)$. It is clear that $1 \wedge \phi \in \mathbb{S}$ for all $\phi \in \mathbb{S}$. Since $1_{X_n} \in \mathbb{S}^+$ and $\sum_{n=1}^\infty 1_{X_n} > 0$ on X,by Remark 11.23 there exists $\chi \in \mathbb{S}_\uparrow$ such that $I(\chi) < \infty$ and $\chi > 0$. So the hypothesis of Theorem 11.22 hold and hence there exits a unique measure μ on $\mathcal{M}$ such that $I(f) = \int_X f d\mu$ for all $f \in \mathbb{S}$. Taking $f = 1_A$ with $A \in \mathcal{A}$ and $\mu_0(A) < \infty$ shows $\mu(A) = \mu_0(A)$. For general $A \in \mathcal{A}$, we have

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) = \lim_{n\to\infty} \mu_0(A \cap X_n) = \mu_0(A).$$

The fact that μ is the only extension of μ_0 to $\mathcal{M}$ follows from Theorem 6.5 or Theorem 6.8. It is also can be proved using Theorem 11.22. Indeed, if ν is another measure on $\mathcal{M}$ such that $\nu = \mu$ on $\mathcal{A}$, then $I_\nu = I$ on $\mathbb{S}$. Therefore by the uniqueness assertion in Theorem 11.22, $\mu = \nu$ on $\mathcal{M}$.

By Eq. (11.13), for $A \in \mathcal{M}$,

$$\begin{aligned}\mu(A) = \bar{I}(1_A) &= \inf\{I(f) : f \in \mathbb{S}_\uparrow \text{ with } 1_A \le f\} \\ &= \inf\left\{\int_X f d\mu : f \in \mathbb{S}_\uparrow \text{ with } 1_A \le f\right\}.\end{aligned}$$

For the moment suppose $\mu(A) < \infty$ and $\epsilon > 0$ is given. Choose $f \in \mathbb{S}_\uparrow$ such that $1_A \le f$ and

$$\mu(f \ge 1) \le \int_X f d\mu = I(f) < \mu(A) + \epsilon.$$

Let $f_n \in \mathbb{S}$ be a sequence such that $f_n \uparrow f$ as $n \to \infty$ and set $B_m := \cup_{n=1}^\infty \{f_n \ge 1 - 1/m\} \in \mathcal{A}_\sigma$. Then $B_m \downarrow \{f \ge 1\}$ as $m \to \infty$,

$$A \subset \{f \ge 1\} \subset B_m \subset \{f \ge 1 - 1/m\}$$

and $\mu(f \ge 1 - 1/m) < \infty$ for $m > 1$ by Chebyshev's inequality. Therefore $\mu(B_m) \downarrow \mu(f \ge 1)$ as $m \to \infty$ and hence, for m sufficiently large, $\mu(B_m) < \mu(A) + \epsilon$. Therefore, there exists $B = B_m \in \mathcal{A}_\sigma$ such that $A \subset B$ and $\mu(B \setminus A) < \epsilon$.

For general $A \in \mathcal{A}$, choose $X_n \uparrow X$ with $X_n \in \mathcal{A}$. Then there exists $B_n \in \mathcal{A}_\sigma$ such that $\mu(B_n \setminus (A_n \cap X_n)) < \epsilon 2^{-n}$. Define $B := \cup_{n=1}^\infty B_n \in \mathcal{A}_\sigma$. Then

$$\begin{aligned}\mu(B \setminus A) = \mu(\cup_{n=1}^\infty (B_n \setminus A)) &\le \sum_{n=1}^\infty \mu((B_n \setminus A)) \\ &\le \sum_{n=1}^\infty \mu((B_n \setminus (A \cap X_n)) < \epsilon.\end{aligned}$$

Eq. (11.14) is an easy consequence of this result and the fact that $\mu(B) = \mu_0(B)$. ∎

Corollary 11.27 (Regularity of μ)**.** *Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra of sets, $\mathcal{M} = \sigma(\mathcal{A})$ and $\mu : \mathcal{M} \to [0, \infty]$ be a measure on $\mathcal{M}$ which is σ – finite on $\mathcal{A}$. Then*

1. *For all $A \in \mathcal{M}$,*

$$\mu(A) = \inf\{\mu(B) : A \subset B \in \mathcal{A}_\sigma\}. \tag{11.16}$$

2. *If $A \in \mathcal{M}$ and $\epsilon > 0$ are given, there exists $B \in \mathcal{A}_\sigma$ such that $A \subset B$ and $\mu(B \setminus A) < \epsilon$.*
3. *For all $A \in \mathcal{M}$ and $\epsilon > 0$ there exists $B \in \mathcal{A}_\delta$ such that $B \subset A$ and $\mu(A \setminus B) < \epsilon$.*
4. *For any $B \in \mathcal{M}$ there exists $A \in \mathcal{A}_{\delta\sigma}$ and $C \in \mathcal{A}_{\sigma\delta}$ such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$.*
5. *The linear space $\mathbb{S} := \mathbb{S}_f(\mathcal{A}, \mu)$ is dense in $L^p(\mu)$ for all $p \in [1, \infty)$, briefly put, $\overline{\mathbb{S}_f(\mathcal{A}, \mu)}^{L^p(\mu)} = L^p(\mu)$.*

Proof. Items 1. and 2. follow by applying Theorem 11.26 to $\mu_0 = \mu|_{\mathcal{A}}$. Items 3. and 4. follow from Items 1. and 2. as in the proof of Corollary 6.41 above.

Item 5. This has already been proved in Theorem 9.3 but we will give yet another proof here. When $p = 1$ and $g \in L^1(\mu;\mathbb{R})$, there exists, by Eq. (11.13), $h \in \mathbb{S}_\uparrow$ such that $g \le h$ and $\|h-g\|_1 = \int_X (h-g)d\mu < \epsilon$. Let $\{h_n\}_{n=1}^\infty \subset \mathbb{S}$ be chosen so that $h_n \uparrow h$ as $n \to \infty$. Then by the dominated convergence theorem, $\|h_n - g\|_1 \to \|h-g\|_1 < \epsilon$ as $n \to \infty$. Therefore for n large we have $h_n \in \mathbb{S}$ with $\|h_n - g\|_1 < \epsilon$. Since $\epsilon > 0$ is arbitrary this shows, $\overline{\mathbb{S}_f(\mathcal{A},\mu)}^{L^1(\mu)} = L^1(\mu)$.

Now suppose $p > 1$, $g \in L^p(\mu;\mathbb{R})$ and $X_n \in \mathcal{A}$ are sets such that $X_n \uparrow X$ and $\mu(X_n) < \infty$. By the dominated convergence theorem, $1_{X_n} \cdot [(g \wedge n) \vee (-n)] \to g$ in $L^p(\mu)$ as $n \to \infty$, so it suffices to consider $g \in L^p(\mu;\mathbb{R})$ with $\{g \ne 0\} \subset X_n$ and $|g| \le n$ for some large $n \in \mathbb{N}$. By Hölder's inequality, such a g is in $L^1(\mu)$. So if $\epsilon > 0$, by the $p = 1$ case, we may find $h \in \mathbb{S}$ such that $\|h-g\|_1 < \epsilon$. By replacing h by $(h \wedge n) \vee (-n) \in \mathbb{S}$, we may assume h is bounded by n as well and hence

$$\begin{aligned}\|h-g\|_p^p &= \int_X |h-g|^p \, d\mu = \int_X |h-g|^{p-1} |h-g| \, d\mu \\ &\le (2n)^{p-1} \int_X |h-g| \, d\mu < (2n)^{p-1} \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this shows $\mathbb{S}$ is dense in $L^p(\mu;\mathbb{R})$. ■

Remark 11.28. If we drop the σ – finiteness assumption on μ_0 we may loose uniqueness assertion in Theorem 11.26. For example, let $X = \mathbb{R}$, $\mathcal{B}_\mathbb{R}$ and $\mathcal{A}$ be the algebra generated by $\mathcal{E} := \{(a,b] \cap \mathbb{R} : -\infty \le a < b \le \infty\}$. Recall $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{E})$. Let $D \subset \mathbb{R}$ be a countable dense set and define $\mu_D(A) := \#(D \cap A)$. Then $\mu_D(A) = \infty$ for all $A \in \mathcal{A}$ such that $A \ne \emptyset$. So if $D' \subset \mathbb{R}$ is another countable dense subset of $\mathbb{R}$, $\mu_{D'} = \mu_D$ on $\mathcal{A}$ while $\mu_D \ne \mu_{D'}$ on $\mathcal{B}_\mathbb{R}$. Also notice that μ_D is σ – finite on $\mathcal{B}_\mathbb{R}$ but **not** on $\mathcal{A}$.

It is now possible to use Theorem 11.26 to give a proof of Theorem 5.8, see subsection 11.8 below. However rather than do this now let us give another application of Theorem 11.26 based on Example 11.16 and use the result to prove Theorem 5.8.

11.4. **Riesz Representation Theorem.**

Definition 11.29. Given a second countable locally compact Hausdorff space (X,τ), let $\mathbb{M}_+$ denote the collection of positive measures, μ, on $\mathcal{B}_X := \sigma(\tau)$ with the property that $\mu(K) < \infty$ for all compact subsets $K \subset X$. Such a measure μ will be called a **Radon** measure on X. For $\mu \in \mathbb{M}_+$ and $f \in C_c(X,\mathbb{R})$ let $I_\mu(f) := \int_X f d\mu$.

Theorem 11.30 (Riesz Representation Theorem)**.** *Let (X,τ) be a second countable*[26] *locally compact Hausdorff space. Then the map $\mu \to I_\mu$ taking $\mathbb{M}_+$ to positive linear functionals on $C_c(X,\mathbb{R})$ is bijective. Moreover every measure $\mu \in \mathbb{M}_+$ has the following properties:*

1. *For all $\epsilon > 0$ and $B \in \mathcal{B}_X$, there exists $F \subset B \subset U$ such that U is open and F is closed and $\mu(U \setminus F) < \epsilon$. If $\mu(B) < \infty$, F may be taken to be a compact subset of X.*

[26] The second countability is assumed here in order to avoid certain technical issues. Recall from Lemma 8.17 that under these assumptions, $\sigma(\mathbb{S}) = \mathcal{B}_X$. Also recall from Uryshon's metrizatoin theorem that X is metrizable. We will later remove the second countability assumption.

2. *For all $B \in \mathcal{B}_X$ there exists $A \in F_\sigma$ and $C \in \tau_\delta$ (τ_δ is more conventionally written as G_δ) such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$.*
3. *For all $B \in \mathcal{B}_X$,*

$$\mu(B) = \inf\{\mu(U) : B \subset U \text{ and } U \text{ is open}\} \tag{11.17}$$

$$= \sup\{\mu(K) : K \subset B \text{ and } K \text{ is compact}\}. \tag{11.18}$$

4. *For all open subsets, $U \subset X$,*

$$\mu(U) = \sup\{\int_X f d\mu : f \prec X\} = \sup\{I_\mu(f) : f \prec X\}. \tag{11.19}$$

5. *For all compact subsets $K \subset X$,*

$$\mu(K) = \inf\{I_\mu(f) : 1_K \le f \prec X\}. \tag{11.20}$$

6. *If $\|I_\mu\|$ denotes the dual norm on $C_c(X, \mathbb{R})^*$, then $\|I_\mu\| = \mu(X)$. In particular I_μ is bounded iff $\mu(X) < \infty$.*
7. *$C_c(X, \mathbb{R})$ is dense in $L^p(\mu; \mathbb{R})$ for all $1 \le p < \infty$.*

Proof. First notice that I_μ is a positive linear functional on $\mathbb{S} := C_c(X, \mathbb{R})$ for all $\mu \in \mathbb{M}_+$ and $\mathbb{S}$ is a lattice such that $1 \wedge f \in \mathbb{S}$ for all $f \in \mathbb{S}$. Example 11.16 shows that any positive linear functional, I, on $\mathbb{S} := C_c(X, \mathbb{R})$ is a Daniell integral on $\mathbb{S}$. By Lemma 8.10, there exists compact sets $K_n \subset X$ such that $K_n \uparrow X$. By Urysohn's lemma, there exists $\phi_n \prec X$ such that $\phi_n = 1$ on K_n. Since $\phi_n \in \mathbb{S}^+$ and $\sum_{n=1}^\infty \phi_n > 0$ on X it follows from Remark 11.23 that there exists $\chi \in \mathbb{S}_\uparrow$ such that $\chi > 0$ on X and $I(\chi) < \infty$. So the hypothesis of the Daniell – Stone Theorem 11.22 hold and hence there exists a unique measure μ on $\sigma(\mathbb{S}) = \mathcal{B}_X$ (Lemma 8.17) such that $I = I_\mu$. Hence the map $\mu \to I_\mu$ taking $\mathbb{M}_+$ to positive linear functionals on $C_c(X, \mathbb{R})$ is bijective. We will now prove the remaining seven assertions of the theorem.

1. Suppose $\epsilon > 0$ and $B \in \mathcal{B}_X$ satisfies $\mu(B) < \infty$. Then $1_B \in L^1(\mu)$ so there exists functions $f_n \in C_c(X, \mathbb{R})$ such that $f_n \uparrow f$, $1_B \le f$, and $\int_X f d\mu < \mu(B) + \epsilon$. The condition $1_B \le f$, implies $1_B \le 1_{\{f \ge 1\}} \le f$ and hence

$$\mu(B) \le \mu(f \ge 1) \le I(f) < \mu(B) + \epsilon. \tag{11.21}$$

Letting $U_m := \cup_{n=1}^\infty \{f_n > 1 - 1/m\} \in \tau_d$, then $U_m \subset \{f > 1 - 1/m\}$ which has finite measure by Chebyshev's inequality, $U_m \downarrow \{f \ge 1\} \supset B$ and hence $\mu(U_m) \downarrow \mu(f \ge 1) \ge \mu(B)$ as $m \to \infty$. Combining this observation with Eq. (11.21), we may choose m sufficiently large so that $B \subset U_m$ and

$$\mu(U_m \setminus B) = \mu(U_m) - \mu(B) < \epsilon.$$

Hence there exists $U \in \tau$ such that $B \subset U$ and $\mu(U \setminus B) < \epsilon$.

For general $B \in \mathcal{B}_X$, by what we have just proved, there exists open sets $U_n \subset X$ such that $B \cap K_n \subset U_n$ and $\mu(U_n \setminus (B \cap K_n)) < \epsilon 2^{-n}$ for all n. Let $U = \cup_{n=1}^\infty U_n$, then $B \subset U \in \tau$ and

$$\mu(U \setminus B) = \mu(\cup_{n=1}^\infty (U_n \setminus B)) \le \sum_{n=1}^\infty \mu(U_n \setminus B)$$
$$\le \sum_{n=1}^\infty \mu(U_n \setminus (B \cap K_n)) \le \sum_{n=1}^\infty \epsilon 2^{-n} = \epsilon.$$

Applying this result to B^c shows there exists a closed set $F \sqsubset X$ such that $B^c \subset F^c$ and

$$\mu(B \setminus F) = \mu(F^c \setminus B^c) < \epsilon.$$

So we have produced $F \subset B \subset U$ such that $\mu(U \setminus F) = \mu(U \setminus B) + \mu(B \setminus F) < 2\epsilon$.

If $\mu(B) < \infty$, using $B \setminus (K_n \cap F) \uparrow B \setminus F$ as $n \to \infty$, we may choose n sufficiently large so that $\mu(B \setminus (K_n \cap F)) < \epsilon$. Hence we may replace F by the compact set $F \cap K_n$ if necessary.

2. Choose $F_n \subset B \subset U_n$ such F_n is closed, U_n is open and $\mu(U_n \setminus F_n) < 1/n$. Let $B = \cup_n F_n \in F_\sigma$ and $C := \cap U_n \in \tau_\delta$. Then $A \subset B \subset C$ and

$$\mu(C \setminus A) \le \mu(F_n \setminus U_n) < \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

3. From Item 1, one easily concludes that

$$\mu(B) = \inf\left\{\mu(U) : B \subset U \subset_o X\right\}$$

for all $B \in \mathcal{B}_X$ and

$$\mu(B) = \sup\left\{\mu(K) : K \sqsubset\sqsubset B\right\}$$

for all $B \in \mathcal{B}_X$ with $\mu(B) < \infty$. So now suppose $B \in \mathcal{B}_X$ and $\mu(B) = \infty$. Using the notation at the end of the proof of Item 1., we have $\mu(F) = \infty$ and $\mu(F \cap K_n) \uparrow \infty$ as $n \to \infty$. This shows $\sup\left\{\mu(K) : K \sqsubset\sqsubset B\right\} = \infty = \mu(B)$ as desired.

4. For $U \subset_o X$, let

$$\nu(U) := \sup\{I_\mu(f) : f \prec U\}.$$

It is evident that $\nu(U) \le \mu(U)$ because $f \prec U$ implies $f \le 1_U$. Let K be a compact subset of U. By Urysohn's Lemma 8.15, there exists $f \prec U$ such that $f = 1$ on K. Therefore,

$$\mu(K) \le \int_X f d\mu \le \nu(U) \tag{11.22}$$

and we have

$$\mu(K) \le \nu(U) \le \mu(U) \text{ for all } U \subset_o X \text{ and } K \sqsubset\sqsubset U. \tag{11.23}$$

By Item 3.,

$$\mu(U) = \sup\{\mu(K) : K \sqsubset\sqsubset U\} \le \nu(U) \le \mu(U)$$

which shows that $\mu(U) = \nu(U)$, i.e. Eq. (11.19) holds.

5. Now suppose K is a compact subset of X. From Eq. (11.22),

$$\mu(K) \le \inf\{I_\mu(f) : 1_K \le f \prec X\} \le \mu(U)$$

for any open subset U such that $K \subset U$. Consequently by Eq. (11.17),

$$\mu(K) \le \inf\{I_\mu(f) : 1_K \le f \prec X\} \le \inf\{\mu(U) : K \subset U \subset_o X\} = \mu(K)$$

which proves Eq. (11.20).

6. For $f \in C_c(X, \mathbb{R})$,

$$|I_\mu(f)| \le \int_X |f|\, d\mu \le \|f\|_u \,\mu(\text{supp}(f)) \le \|f\|_u \,\mu(X) \tag{11.24}$$

which shows $\|I_\mu\| \le \mu(X)$. Let $K \sqsubset\sqsubset X$ and $f \prec X$ such that $f = 1$ on K. By Eq. (11.22),

$$\mu(K) \le \int_X f d\mu = I_\mu(f) \le \|I_\mu\| \|f\|_u = \|I_\mu\|$$

and therefore,

$$\mu(X) = \sup\{\mu(K) : K \sqsubset\sqsubset X\} \le \|I_\mu\| .$$

7. This has already been proved by two methods in Proposition 9.6 but we will give yet another proof here. When $p = 1$ and $g \in L^1(\mu;\mathbb{R})$, there exists, by Eq. (11.13), $h \in \mathbb{S}_\uparrow = C_c(X,\mathbb{R})_\uparrow$ such that $g \le h$ and $\|h-g\|_1 = \int_X (h-g)d\mu < \epsilon$. Let $\{h_n\}_{n=1}^\infty \subset \mathbb{S} = C_c(X,\mathbb{R})$ be chosen so that $h_n \uparrow h$ as $n \to \infty$. Then by the dominated convergence theorem (notice that $|h_n| \le |h_1| + |h|$), $\|h_n - g\|_1 \to \|h-g\|_1 < \epsilon$ as $n \to \infty$. Therefore for n large we have $h_n \in C_c(X,\mathbb{R})$ with $\|h_n - g\|_1 < \epsilon$. Since $\epsilon > 0$ is arbitrary this shows, $\overline{\mathbb{S}_f(\mathcal{A},\mu)}^{L^1(\mu)} = L^1(\mu)$.
   Now suppose $p > 1$, $g \in L^p(\mu;\mathbb{R})$ and $\{K_n\}_{n=1}^\infty$ are as above. By the dominated convergence theorem, $1_{K_n}(g \wedge n) \vee (-n) \to g$ in $L^p(\mu)$ as $n \to \infty$, so it suffices to consider $g \in L^p(\mu;\mathbb{R})$ with $\mathrm{supp}(g) \subset K_n$ and $|g| \le n$ for some large $n \in \mathbb{N}$. By Hölder's inequality, such a g is in $L^1(\mu)$. So if $\epsilon > 0$, by the $p = 1$ case, there exists $h \in \mathbb{S}$ such that $\|h-g\|_1 < \epsilon$. By replacing h by $(h \wedge n) \vee (-n) \in \mathbb{S}$, we may assume h is bounded by n in which case

$$\|h-g\|_p^p = \int_X |h-g|^p \, d\mu = \int_X |h-g|^{p-1} |h-g| \, d\mu$$
$$\le (2n)^{p-1} \int_X |h-g| \, d\mu < (2n)^{p-1} \epsilon.$$

   Since $\epsilon > 0$ was arbitrary, this shows $\mathbb{S}$ is dense in $L^p(\mu;\mathbb{R})$.

■

Remark 11.31. We may give a direct proof of the fact that $\mu \to I_\mu$ is injective. Indeed, suppose $\mu, \nu \in \mathbb{M}_+$ satisfy $I_\mu(f) = I_\nu(f)$ for all $f \in C_c(X,\mathbb{R})$. By Proposition 9.6, if $A \in \mathcal{B}_X$ is a set such that $\mu(A) + \nu(A) < \infty$, there exists $f_n \in C_c(X,\mathbb{R})$ such that $f_n \to 1_A$ in $L^1(\mu+\nu)$. Since $f_n \to 1_A$ in $L^1(\mu)$ and $L^1(\nu)$,

$$\mu(A) = \lim_{n\to\infty} I_\mu(f_n) = \lim_{n\to\infty} I_\nu(f_n) = \nu(A).$$

For general $A \in \mathcal{B}_X$, choose compact subsets $K_n \subset X$ such that $K_n \uparrow X$. Then

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap K_n) = \lim_{n\to\infty} \nu(A \cap K_n) = \nu(A)$$

showing $\mu = \nu$. Therefore the map $\mu \to I_\mu$ is injective.

Theorem 11.32 (Lusin's Theorem). *Suppose (X,τ) is a locally compact and second countable Hausdorff space, $\mathcal{B}_X$ is the Borel σ – algebra on X, and μ is a measure on $(X,\mathcal{B}_X)$ which is finite on compact sets of X. Also let $\epsilon > 0$ be given. If $f : X \to \mathbb{C}$ is a measurable function such that $\mu(f \ne 0) < \infty$, there exists a compact set $K \subset \{f \ne 0\}$ such that $f|_K$ is continuous and $\mu(\{f \ne 0\} \setminus K) < \epsilon$. Moreover there exists $\phi \in C_c(X)$ such that $\mu(f \ne \phi) < \epsilon$ and if f is bounded the function ϕ may be chosen so that $\|\phi\|_u \le \|f\|_u := \sup_{x\in X} |f(x)|$.*

Proof. Suppose first that f is bounded, in which case

$$\int_X |f| \, d\mu \le \|f\|_\mu \, \mu(f \ne 0) < \infty.$$

By Proposition 9.6 or Item 7. of Theorem 11.30, there exists $f_n \in C_c(X)$ such that $f_n \to f$ in $L^1(\mu)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume $\|f - f_n\|_1 < \epsilon n^{-1} 2^{-n}$ for all n and thus $\mu\left(|f - f_n| > n^{-1}\right) < \epsilon 2^{-n}$ for all n. Let $E := \cup_{n=1}^\infty \left\{|f - f_n| > n^{-1}\right\}$, so that $\mu(E) < \epsilon$. On E^c, $|f - f_n| \le 1/n$, i.e. $f_n \to f$ uniformly on E^c and hence $f|_{E^c}$ is continuous.

Let $A := \{ f \neq 0\} \setminus E$. By Theorem 11.30 (or see Exercises 6.4 and 6.5) there exists a compact set K and open set V such that $K \subset A \subset V$ such that $\mu(V \setminus K) < \epsilon$. Notice that

$$\mu(\{f \neq 0\} \setminus K) \le \mu(A \setminus K) + \mu(E) < 2\epsilon.$$

By the Tietze extension Theorem 8.16, there exists $F \in C(X)$ such that $f = F|_K$. By Urysohn's Lemma 8.15 there exists $\psi \prec V$ such that $\psi = 1$ on K. So letting $\phi = \psi F \in C_c(X)$, we have $\phi = f$ on K, $\|\phi\|_u \le \|f\|_u$ and since $\{\phi \neq f\} \subset E \cup (V \setminus K)$, $\mu(\phi \neq f) < 3\epsilon$. This proves the assertions in the theorem when f is bounded.

Suppose that $f : X \to \mathbb{C}$ is (possibly) unbounded. By Lemmas 8.17 and 8.10, there exists compact sets $\{K_N\}_{N=1}^\infty$ of X such that $K_N \uparrow X$. Hence $B_N := K_N \cap \{0 < |f| \le N\} \uparrow \{f \neq 0\}$ as $N \to \infty$. Therefore if $\epsilon > 0$ is given there exists an N such that $\mu(\{f \neq 0\} \setminus B_N) < \epsilon$. We now apply what we have just proved to $1_{B_N} f$ to find a compact set $K \subset \{1_{B_N} f \neq 0\}$, and open set $V \subset X$ and $\phi \in C_c(V) \subset C_c(X)$ such that $\mu(V \setminus K) < \epsilon$, $\mu(\{1_{B_N} f \neq 0\} \setminus K) < \epsilon$ and $\phi = f$ on K. The proof is now complete since

$$\{\phi \neq f\} \subset (\{f \neq 0\} \setminus B_N) \cup (\{1_{B_N} f \neq 0\} \setminus K) \cup (V \setminus K)$$

so that $\mu(\phi \neq f) < 3\epsilon$. ■

To illustrate Theorem 11.32, suppose that $X = (0,1)$, $\mu = m$ is Lebesgue measure and $f = 1_{(0,1)\cap\mathbb{Q}}$. Then Lusin's theorem asserts for any $\epsilon > 0$ there exists a compact set $K \subset (0,1)$ such that $m((0,1) \setminus K) < \epsilon$ and $f|_K$ is continuous. To see this directly, let $\{r_n\}_{n=1}^\infty$ be an enumeration of the rationales in $(0,1)$,

$$J_n = (r_n - \epsilon 2^{-n}, r_n + \epsilon 2^{-n}) \cap (0,1) \text{ and } W = \cup_{n=1}^\infty J_n.$$

Then W is an open subset of X and $\mu(W) < \epsilon$. Therefore $K_n := [1/n, 1 - 1/n] \setminus W$ is a compact subset of X and $m(X \setminus K_n) \le \frac{2}{n} + \mu(W)$. Taking n sufficiently large we have $m(X \setminus K_n) < \epsilon$ and $f|_{K_n} \equiv 0$ is continuous.

11.4.1. *The Riemann – Stieljtes – Lebesgue Integral.*

Notation 11.33. Given an increasing function $F : \mathbb{R} \to \mathbb{R}$, let $F(x-) = \lim_{y \uparrow x} F(y)$, $F(x+) = \lim_{y \downarrow x} F(y)$ and $F(\pm\infty) = \lim_{x \to \pm\infty} F(x) \in \bar{\mathbb{R}}$. Since F is increasing all of theses limits exists.

Theorem 11.34. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and define $G(x) = F(x+)$. Then*

1. *The function G is increasing and right continuous.*
2. *For $x \in \mathbb{R}$, $G(x) = \lim_{y \downarrow x} F(y-)$.*
3. *The set $\{x \in \mathbb{R} : F(x+) > F(x-)\}$ is countable and for each $N > 0$, and moreover,*

$$\sum_{x \in (-N,N]} [F(x+) - F(x-)] \le F(N) - F(-N) < \infty. \tag{11.25}$$

Proof.

1. The following observation shows G is increasing: if $x < y$ then

$$F(x-) \le F(x) \le F(x+) = G(x) \le F(y-) \le F(y) \le F(y+) = G(y). \tag{11.26}$$

Since G is increasing, $G(x) \le G(x+)$. If $y > x$ then $G(x+) \le F(y)$ and hence $G(x+) \le F(x+) = G(x)$, i.e. $G(x+) = G(x)$.

2. Since $G(x) \le F(y-) \le F(y)$ for all $y > x$, it follows that

$$G(x) \le \lim_{y \downarrow x} F(y-) \le \lim_{y \downarrow x} F(y) = G(x)$$

showing $G(x) = \lim_{y \downarrow x} F(y-)$.

3. By Eq. (11.26), if $x \neq y$ then

$$(F(x-), F(x+)] \cap (F(y-), F(y+)] = \emptyset.$$

Therefore, $\{(F(x-), F(x+)]\}_{x \in \mathbb{R}}$ are disjoint possible empty intervals in $\mathbb{R}$. Let $N \in \mathbb{N}$ and $\alpha \subset\subset (-N, N)$ be a finite set, then

$$\coprod_{x \in \alpha} (F(x-), F(x+)] \subset (F(-N), F(N)]$$

and therefore,

$$\sum_{x \in \alpha} [F(x+) - F(x-)] \le F(N) - F(-N) < \infty.$$

Since this is true for all $\alpha \subset\subset (-N, N]$, Eq. (11.25) holds. Eq. (11.25) shows

$$\Gamma_N := \{x \in (-N, N) | F(x+) - F(x-) > 0\}$$

is countable and hence so is

$$\Gamma := \{x \in \mathbb{R} | F(x+) - F(x-) > 0\} = \cup_{N=1}^{\infty} \Gamma_N.$$

■

Theorem 11.35. *If $F : \mathbb{R} \to \mathbb{R}$ is an increasing function, there exists a unique measure $\mu = \mu_F$ on $\mathcal{B}_{\mathbb{R}}$ such that*

$$\int_{-\infty}^{\infty} f dF = \int_{\mathbb{R}} f d\mu \text{ for all } f \in C_c(\mathbb{R}, \mathbb{R}). \tag{11.27}$$

This measure may also be characterized as the unique measure on $\mathcal{B}_{\mathbb{R}}$ such that

$$\mu((a, b]) = F(b+) - F(a+) \text{ for all } -\infty < a < b < \infty. \tag{11.28}$$

Moreover, if $A \in \mathcal{B}_{\mathbb{R}}$ then

$$\begin{aligned} \mu_F(A) &= \inf \left\{ \sum_{i=1}^{\infty} (F(b_i+) - F(a_i+)) : A \subset \cup_{i=1}^{\infty} (a_i, b_i] \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} (F(b_i+) - F(a_i+)) : A \subset \coprod_{i=1}^{\infty} (a_i, b_i] \right\}. \end{aligned}$$

Proof. An application of Theorem 11.30 implies there exists a unique measure μ on $\mathcal{B}_{\mathbb{R}}$ such Eq. (11.27) is valid. Let $-\infty < a < b < \infty$, $\epsilon > 0$ be small and $\phi_\epsilon(x)$ be the function defined in Figure 28, i.e. ϕ_ϵ is one on $[a + 2\epsilon, b + \epsilon]$, linearly interpolates to zero on $[b + \epsilon, b + 2\epsilon]$ and on $[a + \epsilon, a + 2\epsilon]$ and is zero on $(a, b + 2\epsilon)^c$. Since $\phi_\epsilon \to 1_{(a,b]}$ it follows by the dominated convergence theorem that

$$\mu((a, b]) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \phi_\epsilon d\mu = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \phi_\epsilon dF. \tag{11.29}$$

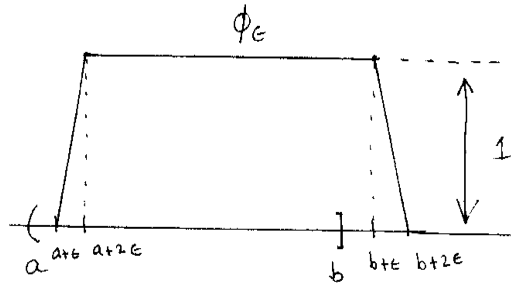

FIGURE 28. .

On the other hand we have $1_{(a+2\epsilon,b+\epsilon]} \le \phi_\epsilon \le 1_{(a+\epsilon,b+2\epsilon]}$ and therefore,

$$\begin{aligned} F(b+\epsilon) - F(a+2\epsilon) &= \int_{\mathbb{R}} 1_{(a+2\epsilon,b+\epsilon]} dF \\ &\le \int_{\mathbb{R}} \phi_\epsilon dF \le \int_{\mathbb{R}} 1_{(a+\epsilon,b+2\epsilon)} dF = F(b+2\epsilon) - F(a+\epsilon). \end{aligned}$$

Letting $\epsilon \downarrow 0$ in this equation and using Eq. (11.29) shows

$$F(b+) - F(a+) \le \mu((a,b]) \le F(b+) - F(a+).$$

The last assertion in the theorem is now a consequence of Corollary 11.27. ∎

11.5. Metric space regularity results resisted.

Proposition 11.36. *Let (X,d) be a metric space and μ be a measure on $\mathcal{M} = \mathcal{B}_X$ which is σ – finite on $\tau := \tau_d$.*

1. *For all $\epsilon > 0$ and $B \in \mathcal{M}$ there exists an open set $V \in \tau$ and a closed set F such that $F \subset B \subset V$ and $\mu(V \setminus F) \le \epsilon$.*
2. *For all $B \in \mathcal{M}$, there exists $A \in F_\sigma$ and $C \in G_\delta$ such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$. Here F_σ denotes the collection of subsets of X which may be written as a countable union of closed sets and $G_\delta = \tau_\delta$ is the collection of subsets of X which may be written as a countable intersection of open sets.*
3. *The space $BC_f(X)$ of bounded continuous functions on X such that $\mu(f \ne 0) < \infty$ is dense in $L^p(\mu)$.*

Proof. Let $\mathbb{S} := BC_f(X)$, $I(f) := \int_X f d\mu$ for $f \in \mathbb{S}$ and $X_n \in \tau$ be chosen so that $\mu(X_n) < \infty$ and $X_n \uparrow X$ as $n \to \infty$. Then $1 \wedge f \in \mathbb{S}$ for all $f \in \mathbb{S}$ and if $\phi_n = 1 \wedge \left(n d_{X_n^c}\right) \in \mathbb{S}^+$, then $\phi_n \uparrow 1$ as $n \to \infty$ and so by Remark 11.23 there exists $\chi \in \mathbb{S}_\uparrow$ such that $\chi > 0$ on X and $I(\chi) < \infty$. Similarly if $V \in \tau$, the function $g_n := 1 \wedge \left(n d_{(X_n \cap V)^c}\right) \in \mathbb{S}$ and $g_n \to 1_V$ as $n \to \infty$ showing $\sigma(\mathbb{S}) = \mathcal{B}_X$. If $f_n \in \mathbb{S}^+$ and $f_n \downarrow 0$ as $n \to \infty$, it follows by the dominated convergence theorem that $I(f_n) \downarrow 0$ as $n \to \infty$. So the hypothesis of the Daniell – Stone Theorem 11.22 hold and hence μ is the unique measure on $\mathcal{B}_X$ such that $I = I_\mu$ and for $B \in \mathcal{B}_X$ and

$$\begin{aligned} \mu(B) = \bar{I}(1_B) &= \inf\{I(f) : f \in \mathbb{S}_\uparrow \text{ with } 1_B \le f\} \\ &= \inf\left\{\int_X f d\mu : f \in \mathbb{S}_\uparrow \text{ with } 1_B \le f\right\}. \end{aligned}$$

Suppose $\epsilon > 0$ and $B \in \mathcal{B}_X$ are given. There exists $f_n \in BC_f(X)$ such that $f_n \uparrow f$, $1_B \leq f$, and $\mu(f) < \mu(B) + \epsilon$. The condition $1_B \leq f$, implies $1_B \leq 1_{\{f \geq 1\}} \leq f$ and hence that

$$\mu(B) \leq \mu(f \geq 1) \leq \mu(f) < \mu(B) + \epsilon. \tag{11.30}$$

Moreover, letting $V_m := \cup_{n=1}^{\infty} \{f_n \geq 1 - 1/m\} \in \tau_d$, we have $V_m \downarrow \{f \geq 1\} \supset B$ hence $\mu(V_m) \downarrow \mu(f \geq 1) \geq \mu(B)$ as $m \to \infty$. Combining this observation with Eq. (11.30), we may choose m sufficiently large so that $B \subset V_m$ and

$$\mu(V_m \setminus B) = \mu(V_m) - \mu(B) < \epsilon.$$

Hence there exists $V \in \tau$ such that $B \subset V$ and $\mu(V \setminus B) < \epsilon$. Applying this result to B^c shows there exists $F \sqsubset X$ such that $B^c \subset F^c$ and

$$\mu(B \setminus F) = \mu(F^c \setminus B^c) < \epsilon.$$

So we have produced $F \subset B \subset V$ such that $\mu(V \setminus F) = \mu(V \setminus B) + \mu(B \setminus F) < 2\epsilon$.

The second assertion is an easy consequence of the first and the third follows in similar manner to any of the proofs of Item 7. in Theorem 11.30. ■

11.6. **Measure on Products of Metric spaces.** Let $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ be a sequence of compact metric spaces, for $N \in \mathbb{N}$ let $X_N := \prod_{n=1}^{N} X_n$ and $\pi_N : X \to X_N$ be the projection map $\pi_N(x) = x|_{\{1,2,\ldots,N\}}$. Recall from Exercise 3.27 and Exercise 4.15 that there is a metric d on $X := \prod_{n\in\mathbb{N}} X_n$ such that $\tau_d = \otimes_{n=1}^{\infty} \tau_{d_n}$ $(= \tau(\pi_n : n \in \mathbb{N})$ – the product topology on $X)$ and X is compact in this topology. Also recall that compact metric spaces are second countable, Exercise 8.4.

Proposition 11.37. *Continuing the notation above, suppose that $\{\mu_N\}_{N\in\mathbb{N}}$ are given probability measures*[27] *on $\mathcal{B}_N := \mathcal{B}_{X_N}$ satisfying the compatibility conditions, $(\pi_N)_* \mu_M = \mu_N$ for all $N \leq M$. Then there exists a unique measure μ on $\mathcal{B}_X = \sigma(\tau_d) = \sigma(\pi_n : n \in \mathbb{N})$ such that $(\pi_N)_* \mu = \mu_N$ for all $N \in \mathbb{N}$, i.e.*

$$\int_X f(\pi_N(x)) d\mu(x) = \int_{X_N} f(y) d\mu_N(y) \tag{11.31}$$

for all $N \in \mathbb{N}$ and $f : X_N \to \mathbb{R}$ bounded a measurable.

Proof. An application of the Stone Weierstrass Theorem 9.43 shows that

$$\mathcal{D} = \{f \in C(X) : f = F \circ \pi_N \text{ with } F \in C(X_N) \text{ and } N \in \mathbb{N}\}$$

is dense in $C(X)$. For $f = F \circ \pi_N \in \mathcal{D}$ let

$$I(f) = \int_{X_N} F \circ \pi_N(x) d\mu_N(x).$$

Let us verify that I is well defined. Suppose that f may also be expressed as $f = G \circ \pi_M$ with $M \in \mathbb{N}$ and $G \in C(X_M)$. By interchanging M and N if necessary we may assume $M \geq N$. By the compatibility assumption,

$$\begin{aligned}\int_{X_M} G(z) d\mu_M(z) &= \int_{X_M} F \circ \pi_N(x) d\mu_M(x) = \int_{X_N} F d\left[(\pi_N)_* \mu_M\right] \\ &= \int_{X_N} F \circ \pi_N d\mu_N.\end{aligned}$$

[27] A typical example of such measures, μ^N, is to set $\mu^N := \mu_1 \otimes \cdots \otimes \mu_N$ where μ_n is a probablity measure on $\mathcal{B}_{X_n}$ for each $n \in \mathbb{N}$.

Since $|I(f)| \le \|f\|_\infty$, the B.L.T. theorem allows us to extend I uniquely to a continuous linear functional on $C(X)$ which we still denote by I. Because I was positive on $\mathcal{D}$, it is easy to check that I is positive on $C(X)$ as well. So by the Riesz Theorem 11.30, there exists a probability measure μ on $\mathcal{B}_X$ such that $I(f) = \int_X f d\mu$ for all $f \in C(X)$. By the definition of I in now follows that

$$\int_{X_N} F d\left(\pi_N\right)_* \mu = \int_{X_N} F \circ \pi_N d\mu = I(F \circ \pi_N) = \int_{X_N} F d\mu_N$$

for all $F \in C(X_N)$ and $N \in \mathbb{N}$. It now follows from Theorem 9.43 or the uniqueness assertion in the Riesz theorem 11.30 (applied with X replaced by X_N) that $\pi_{N*}\mu = \mu_N$. ■

Corollary 11.38. *Keeping the same assumptions from Proposition 11.37. Further assume, for each $n \in \mathbb{N}$, there exists measurable set $Y_n \subset X_n$ such that $\mu_N(Y_N) = 1$ with $Y_N := Y_1 \times \cdots \times Y_N$. Then $\mu(Y) = 1$ where $Y = \prod_{i=1}^\infty Y_i \subset X$.*

Proof. Since $Y = \cap_{N=1}^\infty \pi_N^{-1}(Y_N)$, we have $X \setminus Y = \cup_{N=1}^\infty \pi_N^{-1}(X_N \setminus Y_N)$ and therefore,

$$\mu(X \setminus Y) \le \sum_{N=1}^\infty \mu\left(\pi_N^{-1}(X_N \setminus Y_N)\right) = \sum_{N=1}^\infty \mu_N\left(X_N \setminus Y_N\right) = 0.$$

■

Corollary 11.39. *Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ are probability measures on $\mathcal{B}_{\mathbb{R}^d}$ for all $n \in \mathbb{N}$, $X := \left(\mathbb{R}^d\right)^{\mathbb{N}}$ and $\mathcal{B} := \otimes_{n=1}^\infty \left(\mathcal{B}_{\mathbb{R}^d}\right)$. Then there exists a unique measure μ on $(X, \mathcal{B})$ such that*

$$\int_X f(x_1, x_2, \ldots, x_N) d\mu(x) = \int_{(\mathbb{R}^d)^N} f(x_1, x_2, \ldots, x_N) d\mu_1(x_1) \ldots d\mu_N(x_N) \tag{11.32}$$

for all $N \in \mathbb{N}$ and bounded measurable functions $f : \left(\mathbb{R}^d\right)^N \to \mathbb{R}$.

Proof. Let $\left(\mathbb{R}^d\right)^*$ denote the Alexandrov compactification of $\mathbb{R}^d$. Recall form Exercise 8.11 that $\left(\mathbb{R}^d\right)^*$ is homeomorphic to S^{d+1} and hence $\left(\mathbb{R}^d\right)^*$ is a compact metric space. (Alternatively see Exercise 8.14.) Let $\bar{\mu}_n := i_*\mu_n = \mu_n \circ i^{-1}$ where $i : \mathbb{R}^d \to \left(\mathbb{R}^d\right)^*$ is the inclusion map. Then $\bar{\mu}_n$ is a probability measure on $\mathcal{B}_{(\mathbb{R}^d)^*}$ such that $\bar{\mu}_n\left(\{\infty\}\right) = 0$. An application of Proposition 11.37 and Corollary 11.38 completes the proof. ■

Exercise 11.3. Extend Corollary 11.39 to construct arbitrary (not necessarily countable) products of $\mathbb{R}^d$.

11.7. **Measures on general infinite product spaces.** In this section we drop the topological assumptions used in the last section.

Proposition 11.40. *Let $\{(X_\alpha, \mathcal{M}_\alpha, \mu_\alpha)\}_{\alpha\in A}$ be a collection of probability spaces, that is $\mu_\alpha(X_a) = 1$ for all $\alpha \in A$. Let $X \equiv \prod_{\alpha\in A} X_\alpha$, $\mathcal{M} = \sigma(\pi_\alpha : \alpha \in A)$ and for $\Lambda \subset\subset A$ let $X_\Lambda := \prod_{\alpha\in\Lambda} X_\alpha$ and $\pi_\Lambda : X \to X_\Lambda$ be the projection map $\pi_\Lambda(x) = x|_\Lambda$ and $\mu_\Lambda := \prod_{\alpha\in\Lambda} \mu_\alpha$ be product measure on $\mathcal{M}_\Lambda := \otimes_{\alpha\in\Lambda}\mathcal{M}_\alpha$. Then there exists a* *unique measure μ on $\mathcal{M}$ such that $(\pi_\Lambda)_* \mu = \mu_\Lambda$ for all $\Lambda \subset\subset A$, i.e. if $f : X_\Lambda \to \mathbb{R}$ is a bounded measurable function then*

$$\int_X f(\pi_\Lambda(x))d\mu(x) = \int_{X_\Lambda} f(y)d\mu_\Lambda(y). \tag{11.33}$$

Proof. Let $\mathbb{S}$ denote the collection of functions $f : X \to \mathbb{R}$ such that there exists $\Lambda \subset\subset A$ and a bounded measurable function $F : X_\Lambda \to \mathbb{R}$ such that $f = F \circ \pi_\Lambda$. For $f = F \circ \pi_\Lambda \in \mathbb{S}$, let $I(f) = \int_{X_\Lambda} F d\mu_\Lambda$.

Let us verify that I is well defined. Suppose that f may also be expressed as $f = G \circ \pi_\Gamma$ with $\Gamma \subset\subset A$ and $G : X_\Gamma \to \mathbb{R}$ bounded and measurable. By replacing Γ by $\Gamma \cup \Lambda$ if necessary, we may assume that $\Lambda \subset \Gamma$. Making use of Fubini's theorem we learn

$$\begin{aligned}\int_{X_\Gamma} G(z)\, d\mu_\Gamma(z) &= \int_{X_\Lambda \times X_{\Gamma\setminus\Lambda}} F \circ \pi_\Lambda(x)\, d\mu_\Lambda(x) d\mu_{\Gamma\setminus\Lambda}(y) \\ &= \int_{X_\Lambda} F \circ \pi_\Lambda(x)\, d\mu_\Lambda(x) \cdot \int_{X_{\Gamma\setminus\Lambda}} d\mu_{\Gamma\setminus\Lambda}(y) \\ &= \mu_{\Gamma\setminus\Lambda}\left(X_{\Gamma\setminus\Lambda}\right) \cdot \int_{X_\Lambda} F \circ \pi_\Lambda(x)\, d\mu_\Lambda(x) = \int_{X_\Lambda} F \circ \pi_\Lambda(x)\, d\mu_\Lambda(x),\end{aligned}$$

wherein we have used the fact that $\mu_\Lambda(X_\Lambda) = 1$ for all $\Lambda \subset\subset A$ since $\mu_\alpha(X_\alpha) = 1$ for all $\alpha \in A$. It is now easy to check that I is a positive linear functional on the lattice $\mathbb{S}$. We will now show that I is a Daniel integral.

Suppose that $f_n \in \mathbb{S}^+$ is a decreasing sequence such that $\inf_n I(f_n) = \epsilon > 0$. We need to show $f := \lim_{n\to\infty} f_n$ is not identically zero. As in the proof that I is well defined, there exits $\Lambda_n \subset\subset A$ and bounded measurable functions $F_n : X_{\Lambda_n} \to [0, \infty)$ such that Λ_n is increasing in n and $f_n = F_n \circ \pi_{\Lambda_n}$ for each n. For $k \le n$, let $F_n^k : X_{\Lambda_k} \to [0, \infty)$ be the bounded measurable function

$$F_n^k(x) = \int_{X_{\Lambda_n\setminus\Lambda_k}} F_n(x \times y)d\mu_{\Lambda_n\setminus\Lambda_k}(y)$$

where $x \times y \in X_{\Lambda_n}$ is defined by $(x \times y)(\alpha) = x(\alpha)$ if $\alpha \in \Lambda_k$ and $(x \times y)(\alpha) = y(\alpha)$ for $\alpha \in \Lambda_n \setminus \Lambda_k$. By convention we set $F_n^n = F_n$. Since f_n is decreasing it follows that $F_{n+1}^k \le F_n^k$ for all k and $n \ge k$ and therefore $F^k := \lim_{n\to\infty} F_n^k$ exists. By Fubini's theorem,

$$F_n^k(x) = \int_{X_{\Lambda_n\setminus\Lambda_k}} F_n^{k+1}(x \times y)d\mu_{\Lambda_{k+1}\setminus\Lambda_k}(y) \text{ when } k + 1 \le n$$

and hence letting $n \to \infty$ in this equation shows

$$F^k(x) = \int_{X_{\Lambda_n\setminus\Lambda_k}} F^{k+1}(x \times y)d\mu_{\Lambda_{k+1}\setminus\Lambda_k}(y) \tag{11.34}$$

for all k. Now

$$\int_{X_{\Lambda_1}} F^1(x)d\mu_{\Lambda_1}(x) = \lim_{n\to\infty} \int_{X_{\Lambda_1}} F_n^1(x)d\mu_{\Lambda_1}(x) = \lim_{n\to\infty} I(f_n) = \epsilon > 0$$

so there exists

$$x_1 \in X_{\Lambda_1} \text{ such that } F^1(x_1) \ge \epsilon.$$

From Eq. (11.34) with $k=1$ and $x=x_1$ it follows that

$$\epsilon \leq \int_{X_{\Lambda_2\setminus\Lambda_1}} F^2(x_1\times y)d\mu_{\Lambda_2\setminus\Lambda_1}(y)$$

and hence there exists

$$x_2 \in X_{\Lambda_2\setminus\Lambda_1} \text{ such that } F^2(x_1\times x_2)\geq\epsilon.$$

Working this way inductively using Eq. (11.34) implies there exists

$$x_i \in X_{\Lambda_i\setminus\Lambda_{i-1}} \text{ such that } F^n(x_1\times x_2\times\cdots\times x_n)\geq\epsilon$$

for all n. Now $F_k^n \geq F^n$ for all $k\leq n$ and in particular for $k=n$, thus

$$\begin{aligned} F_n(x_1\times x_2\times\cdots\times x_n) &= F_n^n(x_1\times x_2\times\cdots\times x_n) \\ &\geq F^n(x_1\times x_2\times\cdots\times x_n)\geq\epsilon \end{aligned} \tag{11.35}$$

for all n. Let $x\in X$ be any point such that

$$\pi_{\Lambda_n}(x) = x_1\times x_2\times\cdots\times x_n$$

for all n. From Eq. (11.35) it follows that

$$f_n(x) = F_n\circ\pi_{\Lambda_n}(x) = F_n(x_1\times x_2\times\cdots\times x_n)\geq\epsilon$$

for all n and therefore $f(x) := \lim_{n\to\infty} f_n(x)\geq\epsilon$ showing f is not zero.

Therefore, I is a Daniel integral and there exists by Theorem 11.30 a unique measure μ on $(X,\sigma(\mathbb{S})=\mathcal{M})$ such that

$$I(f) = \int_X f d\mu \text{ for all } f\in\mathbb{S}.$$

Taking $f = 1_A\circ\pi_\Lambda$ in this equation implies

$$\mu_\Lambda(A) = I(f) = \mu\circ\pi_\Lambda^{-1}(A)$$

and the result is proved. ■

Remark 11.41. (Notion of kernel needs more explanation here.) The above theorem may be Jazzed up as follows. Let $\{(X_\alpha,\mathcal{M}_\alpha)\}_{\alpha\in A}$ be a collection of measurable spaces. Suppose for each pair $\Lambda\subset\Gamma\subset\subset A$ there is a kernel $\mu_{\Lambda,\Gamma}(x,dy)$ for $x\in X_\Lambda$ and $y\in X_{\Gamma\setminus\Lambda}$ such that if $\Lambda\subset\Gamma\subset K\subset\subset A$ then

$$\mu_{\Lambda,K}(x,dy\times dz) = \mu_{\Lambda,\Gamma}(x,dy)\mu_{\Gamma,K}(x\times y,dz).$$

Then there exists a unique measure μ on $\mathcal{M}$ such that

$$\int_X f(\pi_\Lambda(x))d\mu(x) = \int_{X_\Lambda} f(y)d\mu_{\emptyset,\Lambda}(y)$$

for all $\Lambda\subset\subset A$ and $f: X_\Lambda\to\mathbb{R}$ bounded and measurable. To prove this assertion, just use the proof of Proposition 11.40 replacing $\mu_{\Gamma\setminus\Lambda}(dy)$ by $\mu_{\Lambda,\Gamma}(x,dy)$ everywhere in the proof.

11.8. Extensions of premeasures to measures II.

Proposition 11.42. *Suppose that $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra of sets and $\mu : \mathcal{A} \to [0, \infty]$ is a* ***finitely*** *additive measure on $\mathcal{A}$. Then if $A, A_i \in \mathcal{A}$ and $A = \coprod_{i=1}^{\infty} A_i$ we have*

$$\sum_{i=1}^{\infty} \mu(A_i) \le \mu(A). \tag{11.36}$$

Proof. Since

$$A = \left(\coprod_{i=1}^{N} A_i \right) \cup \left(A \setminus \bigcup_{i=1}^{N} A_i \right)$$

we find using the finite additivity of μ that

$$\mu(A) = \sum_{i=1}^{N} \mu(A_i) + \mu \left(A \setminus \bigcup_{i=1}^{N} A_i \right) \ge \sum_{i=1}^{N} \mu(A_i).$$

Letting $N \to \infty$ in this last expression shows that $\sum_{i=1}^{\infty} \mu(A_i) \le \mu(A)$. ■

Because of Proposition 11.42, in order to prove that μ is a premeasure on $\mathcal{A}$, it suffices to show μ is subadditive on $\mathcal{A}$, namely

$$\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i) \tag{11.37}$$

whenever $A = \coprod_{i=1}^{\infty} A_i$ with $A \in \mathcal{A}$ and each $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$.

Proposition 11.43. *Suppose that $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family (see Definition 4.11), $\mathcal{A} = \mathcal{A}(\mathcal{E})$ and $\mu : \mathcal{A} \to [0, \infty]$ is an additive measure. Then the following are equivalent:*

1. *μ is a premeasure on $\mathcal{A}$.*
2. *μ is subadditivity on $\mathcal{E}$, i.e. whenever $E \in \mathcal{E}$ is of the form $E = \coprod_{i=1}^{\infty} E_i \in \mathcal{E}$ with $E_i \in \mathcal{E}$ then*

$$\mu(E) \le \sum_{i=1}^{\infty} \mu(E_i). \tag{11.38}$$

Proof. Item 1. trivially implies item 2. For the converse, it suffices to show, by Proposition 11.42, that if $A = \coprod_{n=1}^{\infty} A_n$ with $A \in \mathcal{A}$ and each $A_n \in \mathcal{A}$ then Eq. (11.37) holds. To prove this, write $A = \coprod_{j=1}^{n} E_j$ with $E_j \in \mathcal{E}$ and $A_n = \coprod_{i=1}^{N_n} E_{n,i}$ with $E_{n,i} \in \mathcal{E}$. Then

$$E_j = A \cap E_j = \coprod_{n=1}^{\infty} A_n \cap E_j = \coprod_{n=1}^{\infty} \coprod_{i=1}^{N_n} E_{n,i} \cap E_j$$

which is a countable union and hence by assumption,

$$\mu(E_j) \le \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu \left(E_{n,i} \cap E_j \right).$$

Summing this equation on j and using the additivity of μ shows that

$$\mu(A)=\sum_{j=1}^{n}\mu(E_j)\le\sum_{j=1}^{n}\sum_{n=1}^{\infty}\sum_{i=1}^{N_n}\mu\left(E_{n,i}\cap E_j\right)=\sum_{n=1}^{\infty}\sum_{i=1}^{N_n}\sum_{j=1}^{n}\mu\left(E_{n,i}\cap E_j\right)$$
$$=\sum_{n=1}^{\infty}\sum_{i=1}^{N_n}\mu\left(E_{n,i}\right)=\sum_{n=1}^{\infty}\mu\left(A_n\right)$$

as desired. ■

The following theorem summarizes the results of Proposition 11.2, Proposition 11.43 and Theorem 11.26 above.

Theorem 11.44. *Suppose that $\mathcal{E}\subset\mathcal{P}(X)$ is an elementary family and $\mu_0:\mathcal{E}\to[0,\infty]$ is a function.*

1. *If μ_0 is additive on $\mathcal{E}$, then μ_0 has a unique extension to a finitely additive measure μ_0 on $\mathcal{A}=\mathcal{A}(\mathcal{E})$.*
2. *If we further assume that μ_0 is countably subadditive on $\mathcal{E}$, then μ_0 is a pre-measure on $\mathcal{A}$.*
3. *If we further assume that μ_0 is σ – finite on $\mathcal{E}$, then there exists a **unique** measure μ on $\sigma(\mathcal{E})$ such that $\mu|_{\mathcal{E}}=\mu_0$. Moreover, for $A\in\sigma(\mathcal{E})$,*

$$\mu(A)=\inf\{\mu_0(B):A\subset B\in\mathcal{A}_\sigma\}$$
$$=\inf\{\sum_{n=1}^{\infty}\mu_0(E_n):A\subset\coprod_{n=1}^{\infty}E_n\text{ with }E_n\in\mathcal{E}\}.$$

11.8.1. *"Radon" measures on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ Revisited.* Here we will use Theorem 11.44 to give another proof of Theorem 5.8. The main point is to show that to each right continuous function $F:\mathbb{R}\to\mathbb{R}$ there exists a unique measure μ_F such that $\mu_F((a,b])=F(b)-F(a)$ for all $-\infty<a<b<\infty$. We begin by extending F to a function from $\bar{\mathbb{R}}\to\bar{\mathbb{R}}$ by defining $F(\pm\infty):=\lim_{x\to\pm\infty}F(x)$. As above let $\mathcal{E}=\{(a,b]\cap\mathbb{R}:-\infty\le a\le b\le\infty\}$ and set $\mu_0\left((a,b]\right)=F(b)-F(a)$ for all $a,b\in\bar{\mathbb{R}}$ with $a\le b$. The proof will be finished by Theorem 11.44 if we can show that μ_0 is sub-additive on $\mathcal{E}$.

First suppose that $-\infty<a<b<\infty$, $J=(a,b]$, $J_n=(a_n,b_n]$ such that $J=\coprod_{n=1}^{\infty}J_n$. We wish to show

$$\mu_0(J)\le\sum_{i=1}^{\infty}\mu_0(J_i). \tag{11.39}$$

To do this choose numbers $\tilde{a}>a$, $\tilde{b}_n>b_n$ and set $I=(\tilde{a},b]\subset J$, $\tilde{J}_n=(a_n,\tilde{b}_n]\supset J_n$ and $\tilde{J}_n^o=(a_n,\tilde{b}_n)$. Since $\bar{I}$ is compact and $\bar{I}\subset J\subset\bigcup_{n=1}^{\infty}\tilde{J}_n^o$ there exists $N<\infty$ such that

$$I\subset\bar{I}\subset\bigcup_{n=1}^{N}\tilde{J}_n^o\subset\bigcup_{n=1}^{N}\tilde{J}_n.$$

Hence by finite sub-additivity of μ_0,

$$F(b)-F(\tilde{a})=\mu_0(I)\le\sum_{n=1}^{N}\mu_0(\tilde{J}_n)\le\sum_{n=1}^{\infty}\mu_0(\tilde{J}_n).$$

Using the right continuity of F and letting $\tilde{a} \downarrow a$ in the above inequality shows that

$$\mu_0((a,b]) = F(b) - F(a) \le \sum_{n=1}^{\infty} \mu_0\left(\tilde{J}_n\right)$$

$$= \sum_{n=1}^{\infty} \mu_0\left(J_n\right) + \sum_{n=1}^{\infty} \mu_0(\tilde{J}_n \setminus J_n) \tag{11.40}$$

Given $\epsilon > 0$ we may use the right continuity of F to choose $\tilde{b}_n$ so that

$$\mu_0(\tilde{J}_n \setminus J_n) = F(\tilde{b}_n) - F(b_n) \le \epsilon 2^{-n} \ \forall n.$$

Using this in Eq. (11.40s that

$$\mu_0(J) = \mu_0((a,b]) \le \sum_{n=1}^{\infty} \mu_0\left(J_n\right) + \epsilon$$

and since $\epsilon > 0$ we have verified Eq. (11.39).

We have now done the hard work. We still have to check the cases where $a = -\infty$ or $b = \infty$ or both. For example, suppose that $b = \infty$ so that

$$J = (a, \infty) = \coprod_{n=1}^{\infty} J_n$$

with $J_n = (a_n, b_n] \cap \mathbb{R}$. Then let $I_M := (a, M]$, and notice that

$$I_M = J \cap I_M = \coprod_{n=1}^{\infty} J_n \cap I_M$$

So by what we have already proved,

$$F(M) - F(a) = \mu_0(I_M) \le \sum_{n=1}^{\infty} \mu_0(J_n \cap I_M) \le \sum_{n=1}^{\infty} \mu_0(J_n)$$

Now let $M \to \infty$ in this last inequality to find that

$$\mu_0((a,\infty)) = F(\infty) - F(a) \le \sum_{n=1}^{\infty} \mu_0(J_n).$$

The other cases where $a = -\infty$ and $b \in \mathbb{R}$ and $a = -\infty$ and $b = \infty$ are handled similarly.

11.9. **Supplement: Generalizations of Theorem 11.35 to $\mathbb{R}^n$.**

Theorem 11.45. *Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be algebras. Suppose that*

$$\mu : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$$

is a function such that for each $A \in \mathcal{A}$, the function

$$B \in \mathcal{B} \to \mu(A \times B) \in \mathbb{C}$$

is an additive measure on $\mathcal{B}$ and for each $B \in \mathcal{B}$, the function

$$A \in \mathcal{A} \to \mu(A \times B) \in \mathbb{C}$$

is an additive measure on $\mathcal{A}$. Then μ extends uniquely to an additive measure on the product algebra $\mathcal{C}$ generated by $\mathcal{A} \times \mathcal{B}$.

Proof. The collection

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is an elementary family, see Exercise 4.2. Therefore, it suffices to show μ is additive on $\mathcal{E}$. To check this suppose that $A \times B \in \mathcal{E}$ and

$$A \times B = \coprod_{k=1}^{n} (A_k \times B_k)$$

with $A_k \times B_k \in \mathcal{E}$. We wish to shows

$$\mu(A \times B) = \sum_{k=1}^{n} \mu(A_k \times B_k).$$

For this consider the finite algebras $\mathcal{A}' \subset \mathcal{P}(A)$ and $\mathcal{B}' \subset \mathcal{P}(B)$ generated by $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ respectively. Let $\mathcal{B} \subset \mathcal{A}'$ and $\mathcal{G} \subset \mathcal{B}'$ be partition of A and B respectively as found Proposition 4.18. Then for each k we may write

$$A_k = \coprod_{\alpha \in \mathcal{F}, \alpha \subset A_k} \alpha \text{ and } B_k = \coprod_{\beta \in \mathcal{G}, \beta \subset B_k} \beta.$$

Therefore,

$$\begin{aligned} \mu(A_k \times B_k) &= \mu(A_k \times \bigcup_{\beta \subset B_k} \beta) = \sum_{\beta \subset B_k} \mu(A_k \times \beta) \\ &= \sum_{\beta \subset B_k} \mu(\left(\bigcup_{\alpha \subset A_k} \alpha\right) \times \beta) = \sum_{\alpha \subset A_k, \beta \subset B_k} \mu(\alpha \times \beta) \end{aligned}$$

so that

$$\begin{aligned} \sum_k \mu(A_k \times B_k) &= \sum_k \sum_{\alpha \subset A_k, \beta \subset B_k} \mu(\alpha \times \beta) = \sum_{\alpha \subset A, \beta \subset B} \mu(\alpha \times \beta) \\ &= \sum_{\beta \subset B} \mu(A \times \beta) = \mu(A \times B) \end{aligned}$$

as desired. ■

Proposition 11.46. *Suppose that $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra and for each $t \in \mathbb{R}$, $\mu_t : \mathcal{A} \to \mathbb{C}$ is a finitely additive measure. Let $Y = (u, v] \subset \mathbb{R}$ be a finite interval and $\mathcal{B} \subset \mathcal{P}(Y)$ denote the algebra generated by $\mathcal{E} := \{(a, b] : (a, b] \subset Y\}$. Then there is a unique additive measure μ on $\mathcal{C}$, the algebra generated by $\mathcal{A} \times \mathcal{B}$ such that*

$$\mu(A \times (a, b]) = \mu_b(A) - \mu_a(A) \ \forall \ (a, b] \in \mathcal{E} \text{ and } A \in \mathcal{A}.$$

Proof. By Proposition 11.3, for each $A \in \mathcal{A}$, the function $(a, b] \to \mu(A \times (a, b])$ extends to a unique measure on $\mathcal{B}$ which we continue to denote by μ. Now if $B \in \mathcal{B}$, then $B = \coprod_k I_k$ with $I_k \in \mathcal{E}$, then

$$\mu(A \times B) = \sum_k \mu(A \times I_k)$$

from which we learn that $A \to \mu(A \times B)$ is still finitely additive. The proof is complete with an application of Theorem 11.45. ■

For $a, b \in \mathbb{R}^n$, write $a < b$ $(a \le b)$ if $a_i < b_i$ $(a_i \le b_i)$ for all i. For $a < b$, let $(a, b]$ denote the half open rectangle:

$$(a, b] = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n],$$

$$\mathcal{E} = \{(a,b] : a < b\} \cup \{\mathbb{R}^n\}$$

and $\mathcal{A}(\mathbb{R}^n) \subset \mathcal{P}(\mathbb{R}^n)$ denote the algebra generated by $\mathcal{E}$. Suppose that $F : \mathbb{R}^n \to \mathbb{C}$ is a function, we wish to define a finitely additive complex valued measure μ_F on $\mathcal{A}(\mathbb{R}^n)$ associated to F. Intuitively the definition is to be

$$\begin{aligned}\mu_F((a,b]) &= \int_{(a,b]} F(dt_1, dt_2, \ldots, dt_n) \\ &= \int_{(a,b]} (\partial_1\partial_2 \ldots \partial_n F)(t_1, t_2, \ldots, t_n) dt_1, dt_2, \ldots, dt_n \\ &= \int_{(\tilde{a},\tilde{b}]} (\partial_1\partial_2 \ldots \partial_{n-1} F)(t_1, t_2, \ldots, t_n)|_{t_n=a_n}^{t_n=b_n} dt_1, dt_2, \ldots, dt_{n-1},\end{aligned}$$

where

$$(\tilde{a}, \tilde{b}] = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_{n-1}, b_{n-1}].$$

Using this expression as motivation we are led to define μ_F by induction on n. For $n = 1$, let

$$\mu_F((a,b]) = F(b) - F(a)$$

and then inductively using

$$\mu_F((a,b]) = \mu_{F(\cdot,t)}((\tilde{a},\tilde{b}])|_{t=a_n}^{t=b_n}.$$

Proposition 11.47. *The function μ_F extends uniquely to an additive function on $\mathcal{A}(\mathbb{R}^n)$. Moreover,*

$$\mu_F((a,b]) = \sum_{\Lambda \subset S} (-1)^{|\Lambda|} F(a_\Lambda \times b_{\Lambda^c}) \tag{11.41}$$

where $S = \{1, 2, \ldots, n\}$ and

$$(a_\Lambda \times b_{\Lambda^c})(i) = \begin{cases} a(i) & \text{if} \quad i \in \Lambda \\ b(i) & \text{if} \quad i \notin \Lambda. \end{cases}$$

Proof. Both statements of the proof will be by induction. For $n = 1$ we have $\mu_F((a,b]) = F(b) - F(a)$ so that Eq. (11.41) holds and we have already seen that μ_F extends to a additive measure on $\mathcal{A}(\mathbb{R})$. For general n, notice that $\mathcal{A}(\mathbb{R}^n) = \mathcal{A}(\mathbb{R}^{n-1}) \otimes \mathcal{A}(\mathbb{R})$. For $t \in \mathbb{R}$ and $A \in \mathcal{A}(\mathbb{R}^{n-1})$, let

$$\mu_t(A) = \mu_{F(\cdot,t)}(A)$$

where $\mu_{F(\cdot,t)}$ is defined by the induction hypothesis. Then

$$\mu_F(A \times (a,b]) = \mu_b(A) - \mu_a(A)$$

and by Proposition 11.46 has a unique extension to $\mathcal{A}(\mathbb{R}^{n-1}) \otimes \mathcal{A}(\mathbb{R})$ as a finitely additive measure.

For $n = 1$, Eq. (11.41) says that

$$\mu_F((a,b]) = F(b) - F(a)$$

where the first term corresponds to $\Lambda = \emptyset$ and second to $\Lambda = \{1\}$. This agrees with the definition of μ_F for $n = 1$. Now for the induction step. Let $T = \{1, 2, \ldots, n-1\}$

and suppose that $a, b \in \mathbb{R}^n$, then

$$\begin{aligned}
\mu_F((a,b]) &= \mu_{F(\cdot,t)}((\tilde{a},\tilde{b}])|_{t=a_n}^{t=b_n} \\
&= \sum_{\Lambda\subset T} (-1)^{|\Lambda|} F(\tilde{a}_\Lambda \times \tilde{b}_{\Lambda^c}, t)|_{t=a_n}^{t=b_n} \\
&= \sum_{\Lambda\subset T} (-1)^{|\Lambda|} F(\tilde{a}_\Lambda \times \tilde{b}_{\Lambda^c}, b_n) - \sum_{\Lambda\subset T} (-1)^{|\Lambda|} F(\tilde{a}_\Lambda \times \tilde{b}_{\Lambda^c}, a_n) \\
&= \sum_{\Lambda\subset S: n\in\Lambda^c} (-1)^{|\Lambda|} F(a_\Lambda \times b_{\Lambda^c}) + \sum_{\Lambda\subset S: n\in\Lambda} (-1)^{|\Lambda|} F(a_\Lambda \times b_{\Lambda^c}) \\
&= \sum_{\Lambda\subset S} (-1)^{|\Lambda|} F(a_\Lambda \times b_{\Lambda^c})
\end{aligned}$$

as desired. ∎

11.10. **Exercises.**

Exercise 11.4. Let $(X, \mathcal{A}, \mu)$ be as in Definition 11.4 and Proposition 11.5, Y be a Banach space and $\mathbb{S}(Y) := \mathbb{S}_f(X, \mathcal{A}, \mu; Y)$ be the collection of functions $f : X \to Y$ such that $\#(f(X)) < \infty$, $f^{-1}(\{y\}) \in \mathcal{A}$ for all $y \in Y$ and $\mu(f = y) < \infty$ if $y \neq 0$. We may define a linear functional $I : \mathbb{S}(Y) \to Y$ by

$$I(f) = \sum_{y\in Y} y\mu(f = y).$$

Verify the following statements.

1. Let $\|f\|_\infty = \sup_{x\in X} \|f(x)\|_Y$ be the sup norm on $\ell^\infty(X, Y)$, then for $f \in \mathbb{S}(Y)$,
$$\|I(f)\|_Y \le \|f\|_\infty \mu(f \neq 0).$$
Hence if $\mu(X) < \infty$, I extends to a bounded linear transformation from $\bar{\mathbb{S}}(Y) \subset \ell^\infty(X, Y)$ to Y.
2. Assuming $(X, \mathcal{A}, \mu)$ satisfies the hypothesis in Exercise 11.1, then $C(X, Y) \subset \bar{\mathbb{S}}(Y)$.
3. Now assume the notation in Section 11.4.1, i.e. $X = [-M, M]$ for some $M \in \mathbb{R}$ and μ is determined by an increasing function F. Let $\pi \equiv \{-M = t_0 < t_1 < \cdots < t_n = M\}$ denote a partition of $J := [-M, M]$ along with a choice $c_i \in [t_i, t_{i+1}]$ for $i = 0, 1, 2 \ldots, n-1$. For $f \in C([-M, M], Y)$, set
$$f_\pi \equiv f(c_0)1_{[t_0,t_1]} + \sum_{i=1}^{n-1} f(c_i)1_{(t_i,t_{i+1}]}.$$
Show that $f_\pi \in \mathbb{S}$ and
$$\|f - f_\pi\|_{\mathcal{F}} \to 0 \text{ as } |\pi| \equiv \max\{(t_{i+1} - t_i) : i = 0, 1, 2 \ldots, n-1\} \to 0.$$
Conclude from this that
$$I(f) = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} f(c_i)(F(t_{i+1}) - F(t_i)).$$
As usual we will write this integral as $\int_{-M}^{M} f dF$ and as $\int_{-M}^{M} f(t)dt$ if $F(t) = t$.

Exercise 11.5. Folland problem 1.28.

Exercise 11.6. Suppose that $F \in C^1(\mathbb{R})$ is an increasing function and μ_F is the unique Borel measure on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all $a \le b$. Show that $d\mu_F = \rho dm$ for some function $\rho \ge 0$. Find ρ explicitly in terms of F.

Exercise 11.7. Suppose that $F(x) = e1_{x\ge 3} + \pi 1_{x \ge 7}$ and μ_F is the is the unique Borel measure on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all $a \le b$. Give an explicit description of the measure μ_F.

Exercise 11.8. Let $E \in \mathcal{B}_{\mathbb{R}}$ with $m(E) > 0$. Then for any $\alpha \in (0,1)$ there exists an open interval $J \subset \mathbb{R}$ such that $m(E \cap J) \ge \alpha m(J)$. **Hints:** 1. Reduce to the case where $m(E) \in (0,\infty)$. 2) Approximate E from the outside by an open set $V \subset \mathbb{R}$. 3. Make use of Exercise 3.43, which states that V may be written as a disjoint union of open intervals.

11.10.1. *The Laws of Large Number Exercises.* For the rest of the problems of this section, let ν be a probability measure on $\mathcal{B}_{\mathbb{R}}$ such that $\int_{\mathbb{R}} |x|\, d\nu(x) < \infty$, $\mu_n := \nu$ for $n \in \mathbb{N}$ and μ denote the infinite product measure as constructed in Corollary 11.39. So μ is the unique measure on $(X := \mathbb{R}^{\mathbb{N}}, \mathcal{B} := \mathcal{B}_{\mathbb{R}^{\mathbb{N}}})$ such that

$$\int_X f(x_1, x_2, \dots, x_N) d\mu(x) = \int_{\mathbb{R}^N} f(x_1, x_2, \dots, x_N) d\nu(x_1) \dots d\nu(x_N) \tag{11.42}$$

for all $N \in \mathbb{N}$ and bounded measurable functions $f : \mathbb{R}^N \to \mathbb{R}$. We will also use the following notation:

$$\begin{aligned} S_n(x) &:= \frac{1}{n}\sum_{k=1}^n x_k \text{ for } x \in X, \\ m &:= \int_{\mathbb{R}} x d\nu(x) \text{ the average of } \nu \text{ and} \\ \sigma^2 &:= \int_{\mathbb{R}} (x-m)^2 d\nu(x) \text{ the variance of } \nu. \end{aligned}$$

The variance may also be written as $\sigma^2 = \int_{\mathbb{R}} x^2 d\nu(x) - m^2$.

Exercise 11.9 (Weak Law of Large Numbers)**.** Suppose further that $\sigma^2 < \infty$, show $\int_X S_n d\mu = m$,

$$\|S_n - m\|_2^2 = \int_X (S_n - m)^2 \, d\mu = \frac{\sigma^2}{n}$$

and $\mu(|S_n - m| > \epsilon) \le \frac{\sigma^2}{n\epsilon^2}$ for all $\epsilon > 0$ and $n \in \mathbb{N}$.

Exercise 11.10 (A simple form of the Strong Law of Large Numbers)**.** Suppose now that $\gamma := \int_{\mathbb{R}} x^4 d\mu(x) < \infty$, show for all $\epsilon > 0$ and $n \in \mathbb{N}$ that

$$\begin{aligned} \|S_n - m\|_4^4 &= \int_X (S_n - m)^4 \, d\mu = n^{-2}\left[3\left(1 - n^{-1}\right)\sigma^4 + n^{-1}\gamma\right] \text{ and} \\ \mu(|S_n - m| > \epsilon) &\le \frac{3\left(1 - n^{-1}\right)\sigma^4 + n^{-1}\gamma}{\epsilon^4 n^2}. \end{aligned}$$

Conclude from the last estimate and the first Borel Cantelli Lemma 5.22 that $\lim_{n\to\infty} S_n(x) = m$ for μ – a.e. $x \in X$.

Exercise 11.11. Suppose $m \neq 0$ and again assume $\gamma := \int_{\mathbb{R}} x^4 d\mu(x) < \infty$. For $\lambda \in \mathbb{R}$ let $T_\lambda : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be defined by $T_\lambda(x) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n, \dots)$, $\mu_\lambda = \mu \circ T_\lambda^{-1}$ and

$$X_\lambda := \left\{ x \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_j = \lambda \right\}.$$

Show

$$\mu_\lambda(X_{\lambda'}) = \delta_{\lambda,\lambda'} = \begin{cases} 1 & \text{if} \quad \lambda = \lambda' \\ 0 & \text{if} \quad \lambda \neq \lambda' \end{cases}$$

and use this to show if $\lambda \neq 0$, that $d\mu_\lambda \neq \rho d\mu$ for any measurable function $\rho : \mathbb{R}^{\mathbb{N}} \to [0, \infty]$.

12. Daniell Integral Proofs

(This section follows the exposition in Royden and Loomis.) In this section we let X be a given set. We will be interested in certain spaces of extended real valued functions $f : X \to \bar{\mathbb{R}}$ on X.

Convention: Given functions $f, g : X \to \bar{\mathbb{R}}$, let $f + g$ denote the collection of functions $h : X \to \bar{\mathbb{R}}$ such that $h(x) = f(x) + g(x)$ for all x for which $f(x) + g(x)$ is well defined, i.e. not of the form $\infty - \infty$. For example, if $X = \{1, 2, 3\}$ and $f(1) = \infty$, $f(2) = 2$ and $f(3) = 5$ and $g(1) = g(2) = -\infty$ and $g(3) = 4$, then $h \in f + g$ iff $h(2) = -\infty$ and $h(3) = 7$. The value $h(1)$ may be chosen freely. More generally if $a, b \in \mathbb{R}$ and $f, g : X \to \bar{\mathbb{R}}$ we will write $af + bg$ for the collection of functions $h : X \to \bar{\mathbb{R}}$ such that $h(x) = af(x) + bg(x)$ for those $x \in X$ where $af(x) + bg(x)$ is well defined with the values of $h(x)$ at the remaining points being arbitrary. It will also be useful to have some explicit representatives for $af + bg$ which we define, for $\alpha \in \bar{\mathbb{R}}$, by

$$(af + bg)_\alpha(x) = \begin{cases} af(x) + bg(x) & \text{when defined} \\ \alpha & \text{otherwise.} \end{cases} \tag{12.1}$$

We will make use of this definition with $\alpha = 0$ and $\alpha = \infty$ below.

Definition 12.1. A set, L, of extended real valued functions on X is an **extended vector space** (or a vector space for short) if L is closed under scalar multiplication and addition in the following sense: if $f, g \in L$ and $\lambda \in \mathbb{R}$ then $(f + \lambda g) \subset L$. A vector space L is said to be an **extended lattice** (or a lattice for short) if it is also closed under the lattice operations; $f \vee g = \max(f, g)$ and $f \wedge g = \min(f, g)$. A **linear functional** I on L is a function $I : L \to \mathbb{R}$ such that

$$I(f + \lambda g) = I(f) + \lambda I(g) \text{ for all } f, g \in L \text{ and } \lambda \in \mathbb{R}. \tag{12.2}$$

Eq. (12.2) is to be interpreted as $I(h) = I(f) + \lambda I(g)$ for all $h \in (f + \lambda g)$, and in particular I is required to take the same value on all members of $(f + \lambda g)$. A linear functional I is positive if $I(f) \geq 0$ when $f \in L^+$, where L^+ denotes the non-negative elements of L as in Notation 11.13.

Remark 12.2. Notice that an extended lattice L is closed under the absolute value operation since $|f| = f \vee 0 - f \wedge 0 = f \vee (-f)$. Also if I is positive on L then $I(f) \leq I(g)$ when $f, g \in L$ and $f \leq g$. Indeed, $f \leq g$ implies $(g - f)_0 \geq 0$, so $0 = I(0) = I((g - f)_0) = I(g) - I(f)$ and hence $I(f) \leq I(g)$.

In the remainder of this chapter we fix a lattice, $\mathbb{S}$, of bounded functions, $f : X \to \mathbb{R}$, and a positive linear functional $I : \mathbb{S} \to \mathbb{R}$ satisfying Property (D) of Definition 11.15.

12.1. Extension of Integrals.

Proposition 12.3. *The set $\mathbb{S}_\uparrow$ and the extension of I to $\mathbb{S}_\uparrow$ in Definition 11.20 satisfies:*

1. *(Monotonicity) $I(f) \leq I(g)$ if $f, g \in \mathbb{S}_\uparrow$ with $f \leq g$.*
2. *$\mathbb{S}_\uparrow$ is closed under the lattice operations, i.e. if $f, g \in \mathbb{S}_\uparrow$ then $f \wedge g \in \mathbb{S}_\uparrow$ and $f \vee g \in \mathbb{S}_\uparrow$. Moreover, if $I(f) < \infty$ and $I(g) < \infty$, then $I(f \vee g) < \infty$ and $I(f \wedge g) < \infty$.*
3. *(Positive Linearity) $I(f + \lambda g) = I(f) + \lambda I(g)$ for all $f, g \in \mathbb{S}_\uparrow$ and $\lambda \geq 0$.*

4. *$f \in \mathbb{S}_\uparrow^+$ iff there exists $\phi_n \in \mathbb{S}^+$ such that $f = \sum_{n=1}^\infty \phi_n$. Moreover, $I(f) = \sum_{m=1}^\infty I(\phi_m)$.*
5. *If $f_n \in \mathbb{S}_\uparrow^+$, then $\sum_{n=1}^\infty f_n =: f \in \mathbb{S}_\uparrow^+$ and $I(f) = \sum_{n=1}^\infty I(f_n)$.*

Remark 12.4. Similar results hold for the extension of I to $\mathbb{S}_\downarrow$ in Definition 11.21.

Proof.

1. Monotonicity follows directly from Lemma 11.19.
2. If $f_n, g_n \in \mathbb{S}$ are chosen so that $f_n \uparrow f$ and $g_n \uparrow g$, then $f_n \wedge g_n \uparrow f \wedge g$ and $f_n \vee g_n \uparrow f \vee g$. If we further assume that $I(g) < \infty$, then $f \wedge g \le g$ and hence $I(f \wedge g) \le I(g) < \infty$. In particular it follows that $I(f \wedge 0) \in (-\infty, 0]$ for all $f \in \mathbb{S}_\uparrow$. Combining this with the identity,
$$I(f) = I\left(f \wedge 0 + f \vee 0\right) = I\left(f \wedge 0\right) + I(f \vee 0)\,,$$
shows $I(f) < \infty$ iff $I(f \vee 0) < \infty$. Since $f \vee g \le f \vee 0 + g \vee 0$, if both $I(f) < \infty$ and $I(g) < \infty$ then
$$I(f \vee g) \le I\left(f \vee 0\right) + I\left(g \vee 0\right) < \infty.$$
3. Let $f_n, g_n \in \mathbb{S}$ be chosen so that $f_n \uparrow f$ and $g_n \uparrow g$, then $(f_n + \lambda g_n) \uparrow (f + \lambda g)$ and therefore
$$\begin{aligned} I\left(f + \lambda g\right) &= \lim_{n\to\infty} I\left(f_n + \lambda g_n\right) = \lim_{n\to\infty} I(f_n) + \lambda \lim_{n\to\infty} I(g_n) \\ &= I(f) + \lambda I(g). \end{aligned}$$
4. Let $f \in \mathbb{S}_\uparrow^+$ and $f_n \in \mathbb{S}$ be chosen so that $f_n \uparrow f$. By replacing f_n by $f_n \vee 0$ if necessary we may assume that $f_n \in \mathbb{S}^+$. Now set $\phi_n = f_n - f_{n-1} \in \mathbb{S}$ for $n = 1, 2, 3, \ldots$ with the convention that $f_0 = 0 \in \mathbb{S}$. Then $\sum_{n=1}^\infty \phi_n = f$ and
$$I(f) = \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(\sum_{m=1}^n \phi_m) = \lim_{n\to\infty} \sum_{m=1}^n I(\phi_m) = \sum_{m=1}^\infty I(\phi_m).$$
Conversely, if $f = \sum_{m=1}^\infty \phi_m$ with $\phi_m \in \mathbb{S}^+$, then $f_n := \sum_{m=1}^n \phi_m \uparrow f$ as $n \to \infty$ and $f_n \in \mathbb{S}^+$.
5. Using Item 4., $f_n = \sum_{m=1}^\infty \phi_{n,m}$ with $\phi_{n,m} \in \mathbb{S}^+$. Thus
$$f = \sum_{n=1}^\infty \sum_{m=1}^\infty \phi_{n,m} = \lim_{N\to\infty} \sum_{m,n\le N} \phi_{n,m} \in \mathbb{S}_\uparrow$$
and
$$\begin{aligned} I(f) &= \lim_{N\to\infty} I(\sum_{m,n\le N} \phi_{n,m}) = \lim_{N\to\infty} \sum_{m,n\le N} I(\phi_{n,m}) \\ &= \sum_{n=1}^\infty \sum_{m=1}^\infty I(\phi_{n,m}) = \sum_{n=1}^\infty I(f_n). \end{aligned}$$

■

Definition 12.5. Given an arbitrary function $g : X \to \bar{\mathbb{R}}$, let
$$\bar{I}(g) = \inf\left\{I(f) : g \le f \in \mathbb{S}_\uparrow\right\} \in \bar{\mathbb{R}} \text{ and}$$
$$\underline{I}(g) = \sup\left\{I(f) : \mathbb{S}_\downarrow \ni f \le g\right\} \in \bar{\mathbb{R}}.$$
with the convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Proposition 12.6. *Given functions $f, g : X \to \bar{\mathbb{R}}$, then:*

1. $\bar{I}(\lambda f) = \lambda \bar{I}(f)$ *for all* $\lambda \geq 0$.
2. *(Chebyshev's Inequality.) Suppose* $f : X \to [0, \infty]$ *is a function and* $\alpha \in (0, \infty)$, *then* $\bar{I}(1_{\{f \geq \alpha\}}) \leq \frac{1}{\alpha} \bar{I}(f)$ *and if* $\bar{I}(f) < \infty$ *then* $\bar{I}(1_{\{f=\infty\}}) = 0$.
3. $\bar{I}$ *is subadditive, i.e. if* $\bar{I}(f) + \bar{I}(g)$ *is not of the form* $\infty - \infty$ *or* $-\infty + \infty$, *then*

$$\bar{I}(f + g) \leq \bar{I}(f) + \bar{I}(g). \tag{12.3}$$

This inequality is to be interpreted to mean,

$$\bar{I}(h) \leq \bar{I}(f) + \bar{I}(g) \text{ for all } h \in (f + g).$$

4. $\underline{I}(-g) = -\bar{I}(g)$.
5. $\underline{I}(g) \leq \bar{I}(g)$.
6. *If* $f \leq g$ *then* $\bar{I}(f) \leq \bar{I}(g)$ *and* $\underline{I}(f) \leq \underline{I}(g)$.
7. *If* $g \in \mathbb{S}_\uparrow$ *and* $I(g) < \infty$ *or* $g \in \mathbb{S}_\downarrow$ *and* $I(g) > -\infty$ *then* $\underline{I}(g) = \bar{I}(g) = I(g)$.

Proof.

1. Suppose that $\lambda > 0$ (the $\lambda = 0$ case being trivial), then

$$\begin{aligned} \bar{I}(\lambda f) &= \inf \{I(h) : \lambda f \leq h \in \mathbb{S}_\uparrow\} = \inf \left\{I(h) : f \leq \lambda^{-1} h \in \mathbb{S}_\uparrow\right\} \\ &= \inf \{I(\lambda g) : f \leq g \in \mathbb{S}_\uparrow\} = \lambda \inf \{I(g) : f \leq g \in \mathbb{S}_\uparrow\} = \lambda \bar{I}(f). \end{aligned}$$

2. For $\alpha \in (0, \infty)$, $\alpha 1_{\{f \geq \alpha\}} \leq f$ and therefore,

$$\alpha \bar{I}(1_{\{f \geq \alpha\}}) = \bar{I}(\alpha 1_{\{f \geq \alpha\}}) \leq \bar{I}(f).$$

Since $N 1_{\{f=\infty\}} \leq f$ for all $N \in (0, \infty)$,

$$N \bar{I}(1_{\{f=\infty\}}) = \bar{I}(N 1_{\{f=\infty\}}) \leq \bar{I}(f).$$

So if $\bar{I}(f) < \infty$, this inequality implies $\bar{I}(1_{\{f=\infty\}}) = 0$ because N is arbitrary.

3. If $\bar{I}(f) + \bar{I}(g) = \infty$ the inequality is trivial so we may assume that $\bar{I}(f), \bar{I}(g) \in [-\infty, \infty)$. If $\bar{I}(f) + \bar{I}(g) = -\infty$ then we may assume, by interchanging f and g if necessary, that $\bar{I}(f) = -\infty$ and $\bar{I}(g) < \infty$. By definition of $\bar{I}$, there exists $f_n \in \mathbb{S}_\uparrow$ and $g_n \in \mathbb{S}_\uparrow$ such that $f \leq f_n$ and $g \leq g_n$ and $I(f_n) \downarrow -\infty$ and $I(g_n) \downarrow \bar{I}(g)$. Since $f + g \leq f_n + g_n \in \mathbb{S}_\uparrow$, (i.e. $h \leq f_n + g_n$ for all $h \in (f + g)$ which holds because $f_n, g_n > -\infty$) and

$$I(f_n + g_n) = I(f_n) + I(g_n) \downarrow -\infty + \bar{I}(g) = -\infty,$$

it follows that $\bar{I}(f+g) = -\infty$, i.e. $\bar{I}(h) = -\infty$ for all $h \in f+g$. Henceforth we may assume $\bar{I}(f), \bar{I}(g) \in \mathbb{R}$. Let $k \in (f + g)$ and $f \leq h_1 \in \mathbb{S}_\uparrow$ and $g \leq h_2 \in \mathbb{S}_\uparrow$. Then $k \leq h_1 + h_2 \in \mathbb{S}_\uparrow$ because if (for example) $f(x) = \infty$ and $g(x) = -\infty$, then $h_1(x) = \infty$ and $h_2(x) > -\infty$ since $h_2 \in \mathbb{S}_\uparrow$. Thus $h_1(x) + h_2(x) = \infty \geq k(x)$ no matter the value of $k(x)$. It now follows from the definitions that $\bar{I}(k) \leq I(h_1) + I(h_2)$ for all $f \leq h_1 \in \mathbb{S}_\uparrow$ and $g \leq h_2 \in \mathbb{S}_\uparrow$. Therefore,

$$\begin{aligned} \bar{I}(k) &\leq \inf \{I(h_1) + I(h_2) : f \leq h_1 \in \mathbb{S}_\uparrow \text{ and } g \leq h_2 \in \mathbb{S}_\uparrow\} \\ &= \bar{I}(f) + \bar{I}(g) \end{aligned}$$

and since $k \in (f + g)$ is arbitrary we have proven Eq. (12.3).

4. From the definitions and Exercise 11.2,

$$\begin{aligned} \underline{I}(-g) &= \sup \{I(f) : f \leq -g \in \mathbb{S}_\downarrow\} = \sup \{I(f) : g \leq -f \in \mathbb{S}_\uparrow\} \\ &= \sup \{I(-h) : g \leq h \in \mathbb{S}_\uparrow\} = -\inf \{I(h) : g \leq h \in \mathbb{S}_\uparrow\} = -\bar{I}(g). \end{aligned}$$

5. The assertion is trivially true if $\bar{I}(g) = \underline{I}(g) = \infty$ or $\bar{I}(g) = \underline{I}(g) = -\infty$. So we now assume that $\bar{I}(g)$ and $\underline{I}(g)$ are not both ∞ or $-\infty$. Since $0 \in (g-g)$ and $\bar{I}(g-g) \le \bar{I}(g) + \bar{I}(-g)$ (by Item 1),
$$0 = \bar{I}(0) \le \bar{I}(g) + \bar{I}(-g) = \bar{I}(g) - \underline{I}(g)$$
provided the right side is well defined which it is by assumption. So again we deduce that $\underline{I}(g) \le \bar{I}(g)$.
6. If $f \le g$ then
$$\bar{I}(f) = \inf\{I(h) : f \le h \in \mathbb{S}_\uparrow\} \le \inf\{I(h) : g \le h \in \mathbb{S}_\uparrow\} = \bar{I}(g)$$
and
$$\underline{I}(f) = \sup\{I(h) : \mathbb{S}_\downarrow \ni h \le f\} \le \sup\{I(h) : \mathbb{S}_\downarrow \ni h \le g\} = \underline{I}(g).$$
7. Let $g \in \mathbb{S}_\uparrow$ with $I(g) < \infty$ and choose $g_n \in \mathbb{S}$ such that $g_n \uparrow g$. Then
$$\bar{I}(g) \ge \underline{I}(g) \ge I(g_n) \to I(g) \text{ as } n \to \infty.$$
Combining this with
$$\bar{I}(g) = \inf\{I(f) : g \le f \in \mathbb{S}_\uparrow\} = I(g)$$
shows
$$\bar{I}(g) \ge \underline{I}(g) \ge I(g) = \bar{I}(g)$$
and hence $\underline{I}(g) = I(g) = \bar{I}(g)$. If $g \in \mathbb{S}_\downarrow$ and $I(g) > -\infty$, then by what we have just proved,
$$\underline{I}(-g) = I(-g) = \bar{I}(-g).$$
This finishes the proof since $\underline{I}(-g) = -\bar{I}(g)$ and $I(-g) = -I(g)$.

■

Lemma 12.7. *Let $f_n : X \to [0,\infty]$ be a sequence of functions and $F := \sum_{n=1}^\infty f_n$. Then*

$$\bar{I}(F) = \bar{I}(\sum_{n=1}^\infty f_n) \le \sum_{n=1}^\infty \bar{I}(f_n). \tag{12.4}$$

Proof. Suppose $\sum_{n=1}^\infty \bar{I}(f_n) < \infty$, for otherwise the result is trivial. Let $\epsilon > 0$ be given and choose $g_n \in \mathbb{S}_\uparrow^+$ such that $f_n \le g_n$ and $I(g_n) = \bar{I}(f_n) + \epsilon_n$ where $\sum_{n=1}^\infty \epsilon_n \le \epsilon$. (For example take $\epsilon_n \le 2^{-n}\epsilon$.) Then $\sum_{n=1}^\infty g_n =: G \in \mathbb{S}_\uparrow^+$, $F \le G$ and so
$$\bar{I}(F) \le \bar{I}(G) = I(G) = \sum_{n=1}^\infty I(g_n) = \sum_{n=1}^\infty \left(\bar{I}(f_n) + \epsilon_n\right) \le \sum_{n=1}^\infty \bar{I}(f_n) + \epsilon.$$
Since $\epsilon > 0$ is arbitrary, the proof is complete. ■

Definition 12.8. A function $g : X \to \bar{\mathbb{R}}$ is **integrable** if $\underline{I}(g) = \bar{I}(g) \in \mathbb{R}$. Let
$$L^1(I) := \left\{g : X \to \bar{\mathbb{R}} : \underline{I}(g) = \bar{I}(g) \in \mathbb{R}\right\}$$
and for $g \in L^1(I)$, let $\hat{I}(g)$ denote the common value $\underline{I}(g) = \bar{I}(g)$.

Remark 12.9. A function $g : X \to \bar{\mathbb{R}}$ is integrable iff there exists $f \in \mathbb{S}_\downarrow \cap L^1(I)$ and $h \in \mathbb{S}_\uparrow \cap L^1(I)$[28] such that $f \le g \le h$ and $I(h - f) < \epsilon$. Indeed if g is integrable, then $\underline{I}(g) = \bar{I}(g)$ and there exists $f \in \mathbb{S}_\downarrow \cap L^1(I)$ and $h \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $f \le g \le h$ and $0 \le \underline{I}(g) - I(f) < \epsilon/2$ and $0 \le I(h) - \bar{I}(g) < \epsilon/2$. Adding these two inequalities implies $0 \le I(h) - I(f) = I(h - f) < \epsilon$. Conversely, if there exists $f \in \mathbb{S}_\downarrow \cap L^1(I)$ and $h \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $f \le g \le h$ and $I(h - f) < \epsilon$, then

$$\begin{aligned} I(f) &= \underline{I}(f) \le \underline{I}(g) \le \underline{I}(h) = I(h) \text{ and} \\ I(f) &= \bar{I}(f) \le \bar{I}(g) \le \bar{I}(h) = I(h) \end{aligned}$$

and therefore

$$0 \le \bar{I}(g) - \underline{I}(g) \le I(h) - I(f) = I(h - f) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows $\bar{I}(g) = \underline{I}(g)$.

Proposition 12.10. *The space $L^1(I)$ is an extended lattice and $\hat{I} : L^1(I) \to \mathbb{R}$ is linear in the sense of Definition 12.1.*

Proof. Let us begin by showing that $L^1(I)$ is a vector space. Suppose that $g_1, g_2 \in L^1(I)$, and $g \in (g_1 + g_2)$. Given $\epsilon > 0$ there exists $f_i \in \mathbb{S}_\downarrow \cap L^1(I)$ and $h_i \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $f_i \le g_i \le h_i$ and $I(h_i - f_i) < \epsilon/2$. Let us now show

$$f_1(x) + f_2(x) \le g(x) \le h_1(x) + h_2(x) \ \forall x \in X. \tag{12.5}$$

This is clear at points $x \in X$ where $g_1(x) + g_2(x)$ is well defined. The other case to consider is where $g_1(x) = \infty = -g_2(x)$ in which case $h_1(x) = \infty$ and $f_2(x) = -\infty$ while , $h_2(x) > -\infty$ and $f_1(x) < \infty$ because $h_2 \in \mathbb{S}_\uparrow$ and $f_1 \in \mathbb{S}_\downarrow$. Therefore, $f_1(x) + f_2(x) = -\infty$ and $h_1(x) + h_2(x) = \infty$ so that Eq. (12.5) is valid no matter how $g(x)$ is chosen.

Since $f_1 + f_2 \in \mathbb{S}_\downarrow \cap L^1(I)$, $h_1 + h_2 \in \mathbb{S}_\uparrow \cap L^1(I)$ and

$$\hat{I}(g_i) \le I(f_i) + \epsilon/2 \text{ and } -\epsilon/2 + I(h_i) \le \hat{I}(g_i),$$

we find

$$\begin{aligned} \hat{I}(g_1) + \hat{I}(g_2) - \epsilon &\le I(f_1) + I(f_2) = I(f_1 + f_2) \le \underline{I}(g) \le \bar{I}(g) \\ &\le I(h_1 + h_2) = I(h_1) + I(h_2) \le \hat{I}(g_1) + \hat{I}(g_2) + \epsilon. \end{aligned}$$

Because $\epsilon > 0$ is arbitrary, we have shown that $g \in L^1(I)$ and $\hat{I}(g_1) + \hat{I}(g_2) = \hat{I}(g)$, i.e. $\hat{I}(g_1 + g_2) = \hat{I}(g_1) + \hat{I}(g_2)$.

It is a simple matter to show $\lambda g \in L^1(I)$ and $\hat{I}(\lambda g) = \lambda \hat{I}(g)$ for all $g \in L^1(I)$ and $\lambda \in \mathbb{R}$. For example if $\lambda = -1$ (the most interesting case), choose $f \in \mathbb{S}_\downarrow \cap L^1(I)$ and $h \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $f \le g \le h$ and $I(h - f) < \epsilon$. Therefore,

$$\mathbb{S}_\downarrow \cap L^1(I) \ni -h \le -g \le -f \in \mathbb{S}_\uparrow \cap L^1(I)$$

with $I(-f - (-h)) = I(h - f) < \epsilon$ and this shows that $-g \in L^1(I)$ and $\hat{I}(-g) = -\hat{I}(g)$. We have now shown that $L^1(I)$ is a vector space of extended real valued functions and $\hat{I} : L^1(I) \to \mathbb{R}$ is linear.

To show $L^1(I)$ is a lattice, let $g_1, g_2 \in L^1(I)$ and $f_i \in \mathbb{S}_\downarrow \cap L^1(I)$ and $h_i \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $f_i \le g_i \le h_i$ and $I(h_i - f_i) < \epsilon/2$ as above. Then using Proposition 12.3 and Remark 12.4,

$$\mathbb{S}_\downarrow \cap L^1(I) \ni f_1 \wedge f_2 \le g_1 \wedge g_2 \le h_1 \wedge h_2 \in \mathbb{S}_\uparrow \cap L^1(I).$$

[28] Equivalently, $f \in \mathbb{S}_\downarrow$ with $I(f) > -\infty$ and $h \in \mathbb{S}_\uparrow$ with $I(h) < \infty$.

Moreover,

$$0 \le h_1 \wedge h_2 - f_1 \wedge f_2 \le h_1 - f_1 + h_2 - f_2,$$

because, for example, if $h_1 \wedge h_2 = h_1$ and $f_1 \wedge f_2 = f_2$ then

$$h_1 \wedge h_2 - f_1 \wedge f_2 = h_1 - f_2 \le h_2 - f_2.$$

Therefore,

$$I\left(h_1 \wedge h_2 - f_1 \wedge f_2\right) \le I\left(h_1 - f_1 + h_2 - f_2\right) < \epsilon$$

and hence by Remark 12.9, $g_1 \wedge g_2 \in L^1(I)$. Similarly

$$0 \le h_1 \vee h_2 - f_1 \vee f_2 \le h_1 - f_1 + h_2 - f_2,$$

because, for example, if $h_1 \vee h_2 = h_1$ and $f_1 \vee f_2 = f_2$ then

$$h_1 \vee h_2 - f_1 \vee f_2 = h_1 - f_2 \le h_1 - f_1.$$

Therefore,

$$I\left(h_1 \vee h_2 - f_1 \vee f_2\right) \le I\left(h_1 - f_1 + h_2 - f_2\right) < \epsilon$$

and hence by Remark 12.9, $g_1 \vee g_2 \in L^1(I)$. ■

Theorem 12.11 (Monotone convergence theorem)**.** *If $f_n \in L^1(I)$ and $f_n \uparrow f$, then $f \in L^1(I)$ iff $\lim_{n\to\infty} \hat{I}(f_n) = \sup_n \hat{I}(f_n) < \infty$ in which case $\hat{I}(f) = \lim_{n\to\infty} \hat{I}(f_n)$.*

Proof. If $f \in L^1(I)$, then by monotonicity $\hat{I}(f_n) \le \hat{I}(f)$ for all n and therefore $\lim_{n\to\infty} \hat{I}(f_n) \le \hat{I}(f) < \infty$. Conversely, suppose $\ell := \lim_{n\to\infty} \hat{I}(f_n) < \infty$ and let $g := \sum_{n=1}^\infty (f_{n+1} - f_n)_0$. The reader should check that $f \le (f_1 + g)_\infty \in (f_1 + g)$. So by Lemma 12.7,

$$\begin{aligned}
\bar{I}(f) &\le \bar{I}((f_1 + g)_\infty) \le \bar{I}(f_1) + \bar{I}(g) \\
&\le \bar{I}(f_1) + \sum_{n=1}^\infty \bar{I}\left((f_{n+1} - f_n)_0\right) = \hat{I}(f_1) + \sum_{n=1}^\infty \hat{I}\left(f_{n+1} - f_n\right) \\
&= \hat{I}(f_1) + \sum_{n=1}^\infty \left[\hat{I}(f_{n+1}) - \hat{I}(f_n)\right] = \hat{I}(f_1) + \ell - \hat{I}(f_1) = \ell. \qquad (12.6)
\end{aligned}$$

Because $f_n \le f$, it follows that $\hat{I}(f_n) = \underline{I}(f_n) \le \underline{I}(f)$ which upon passing to limit implies $\ell \le \underline{I}(f)$. This inequality and the one in Eq. (12.6) shows $\bar{I}(f) \le \ell \le \underline{I}(f)$ and therefore, $f \in L^1(I)$ and $\hat{I}(f) = \ell = \lim_{n\to\infty} \hat{I}(f_n)$. ■

Lemma 12.12 (Fatou's Lemma)**.** *Suppose $\{f_n\} \subset \left[L^1(I)\right]^+$, then $\inf f_n \in L^1(I)$. If $\liminf_{n\to\infty} \hat{I}(f_n) < \infty$, then $\liminf_{n\to\infty} f_n \in L^1(I)$ and in this case*

$$\hat{I}(\liminf_{n\to\infty} f_n) \le \liminf_{n\to\infty} \hat{I}(f_n).$$

Proof. Let $g_k := f_1 \wedge \cdots \wedge f_k \in L^1(I)$, then $g_k \downarrow g := \inf_n f_n$. Since $-g_k \uparrow -g$, $-g_k \in L^1(I)$ for all k and $\hat{I}(-g_k) \le \hat{I}(0) = 0$, it follow from Theorem 12.11 that $-g \in L^1(I)$ and hence so is $\inf_n f_n = g \in L^1(I)$.

By what we have just proved, $u_k := \inf_{n\ge k} f_n \in L^1(I)$ for all k. Notice that $u_k \uparrow \liminf_{n\to\infty} f_n$, and by monotonicity that $\hat{I}(u_k) \le \hat{I}(f_k)$ for all k. Therefore,

$$\lim_{k\to\infty} \hat{I}(u_k) = \liminf_{k\to\infty} \hat{I}(u_k) \le \liminf_{k\to\infty} \hat{I}(f_n) < \infty$$

and by the monotone convergence Theorem 12.11, $\liminf_{n\to\infty} f_n = \lim_{k\to\infty} u_k \in L^1(I)$ and

$$\hat{I}(\liminf_{n\to\infty} f_n) = \lim_{k\to\infty} \hat{I}(u_k) \le \liminf_{n\to\infty} \hat{I}(f_n).$$

■

Before stating the dominated convergence theorem, it is helpful to remove some of the annoyances of dealing with extended real valued functions. As we have done when studying integrals associated to a measure, we can do this by modifying integrable functions by a "null" function.

Definition 12.13. A function $n : X \to \bar{\mathbb{R}}$ is a **null function** if $\bar{I}(|n|) = 0$. A subset $E \subset X$ is said to be a **null set** if 1_E is a null function. Given two functions $f, g : X \to \bar{\mathbb{R}}$ we will write $f = g$ a.e. if $\{f \ne g\}$ is a null set.

Here are some basic properties of null functions and null sets.

Proposition 12.14. *Suppose that $n : X \to \bar{\mathbb{R}}$ is a null function and $f : X \to \bar{\mathbb{R}}$ is an arbitrary function. Then*

1. $n \in L^1(I)$ *and* $\hat{I}(n) = 0$.
2. *The function* $n \cdot f$ *is a null function.*
3. *The set* $\{x \in X : n(x) \ne 0\}$ *is a null set.*
4. *If E is a null set and $f \in L^1(I)$, then $1_{E^c} f \in L^1(I)$ and $\hat{I}(f) = \hat{I}(1_{E^c} f)$.*
5. *If $g \in L^1(I)$ and $f = g$ a.e. then $f \in L^1(I)$ and $\hat{I}(f) = \hat{I}(g)$.*
6. *If $f \in L^1(I)$, then $\{|f| = \infty\}$ is a null set.*

Proof.

1. If n is null, using $\pm n \le |n|$ we find $\bar{I}(\pm n) \le \bar{I}(|n|) = 0$, i.e. $\bar{I}(n) \le 0$ and $-\underline{I}(n) = \bar{I}(-n) \le 0$. Thus it follows that $\bar{I}(n) \le 0 \le \underline{I}(n)$ and therefore $n \in L^1(I)$ with $\hat{I}(n) = 0$.
2. Since $|n \cdot f| \le \infty \cdot |n|$, $\bar{I}(|n \cdot f|) \le \bar{I}(\infty \cdot |n|)$. For $k \in \mathbb{N}$, $k|n| \in L^1(I)$ and $\hat{I}(k|n|) = kI(|n|) = 0$, so $k|n|$ is a null function. By the monotone convergence Theorem 12.11 and the fact $k|n| \uparrow \infty \cdot |n| \in L^1(I)$ as $k \uparrow \infty$, $\hat{I}(\infty \cdot |n|) = \lim_{k\to\infty} \hat{I}(k|n|) = 0$. Therefore $\infty \cdot |n|$ is a null function and hence so is $n \cdot f$.
3. Since $1_{\{n\ne 0\}} \le \infty \cdot 1_{\{n\ne 0\}} = \infty \cdot |n|$, $\bar{I}\left(1_{\{n\ne 0\}}\right) \le \bar{I}(\infty \cdot |n|) = 0$ showing $\{n \ne 0\}$ is a null set.
4. Since $1_E f \in L^1(I)$ and $\hat{I}(1_E f) = 0$,
$$f 1_{E^c} = (f - 1_E f)_0 \in (f - 1_E f) \subset L^1(I)$$
and $\hat{I}(f 1_{E^c}) = \hat{I}(f) - \hat{I}(1_E f) = \hat{I}(f)$.
5. Letting E be the null set $\{f \ne g\}$, then $1_{E^c} f = 1_{E^c} g \in L^1(I)$ and $1_E f$ is a null function and therefore, $f = 1_E f + 1_{E^c} f \in L^1(I)$ and
$$\hat{I}(f) = \hat{I}(1_E f) + \hat{I}(f 1_{E^c}) = \hat{I}(1_{E^c} f) = \hat{I}(1_{E^c} g) = \hat{I}(g).$$
6. By Proposition 12.10, $|f| \in L^1(I)$ and so by Chebyshev's inequality (Item 2 of Proposition 12.6), $\{|f| = \infty\}$ is a null set.

■

Theorem 12.15 (Dominated Convergence Theorem)**.** *Suppose that* $\{f_n : n \in \mathbb{N}\} \subset L^1(I)$ *such that* $f := \lim f_n$ *exits pointwise and there exists* $g \in L^1(I)$ *such that* $|f_n| \le g$ *for all* n*. Then* $f \in L^1(I)$ *and*

$$\lim_{n\to\infty} \hat{I}(f_n) = \hat{I}(\lim_{n\to\infty} f_n) = \hat{I}(f).$$

Proof. By Proposition 12.14, the set $E := \{g = \infty\}$ is a null set and $\hat{I}(1_{E^c} f_n) = \hat{I}(f_n)$ and $\hat{I}(1_{E^c} g) = \hat{I}(g)$. Since

$$\hat{I}(1_{E^c}(g \pm f_n)) \le 2\hat{I}(1_{E^c} g) = 2\hat{I}(g) < \infty,$$

we may apply Fatou's Lemma 12.12 to find $1_{E^c}(g \pm f) \in L^1(I)$ and

$$\begin{aligned}\hat{I}(1_{E^c}(g \pm f)) &\le \liminf_{n\to\infty} \hat{I}(1_{E^c}(g \pm f_n)) \\ &= \liminf_{n\to\infty} \left\{\hat{I}(1_{E^c} g) \pm \hat{I}(1_{E^c} f_n)\right\} = \liminf_{n\to\infty} \left\{\hat{I}(g) \pm \hat{I}(f_n)\right\}.\end{aligned}$$

Since $f = 1_{E^c} f$ a.e. and $1_{E^c} f = \frac{1}{2} 1_{E^c}(g + f - (g + f)) \in L^1(I)$, Proposition 12.14 implies $f \in L^1(I)$. So the previous inequality may be written as

$$\begin{aligned}\hat{I}(g) \pm \hat{I}(f) &= \hat{I}(1_{E^c} g) \pm \hat{I}(1_{E^c} f) \\ &= \hat{I}(1_{E^c}(g \pm f)) \le \hat{I}(g) + \begin{cases} \liminf_{n\to\infty} \hat{I}(f_n) \\ -\limsup_{n\to\infty} \hat{I}(f_n), \end{cases}\end{aligned}$$

wherein we have used $\liminf_{n\to\infty}(-a_n) = -\limsup a_n$. These two inequalities imply $\limsup_{n\to\infty} \hat{I}(f_n) \le \hat{I}(f) \le \liminf_{n\to\infty} \hat{I}(f_n)$ which shows that $\lim\limits_{n\to\infty} \hat{I}(f_n)$ exists and is equal to $\hat{I}(f)$. ■

12.2. **The Structure of $L^1(I)$.** Let $\mathbb{S}_{\uparrow\downarrow}$ denote the collections of functions $f : X \to \bar{\mathbb{R}}$ for which there exists $f_n \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $f_n \downarrow f$ as $n \to \infty$ and $\lim_{n\to\infty} \hat{I}(f_n) > -\infty$. Applying the monotone convergence theorem to $f_1 - f_n$, it follows that $f_1 - f \in L^1(I)$ and hence $-f \in L^1(I)$ so that $\mathbb{S}_{\uparrow\downarrow} \subset L^1(I)$.

Lemma 12.16. *Let* $f : X \to \bar{\mathbb{R}}$ *be a function. If* $\bar{I}(f) \in \mathbb{R}$*, then there exists* $g \in \mathbb{S}_{\uparrow\downarrow}$ *such that* $f \le g$ *and* $\bar{I}(f) = \hat{I}(g)$*. (Consequently,* $n : X \to [0,,\infty)$ *is a positive null function iff there exists* $g \in \mathbb{S}_{\uparrow\downarrow}$ *such that* $g \ge n$ *and* $\hat{I}(g) = 0$*.) Moreover,* $f \in L^1(I)$ *iff there exists* $g \in \mathbb{S}_{\uparrow\downarrow}$ *such that* $g \ge f$ *and* $f = g$ *a.e.*

Proof. By definition of $\bar{I}(f)$ we may choose a sequence of functions $g_k \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $g_k \ge f$ and $\hat{I}(g_k) \downarrow \bar{I}(f)$. By replacing g_k by $g_1 \wedge \cdots \wedge g_k$ if necessary ($g_1 \wedge \cdots \wedge g_k \in \mathbb{S}_\uparrow \cap L^1(I)$ by Proposition 12.3), we may assume that g_k is a decreasing sequence. Then $\lim_{k\to\infty} g_k =: g \ge f$ and, since $\lim_{k\to\infty} \hat{I}(g_k) = \bar{I}(f) > -\infty$, $g \in \mathbb{S}_{\uparrow\downarrow}$. By the monotone convergence theorem applied to $g_1 - g_k$,

$$\hat{I}(g_1 - g) = \lim_{k\to\infty} \hat{I}(g_1 - g_k) = \hat{I}(g_1) - \bar{I}(f),$$

so $\hat{I}(g) = \bar{I}(f)$.

Now suppose that $f \in L^1(I)$, then $(g - f)_0 \ge 0$ and

$$\hat{I}\left((g - f)_0\right) = \hat{I}\left(g\right) - \hat{I}(f) = \hat{I}(g) - \bar{I}(f) = 0.$$

Therefore $(g - f)_0$ is a null functions and hence so is $\infty \cdot (g - f)_0$. Because

$$1_{\{f \ne g\}} = 1_{\{f < g\}} \le \infty \cdot (g - f)_0,$$

$\{f \neq g\}$ is a null set so if $f \in L^1(I)$ there exists $g \in \mathbb{S}_{\uparrow\downarrow}$ such that $f = g$ a.e. The converse statement has already been proved in Proposition 12.14. ∎

Proposition 12.17. *Suppose that I and $\mathbb{S}$ are as above and J is another Daniell integral on a vector lattice $\mathbb{T}$ such that $\mathbb{S} \subset \mathbb{T}$ and $I = J|_{\mathbb{S}}$. (We abbreviate this by writing $I \subset J$.) Then $L^1(I) \subset L^1(J)$ and $\hat{I} = \hat{J}$ on $L^1(I)$, or in abbreviated form: if $I \subset J$ then $\hat{I} \subset \hat{J}$.*

Proof. From the construction of the extensions, it follows that $\mathbb{S}_\uparrow \subset \mathbb{T}_\uparrow$ and the $I = J$ on $\mathbb{S}_\uparrow$. Similarly, it follows that $\mathbb{S}_{\uparrow\downarrow} \subset \mathbb{T}_{\uparrow\downarrow}$ and $\hat{I} = \hat{J}$ on $\mathbb{S}_{\uparrow\downarrow}$. From Lemma 12.16 we learn, if $n \geq 0$ is an $I-$null function then there exists $g \in \mathbb{S}_{\uparrow\downarrow} \subset \mathbb{T}_{\uparrow\downarrow}$ such that $n \leq g$ and $0 = I(g) = J(g)$. This shows that n is also a $J-$null function and in particular every $I-$null set is a $J-$null set. Again by Lemma 12.16, if $f \in L^1(I)$ there exists $g \in \mathbb{S}_{\uparrow\downarrow} \subset \mathbb{T}_{\uparrow\downarrow}$ such that $\{f \neq g\}$ is an $I-$null set and hence a $J-$null set. So by Proposition 12.14, $f \in L^1(J)$ and $I(f) = I(g) = J(g) = J(f)$. ∎

12.3. **Relationship to Measure Theory.**

Definition 12.18. A function $f : X \to [0, \infty]$ is said to measurable if $f \wedge g \in L^1(I)$ for all $g \in L^1(I)$.

Lemma 12.19. *The set of non-negative measurable functions is closed under pairwise minimums and maximums and pointwise limits.*

Proof. Suppose that $f, g : X \to [0, \infty]$ are measurable functions. The fact that $f \wedge g$ and $f \vee g$ are measurable (i.e. $(f \wedge g) \wedge h$ and $(f \vee g) \vee h$ are in $L^1(I)$ for all $h \in L^1(I)$) follows from the identities

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) \text{ and } (f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h)$$

and the fact that $L^1(I)$ is a lattice. If $f_n : X \to [0, \infty]$ is a sequence of measurable functions such that $f = \lim_{n\to\infty} f_n$ exists pointwise, then for $h \in L^1(I)$, we have $h \wedge f_n \to h \wedge f$. By the dominated convergence theorem (using $|h \wedge f_n| \leq |h|$) it follows that $h \wedge f \in L^1(I)$. Since $h \in L^1(I)$ is arbitrary we conclude that f is measurable as well. ∎

Lemma 12.20. *A non-negative function f on X is measurable iff $\phi \wedge f \in L^1(I)$ for all $\phi \in \mathbb{S}$.*

Proof. Suppose $f : X \to [0, \infty]$ is a function such that $\phi \wedge f \in L^1(I)$ for all $\phi \in \mathbb{S}$ and let $g \in \mathbb{S}_\uparrow \cap L^1(I)$. Choose $\phi_n \in \mathbb{S}$ such that $\phi_n \uparrow g$ as $n \to \infty$, then $\phi_n \wedge f \in L^1(I)$ and by the monotone convergence Theorem 12.11, $\phi_n \wedge f \uparrow g \wedge f \in L^1(I)$. Similarly, using the dominated convergence Theorem 12.15, it follows that $g \wedge f \in L^1(I)$ for all $g \in \mathbb{S}_{\uparrow\downarrow}$. Finally for any $h \in L^1(I)$, there exists $g \in \mathbb{S}_{\uparrow\downarrow}$ such that $h = g$ a.e. and hence $h \wedge f = g \wedge f$ a.e. and therefore by Proposition 12.14, $h \wedge f \in L^1(I)$. This completes the proof since the converse direction is trivial. ∎

Definition 12.21. A set $A \subset X$ is **measurable** if 1_A is measurable and A **integrable** if $1_A \in L^1(I)$. Let $\mathcal{R}$ denote the collection of measurable subsets of X.

Remark 12.22. Suppose that $f \geq 0$, then $f \in L^1(I)$ iff f is measurable and $\bar{I}(f) < \infty$. Indeed, if f is measurable and $\bar{I}(f) < \infty$, there exists $g \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $f \leq g$. Since f is measurable, $f = f \wedge g \in L^1(I)$. In particular if $A \in \mathcal{R}$, then A is integrable iff $\bar{I}(1_A) < \infty$.

Lemma 12.23. *The set $\mathcal{R}$ is a ring which is a σ – algebra if 1 is measurable. (Notice that 1 is measurable iff $1 \wedge \phi \in L^1(I)$ for all $\phi \in \mathbb{S}$. This condition is clearly implied by assuming $1 \wedge \phi \in \mathbb{S}$ for all $\phi \in \mathbb{S}$. This will be the typical case in applications.)*

Proof. Suppose that $A, B \in \mathcal{R}$, then $A \cap B$ and $A \cup B$ are in $\mathcal{R}$ by Lemma 12.19 because

$$1_{A\cap B} = 1_A \wedge 1_B \text{ and } 1_{A\cup B} = 1_A \vee 1_B.$$

If $A_k \in \mathcal{R}$, then the identities,

$$1_{\cup_{k=1}^\infty A_k} = \lim_{n\to\infty} 1_{\cup_{k=1}^n A_k} \text{ and } 1_{\cap_{k=1}^\infty A_k} = \lim_{n\to\infty} 1_{\cap_{k=1}^n A_k}$$

along with Lemma 12.19 shows that $\cup_{k=1}^\infty A_k$ and $\cap_{k=1}^\infty A_k$ are in $\mathcal{R}$ as well. Also if $A, B \in \mathcal{R}$ and $g \in \mathbb{S}$, then

$$g \wedge 1_{A\setminus B} = g \wedge 1_A - g \wedge 1_{A\cap B} + g \wedge 0 \in L^1(I) \tag{12.7}$$

showing the $A \setminus B \in \mathcal{R}$ as well.[29] Thus we have shown that $\mathcal{R}$ is a ring. If $1 = 1_X$ is measurable it follows that $X \in \mathcal{R}$ and $\mathcal{R}$ becomes a σ – algebra. ■

Lemma 12.24 (Chebyshev's Inequality)**.** *Suppose that 1 is measurable.*

1. *If $f \in \left[L^1(I)\right]^+$ then, for all $\alpha \in \mathbb{R}$, the set $\{f > \alpha\}$ is measurable. Moreover, if $\alpha > 0$ then $\{f > \alpha\}$ is integrable and $\hat{I}(1_{\{f>\alpha\}}) \le \alpha^{-1}\hat{I}(f)$.*
2. *$\sigma(\mathbb{S}) \subset \mathcal{R}$.*

Proof.

1. If $\alpha < 0$, $\{f > \alpha\} = X \in \mathcal{R}$ since 1 is measurable. So now assume that $\alpha \ge 0$. If $\alpha = 0$ let $g = f \in L^1(I)$ and if $\alpha > 0$ let $g = \alpha^{-1}f - \left(\alpha^{-1}f\right)\wedge 1$. (Notice that g is a difference of two $L^1(I)$ – functions and hence in $L^1(I)$.) The function $g \in \left[L^1(I)\right]^+$ has been manufactured so that $\{g > 0\} = \{f > \alpha\}$. Now let $\phi_n := (ng) \wedge 1 \in \left[L^1(I)\right]^+$ then $\phi_n \uparrow 1_{\{f>\alpha\}}$ as $n \to \infty$ showing $1_{\{f>\alpha\}}$ is measurable and hence that $\{f > \alpha\}$ is measurable. Finally if $\alpha > 0$,
$$1_{\{f>\alpha\}} = 1_{\{f>\alpha\}} \wedge \left(\alpha^{-1}f\right) \in L^1(I)$$
showing the $\{f > \alpha\}$ is integrable and
$$\hat{I}(1_{\{f>\alpha\}}) = \hat{I}(1_{\{f>\alpha\}} \wedge \left(\alpha^{-1}f\right)) \le \hat{I}(\alpha^{-1}f) = \alpha^{-1}\hat{I}(f).$$
2. Since $f \in \mathbb{S}_+$ is $\mathcal{R}$ measurable by (1) and $\mathbb{S} = \mathbb{S}_+ - \mathbb{S}_+$, it follows that any $f \in \mathbb{S}$ is $\mathcal{R}$ measurable, $\sigma(\mathbb{S}) \subset \mathcal{R}$.

■

Lemma 12.25. *Let 1 be measurable. Define $\mu_\pm : \mathcal{R} \to [0, \infty]$ by*

$$\mu_+(A) = \bar{I}(1_A) \text{ and } \mu_-(A) = \underline{I}(1_A)$$

Then $\mu_\pm$ are measures on $\mathcal{R}$ such that $\mu_- \le \mu_+$ and $\mu_-(A) = \mu_+(A)$ whenever $\mu_+(A) < \infty$.

---

[29] Indeed, for $x \in A \cap B$, $x \in A \setminus B$ and $x \in A^c$, Eq. (12.7) evaluated at x states, respectively, that

$$\begin{aligned} g \wedge 0 &= g \wedge 1 - g \wedge 1 + g \wedge 0, \\ g \wedge 1 &= g \wedge 1 - g \wedge 0 + g \wedge 0 \text{ and} \\ g \wedge 0 &= g \wedge 0 - g \wedge 0 + g \wedge 0, \end{aligned}$$

all of which are true.

Notice by Remark 12.22 that

$$\mu_+(A) = \begin{cases} \hat{I}(1_A) & \text{if} \quad A \text{ is integrable} \\ \infty & \text{if} \quad A \in \mathcal{R} \text{ but } A \text{ is not integrable.} \end{cases}$$

Proof. Since $1_\emptyset = 0$, $\mu_\pm(\emptyset) = \hat{I}(0) = 0$ and if $A, B \in \mathcal{R}$, $A \subset B$ then $\mu_+(A) = \bar{I}(1_A) \le \bar{I}(1_B) = \mu_+(B)$ and similarly, $\mu_-(A) = \underline{I}(1_A) \le \underline{I}(1_B) = \mu_-(B)$. Hence $\mu_\pm$ are monotonic. By Remark 12.22 if $\mu_+(A) < \infty$ then A is integrable so

$$\mu_-(A) = \underline{I}(1_A) = \hat{I}(1_A) = \bar{I}(1_A) = \mu_+(A).$$

Now suppose that $\{E_j\}_{j=1}^\infty \subset \mathcal{R}$ is a sequence of pairwise disjoint sets and let $E := \cup_{j=1}^\infty E_j \in \mathcal{R}$. If $\mu_+(E_i) = \infty$ for some i then by monotonicity $\mu_+(E) = \infty$ as well. If $\mu_+(E_j) < \infty$ for all j then $f_n := \sum_{j=1}^n 1_{E_j} \in \left[L^1(I)\right]^+$ with $f_n \uparrow 1_E$. Therefore, by the monotone convergence theorem, 1_E is integrable iff

$$\lim_{n\to\infty} \hat{I}(f_n) = \sum_{j=1}^\infty \mu_+(E_j) < \infty$$

in which case $1_E \in L^1(I)$ and $\lim_{n\to\infty} \hat{I}(f_n) = \hat{I}(1_E) = \mu_+(E)$. Thus we have shown that μ_+ is a measure and $\mu_-(E) = \mu_+(E)$ whenever $\mu_+(E) < \infty$. The fact the μ_- is a measure will be shown in the course of the proof of Theorem 12.28. ■

Example 12.26. Suppose X is a set, $\mathbb{S} = \{0\}$ is the trivial vector space and $I(0) = 0$. Then clearly I is a Daniel integral,

$$\bar{I}(g) = \begin{cases} \infty & \text{if} \quad g(x) > 0 \text{ for some } x \\ 0 & \text{if} \quad g \le 0 \end{cases}$$

and similarly,

$$\underline{I}(g) = \begin{cases} -\infty & \text{if} \quad g(x) < 0 \text{ for some } x \\ 0 & \text{if} \quad g \ge 0. \end{cases}$$

Therefore, $L^1(I) = \{0\}$ and for any $A \subset X$ we have $1_A \wedge 0 = 0 \in \mathbb{S}$ so that $\mathcal{R} = 2^X$. Since $1_A \notin L^1(I) = \{0\}$ unless $A = \emptyset$ set, the measure μ_+ in Lemma 12.25 is given by $\mu_+(A) = \infty$ if $A \neq \emptyset$ and $\mu_+(\emptyset) = 0$, i.e. $\mu_+(A) = \bar{I}(1_A)$ while $\mu_- \equiv 0$.

Lemma 12.27. *For $A \in \mathcal{R}$, let*

$$\alpha(A) := \sup\{\mu_+(B) : B \in \mathcal{R},\ B \subset A \text{ and } \mu_+(B) < \infty\},$$

then α is a measure on $\mathcal{R}$ such that $\alpha(A) = \mu_+(A)$ whenever $\mu_+(A) < \infty$. If ν is any measure on $\mathcal{R}$ such that $\nu(B) = \mu_+(B)$ when $\mu_+(B) < \infty$, then $\alpha \le \nu$. Moreover, $\alpha \le \mu_-$.

Proof. Clearly $\alpha(A) = \mu_+(A)$ whenever $\mu_+(A) < \infty$. Now let $A = \cup_{n=1}^\infty A_n$ with$\{A_n\}_{n=1}^\infty \subset \mathcal{R}$ being a collection of pairwise disjoint subsets. Let $B_n \subset A_n$ with $\mu_+(B_n) < \infty$, then $B^N := \cup_{n=1}^N B_n \subset A$ and $\mu_+(B^N) < \infty$ and hence

$$\alpha(A) \ge \mu_+(B^N) = \sum_{n=1}^N \mu_+(B_n)$$

and since $B_n \subset A_n$ with $\mu_+(B_n) < \infty$ is arbitrary it follows that $\alpha(A) \ge \sum_{n=1}^N \alpha(A_n)$ and hence letting $N \to \infty$ implies $\alpha(A) \ge \sum_{n=1}^\infty \alpha(A_n)$. Conversely, if $B \subset A$ with $\mu_+(B) < \infty$, then $B \cap A_n \subset A_n$ and $\mu_+(B \cap A_n) < \infty$. Therefore,

$$\mu_+(B) = \sum_{n=1}^{\infty} \mu_+(B \cap A_n) \leq \sum_{n=1}^{\infty} \alpha(A_n)$$

for all such B and hence $\alpha(A) \leq \sum_{n=1}^{\infty} \alpha(A_n)$.

Using the definition of α and the assumption that $\nu(B) = \mu_+(B)$ when $\mu_+(B) < \infty$,

$$\alpha(A) = \sup\{\nu(B) : B \in \mathcal{R},\ B \subset A \text{ and } \mu_+(B) < \infty\} \leq \nu(A),$$

showing $\alpha \leq \nu$. Similarly,

$$\begin{aligned}\alpha(A) &= \sup\{\hat{I}(1_B) : B \in \mathcal{R},\ B \subset A \text{ and } \mu_+(B) < \infty\} \\ &= \sup\{\underline{I}(1_B) : B \in \mathcal{R},\ B \subset A \text{ and } \mu_+(B) < \infty\} \leq \underline{I}(1_A) = \mu_-(A).\end{aligned}$$

■

Theorem 12.28 (Stone)**.** *Suppose that 1 is measurable and μ_+ and μ_- are as defined in Lemma 12.25, then:*

1. *$L^1(I) = L^1(X, \mathcal{R}, \mu_+) = L^1(\mu_+)$ and for integrable $f \in L^1(\mu_+)$,*

$$\hat{I}(f) = \int_X f d\mu_+. \tag{12.8}$$

2. *If ν is any measure on $\mathcal{R}$ such that $\mathbb{S} \subset L^1(\nu)$ and*

$$\hat{I}(f) = \int_X f d\nu \text{ for all } f \in \mathbb{S} \tag{12.9}$$

then $\mu_-(A) \leq \nu(A) \leq \mu_+(A)$ for all $A \in \mathcal{R}$ with $\mu_-(A) = \nu(A) = \mu_+(A)$ whenever $\mu_+(A) < \infty$.

3. *Letting α be as defined in Lemma 12.27, $\mu_- = \alpha$ and hence μ_- is a measure. (So μ_+ is the maximal and μ_- is the minimal measure for which Eq. (12.9) holds.)*
4. *Conversely if ν is any measure on $\sigma(\mathbb{S})$ such that $\nu(A) = \mu_+(A)$ when $A \in \sigma(\mathbb{S})$ and $\mu_+(A) < \infty$, then Eq. (12.9) is valid.*

Proof.

1. Suppose that $f \in \left[L^1(I)\right]^+$, then Lemma 12.24 implies that f is $\mathcal{R}$ measurable. Given $n \in \mathbb{N}$, let

$$\phi_n := \sum_{k=1}^{2^{2n}} \frac{k}{2^n} 1_{\{\frac{k}{2^n} < f \leq \frac{k+1}{2^n}\}} = 2^{-n} \sum_{k=1}^{2^{2n}} 1_{\{\frac{k}{2^n} < f\}}. \tag{12.10}$$

Then we know $\{\frac{k}{2^n} < f\} \in \mathcal{R}$ and that $1_{\{\frac{k}{2^n} < f\}} = 1_{\{\frac{k}{2^n} < f\}} \wedge \left(\frac{2^n}{k} f\right) \in L^1(I)$, i.e. $\mu_+\left(\frac{k}{2^n} < f\right) < \infty$. Therefore $\phi_n \in \left[L^1(I)\right]^+$ and $\phi_n \uparrow f$. Suppose that ν is any measure such that $\nu(A) = \mu_+(A)$ when $\mu_+(A) < \infty$, then by the monotone convergence theorems for $\hat{I}$ and the Lebesgue integral,

$$\hat{I}(f) = \lim_{n\to\infty} \hat{I}(\phi_n) = \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{2^{2n}} \hat{I}(1_{\{\frac{k}{2^n} < f\}}) = \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{2^{2n}} \mu_+\left(\frac{k}{2^n} < f\right)$$

$$= \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{2^{2n}} \nu\left(\frac{k}{2^n} < f\right) = \lim_{n\to\infty} \int_X \phi_n d\nu = \int_X f d\nu. \tag{12.11}$$

This shows that $f \in \left[L^1(\nu)\right]^+$ and that $\hat{I}(f) = \int_X f d\nu$. Since every $f \in L^1(I)$ is of the form $f = f^+ - f^-$ with $f^\pm \in \left[L^1(I)\right]^+$, it follows that $L^1(I) \subset L^1(\mu_+) \subset L^1(\nu) \subset L^1(\alpha)$ and Eq. (12.9) holds for all $f \in L^1(I)$.

Conversely suppose that $f \in \left[L^1(\mu_+)\right]^+$. Define ϕ_n as in Eq. (12.10). Chebyshev's inequality implies that $\mu_+(\frac{k}{2^n} < f) < \infty$ and hence $\{\frac{k}{2^n} < f\}$ is I – integrable. Again by the monotone convergence for Lebesgue integrals and the computations in Eq. (12.11),

$$\infty > \int_X f d\mu_+ = \lim_{n\to\infty} \hat{I}(\phi_n)$$

and therefore by the monotone convergence theorem for $\hat{I}$, $f \in L^1(I)$ and

$$\int_X f d\mu_+ = \lim_{n\to\infty} \hat{I}(\phi_n) = \hat{I}(f).$$

2. Suppose that ν is any measure such that Eq. (12.9) holds. Then by the monotone convergence theorem,

$$I(f) = \int_X f d\nu \text{ for all } f \in \mathbb{S}_\uparrow \cup \mathbb{S}_\downarrow.$$

Let $A \in \mathcal{R}$ and assume that $\mu_+(A) < \infty$, i.e. $1_A \in L^1(I)$. Then there exists $f \in \mathbb{S}_\uparrow \cap L^1(I)$ such that $1_A \le f$ and integrating this inequality relative to ν implies

$$\nu(A) = \int_X 1_A d\nu \le \int_X f d\nu = \hat{I}(f).$$

Taking the infinum of this equation over those $f \in \mathbb{S}_\uparrow$ such that $1_A \le f$ implies $\nu(A) \le \bar{I}(1_A) = \mu_+(A)$. If $\mu_+(A) = \infty$ in this inequality holds trivially.

Similarly, if $A \in \mathcal{R}$ and $f \in \mathbb{S}_\downarrow$ such that $0 \le f \le 1_A$, then

$$\nu(A) = \int_X 1_A d\nu \ge \int_X f d\nu = \hat{I}(f).$$

Taking the supremum of this equation over those $f \in \mathbb{S}_\downarrow$ such that $0 \le f \le 1_A$ then implies $\nu(A) \ge \mu_-(A)$. So we have shown that $\mu_- \le \nu \le \mu_+$.

3. By Lemma 12.27, $\nu = \alpha$ is a measure as in (2) satisfying $\alpha \le \mu_-$ and therefore $\mu_- \le \alpha$ and hence we have shown that $\alpha = \mu_-$. This also shows that μ_- is a measure.
4. This can be done by the same type of argument used in the proof of (1).

■

Proposition 12.29 (Uniqueness)**.** *Suppose that* 1 *is measurable and there exists a function* $\chi \in L^1(I)$ *such that* $\chi(x) > 0$ *for all* x. *Then there is only one measure* μ *on* $\sigma(\mathbb{S})$ *such that*

$$\hat{I}(f) = \int_X f d\mu \text{ for all } f \in \mathbb{S}.$$

Remark 12.30. The existence of a function $\chi \in L^1(I)$ such that $\chi(x) > 0$ for all x is equivalent to the existence of a function $\chi \in \mathbb{S}_\uparrow$ such that $\hat{I}(\chi) < \infty$ and $\chi(x) > 0$ for all $x \in X$. Indeed by Lemma 12.16, if $\chi \in L^1(I)$ there exists $\tilde{\chi} \in \mathbb{S}_\uparrow \cap L^1(I)$ such $\tilde{\chi} \ge \chi$.

Proof. As in Remark 12.30, we may assume $\chi \in \mathbb{S}_\uparrow \cap L^1(I)$. The sets $X_n := \{\chi > 1/n\} \in \sigma(\mathbb{S}) \subset \mathcal{R}$ satisfy $\mu(X_n) \leq n\hat{I}(\chi) < \infty$. The proof is completed using Theorem 12.28 to conclude, for any $A \in \sigma(\mathbb{S})$, that

$$\mu_+(A) = \lim_{n\to\infty} \mu_+(A \cap X_n) = \lim_{n\to\infty} \mu_-(A \cap X_n) = \mu_-(A).$$

Since $\mu_- \leq \mu \leq \mu_+ = \mu_-$, we see that $\mu = \mu_+ = \mu_-$. ■