15. More Point Set Topology

15.1. Connectedness.

Definition 15.1. (X,τ) is **disconnected** if there exists non-empty open sets U and V of X such that $U\cap V=\emptyset$ and $X=U\cup V$. We say $\{U,V\}$ is a **disconnection** of X. The topological space (X,τ) is called **connected** if it is not disconnected, i.e. if there are no disconnection of X. If $A\subset X$ we say A is connected iff (A,τ_A) is connected where τ_A is the relative topology on A. Explicitly, A is disconnected in (X,τ) iff there exists $U,V\in\tau$ such that $U\cap A\neq\emptyset$, $U\cap A\neq\emptyset$, $A\cap U\cap V=\emptyset$ and $A\subset U\cup V$.

The reader should check that the following statement is an equivalent definition of connectivity. A topological space (X,τ) is connected iff the only sets $A\subset X$ which are both open and closed are the sets X and $\emptyset$.

Remark 15.2. Let $A\subset Y\subset X$. Then A is connected in X iff A is connected in Y.

Proof. Since

$$\tau_A\equiv\{V\cap A:V\subset X\}=\{V\cap A\cap Y:V\subset X\}=\{U\cap A:U\subset_o Y\},$$

the relative topology on A inherited from X is the same as the relative topology on A inherited from Y. Since connectivity is a statement about the relative topologies on A, A is connected in X iff A is connected in Y. ■

The following elementary but important lemma is left as an exercise to the reader.

Lemma 15.3. *Suppose that $f:X\to Y$ is a continuous map between topological spaces. Then $f(X)\subset Y$ is connected if X is connected.*

Here is a typical way these connectedness ideas are used.

Example 15.4. Suppose that $f:X\to Y$ is a continuous map between topological spaces, X is connected, Y is Hausdorff, and f is locally constant, i.e. for all $x\in X$ there exists an open neighborhood V of x in X such that $f|_V$ is constant. Then f is constant, i.e. $f(X)=\{y_0\}$ for some $y_0\in Y$. To prove this, let $y_0\in f(X)$ and let $W:=f^{-1}(\{y_0\})$. Since Y is Hausdorff, $\{y_0\}\subset Y$ is a closed set and since f is continuous $W\subset X$ is also closed. Since f is locally constant, W is open as well and since X is connected it follows that $W=X$, i.e. $f(X)=\{y_0\}$.

Proposition 15.5. *Let (X,τ) be a topological space.*

1. *If $B\subset X$ is a connected set and X is the disjoint union of two open sets U and V, then either $B\subset U$ or $B\subset V$.*
2. *a) If $A\subset X$ is connected, then $\bar{A}$ is connected.*
   b) More generally, if A is connected and $B\subset\operatorname{acc}(A)$, then $A\cup B$ is connected as well. (Recall that $\operatorname{acc}(A)$ – the set of accumulation points of A was defined in Defintion 3.19 above.)
3. *If $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected sets such that $\bigcap_{\alpha\in A}E_\alpha\neq\emptyset$, then $Y:=\bigcup_{\alpha\in A}E_\alpha$ is connected as well.*
4. *Suppose $A,B\subset X$ are non-empty connected subsets of X such that $\bar{A}\cap B\neq\emptyset$, then $A\cup B$ is connected in X.*
5. *Every point $x\in X$ is contained in a unique maximal connected subset C_x of X and this subset is closed. The set C_x is called the **connected component** of x.*

Proof.

(1) Since B is the disjoint union of the relatively open sets $B\cap U$ and $B\cap V$, we must have $B\cap U = B$ or $B\cap V = B$ for otherwise $\{B\cap U, B\cap V\}$ would be a disconnection of B.

(2) a. Let $Y = \bar{A}$ equipped with the relative topology from X. Suppose that $U, V \subset_o Y$ form a disconnection of $Y = \bar{A}$. Then by 1. either $A \subset U$ or $A \subset V$. Say that $A \subset U$. Since U is both open an closed in Y, it follows that $Y = \bar{A} \subset U$. Therefore $V = \emptyset$ and we have a contradiction to the assumption that $\{U, V\}$ is a disconnection of $Y = \bar{A}$. Hence we must conclude that $Y = \bar{A}$ is connected as well.

b. Now let $Y = A \cup B$ with $B \subset \mathrm{acc}(A)$, then

$$\bar{A}^Y = \bar{A}\cap Y = (A\cup \mathrm{acc}(A))\cap Y = A\cup B.$$

Because A is connected in Y, by (2) b. $Y = A\cup B = \bar{A}^Y$ is also connected.

(3) Let $Y := \bigcup_{\alpha\in A} E_\alpha$. By Remark 15.2, we know that E_α is connected in Y for each $\alpha \in A$. If $\{U, V\}$ were a disconnection of Y, By item (1), either $E_\alpha \subset U$ or $E_\alpha \subset V$ for all α. Let $\Lambda = \{\alpha \in A : E_\alpha \subset U\}$ then $U = \cup_{\alpha\in\Lambda} E_\alpha$ and $V = \cup_{\alpha\in A\setminus\Lambda} E_\alpha$. (Notice that neither Λ or $A\setminus\Lambda$ can be empty since U and V are not empty.) Since

$$\emptyset = U\cap V = \bigcup\nolimits_{\alpha\in\Lambda,\beta\in\Lambda^c} (E_\alpha \cap E_\beta) \supset \bigcap_{\alpha\in A} E_\alpha \neq \emptyset.$$

we have reached a contradiction and hence no such disconnection exists.

(4) (A good example to keep in mind here is $X = \mathbb{R}$, $A = (0,1)$ and $B = [1,2)$.) For sake of contradiction suppose that $\{U, V\}$ were a disconnection of $Y = A\cup B$. By item (1) either $A \subset U$ or $A \subset V$, say $A \subset U$ in which case $B \subset V$. Since $Y = A\cup B$ we must have $A = U$ and $B = V$ and so we may conclude: A and B are disjoint subsets of Y which are both open and closed. This implies

$$A = \bar{A}^Y = \bar{A}\cap Y = \bar{A}\cap (A\cup B) = A\cup \left(\bar{A}\cap B\right)$$

and therefore

$$\emptyset \neq \bar{A}\cap B \subset A\cap B = \emptyset,$$

which gives us the desired contradiction.

(5) Let $\mathcal{C}$ denote the collection of connected subsets $C \subset X$ such that $x \in C$. Then by item 3., the set $C_x := \cup\mathcal{C}$ is also a connected subset of X which contains x and clearly this is the unique maximal connected set containing x. Since $\bar{C}_x$ is also connected by item (2) and C_x is maximal, $C_x = \bar{C}_x$, i.e. C_x is closed.

■

Theorem 15.6. *The connected subsets of* $\mathbb{R}$ *are intervals.*

Proof. Suppose that $A \subset \mathbb{R}$ is a connected subset and that $a, b \in A$ with $a < b$. If there exists $c \in (a,b)$ such that $c \notin A$, then $U := (-\infty, c)\cap A$ and $V := (c,\infty)\cap A$ would form a disconnection of A. Hence $(a,b) \subset A$. Let $\alpha := \inf(A)$ and $\beta := \sup(A)$ and choose $\alpha_n, \beta_n \in A$ such that $\alpha_n < \beta_n$ and $\alpha_n \downarrow \alpha$ and $\beta_n \uparrow \beta$ as $n \to \infty$. By what we have just shown, $(\alpha_n, \beta_n) \subset A$ for all n and hence $(\alpha, \beta) = \cup_{n=1}^\infty (\alpha_n, \beta_n) \subset A$. From this it follows that $A = (\alpha,\beta)$, $[\alpha,\beta)$, $(\alpha,\beta]$ or $[\alpha,\beta]$, i.e. A is an interval.

Conversely suppose that A is an interval, and for sake of contradiction, suppose that $\{U, V\}$ is a disconnection of A with $a \in U$, $b \in V$. After relabeling U and V if necessary we may assume that $a < b$. Since A is an interval $[a, b] \subset A$. Let $p = \sup\left([a, b] \cap U\right)$, then because U and V are open, $a < p < b$. Now p can not be in U for otherwise $\sup\left([a, b] \cap U\right) > p$ and p can not be in V for otherwise $p < \sup\left([a, b] \cap U\right)$. From this it follows that $p \notin U \cup V$ and hence $A \neq U \cup V$ contradicting the assumption that $\{U, V\}$ is a disconnection. ■

Definition 15.7. A topological space X is **path connected** if to every pair of points $\{x_0, x_1\} \subset X$ there exists a continuous path $\sigma \in C([0, 1], X)$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. The space X is said to be **locally path connected** if for each $x \in X$, there is an open neighborhood $V \subset X$ of x which is path connected.

Proposition 15.8. *Let X be a topological space.*

(1) *If X is path connected then X is connected.*
(2) *If X is connected and locally path connected, then X is path connected.*
(3) *If X is any connected open subset of $\mathbb{R}^n$, then X is path connected.*

Proof. The reader is asked to prove this proposition in Exercises 15.1 – 15.3 below. ■

15.2. **Product Spaces.** Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a collection of topological spaces (we assume $X_\alpha \neq \emptyset$) and let $X_A = \prod_{\alpha \in A} X_\alpha$. Recall that $x \in X_A$ is a function

$$x : A \to \coprod_{\alpha \in A} X_\alpha$$

such that $x_\alpha := x(\alpha) \in X_\alpha$ for all $\alpha \in A$. An element $x \in X_A$ is called a choice function and the **axiom of choice** states that $X_A \neq \emptyset$ provided that $X_\alpha \neq \emptyset$ for each $\alpha \in A$. If each X_α above is the same set X, we will denote $X_A = \prod_{\alpha \in A} X_\alpha$ by X^A. So $x \in X^A$ is a function from A to X.

Notation 15.9. For $\alpha \in A$, let $\pi_\alpha : X_A \to X_\alpha$ be the canonical projection map, $\pi_\alpha(x) = x_\alpha$. The **product topology** $\tau = \otimes_{\alpha \in A} \tau_\alpha$ is the smallest topology on X_A such that each projection π_α is continuous. Explicitly, τ is the topology generated by

$$\mathcal{E} = \{\pi_\alpha^{-1}(V_\alpha) : \alpha \in A, V_\alpha \in \tau_\alpha\}. \tag{15.1}$$

A "basic" open set in this topology is of the form

$$V = \{x \in X_A : \pi_\alpha(x) \in V_\alpha \text{ for } \alpha \in \Lambda\} \tag{15.2}$$

where Λ is a finite subset of A and $V_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda$. We will sometimes write V above as

$$V = \prod_{\alpha \in \Lambda} V_\alpha \times \prod_{\alpha \notin \Lambda} X_\alpha = V_\Lambda \times X_{A \setminus \Lambda}.$$

Proposition 15.10. *Suppose Y is a topological space and $f : Y \to X_A$ is a map. Then f is continuous iff $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous for all $\alpha \in A$.*

Proof. If f is continuous then $\pi_\alpha \circ f$ is the composition of two continuous functions and hence is continuous. Conversely if $\pi_\alpha \circ f$ is continuous for all $\alpha \in A$, the $(\pi_\alpha \circ f)^{-1}(V_\alpha) = f^{-1}(\pi_\alpha^{-1}(V_\alpha))$ is open in Y for all $\alpha \in A$ and $V_\alpha \subset_o X_\alpha$. That is to say, $f^{-1}(\mathcal{E})$ consists of open sets, and therefore f is continuous since $\mathcal{E}$ is a sub-basis for the product topology. ■

Proposition 15.11. *Suppose that (X,τ) is a topological space and $\{f_n\} \subset X^A$ is a sequence. Then $f_n \to f$ in the product topology of X^A iff $f_n(\alpha) \to f(\alpha)$ for all $\alpha \in A$.*

Proof. Since π_α is continuous, if $f_n \to f$ then $f_n(\alpha) = \pi_\alpha(f_n) \to \pi_\alpha(f) = f(\alpha)$ for all $\alpha \in A$. Conversely, $f_n(\alpha) \to f(\alpha)$ for all $\alpha \in A$ iff $\pi_\alpha(f_n) \to \pi_\alpha(f)$ for all $\alpha \in A$. Therefore if $V = \pi_\alpha^{-1}(V_\alpha) \in \mathcal{E}$ and $f \in V$, then $\pi_\alpha(f) \in V_\alpha$ and $\pi_\alpha(f_n) \in V_\alpha$ a.a. and hence $f_n \in V$ a.a.. This shows that $f_n \to f$ as $n \to \infty$. ■

Proposition 15.12. *Let (X_α, τ_α) be topological spaces and X_A be the product space with the product topology.*

(1) *If X_α is Hausdorff for all $\alpha \in A$, then so is X_A.*
(2) *If each X_α is connected for all $\alpha \in A$, then so is X_A.*

Proof.

(1) Let $x, y \in X_A$ be distinct points. Then there exists $\alpha \in A$ such that $\pi_\alpha(x) = x_\alpha \neq y_\alpha = \pi_\alpha(y)$. Since X_α is Hausdorff, there exists disjoint open sets $U, V \subset X_\alpha$ such $\pi_\alpha(x) \in U$ and $\pi_\alpha(y) \in V$. Then $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint open sets in X_A containing x and y respectively.
(2) Let us begin with the case of two factors, namely assume that X and Y are connected topological spaces, then we will show that $X \times Y$ is connected as well. To do this let $p = (x_0, y_0) \in X \times Y$ and E denote the connected component of p. Since $\{x_0\} \times Y$ is homeomorphic to Y, $\{x_0\} \times Y$ is connected in $X \times Y$ and therefore $\{x_0\} \times Y \subset E$, i.e. $(x_0, y) \in E$ for all $y \in Y$. A similar argument now shows that $X \times \{y\} \subset E$ for any $y \in Y$, that is to $X \times Y = E$. By induction the theorem holds whenever A is a finite set.

For the general case, again choose a point $p \in X_A = X^A$ and let $C = C_p$ be the connected component of p in X_A. Recall that C_p is closed and therefore if C_p is a proper subset of X_A, then $X_A \setminus C_p$ is a non-empty open set. By the definition of the product topology, this would imply that $X_A \setminus C_p$ contains an open set of the form

$$V := \cap_{\alpha \in \Lambda} \pi_\alpha^{-1}(V_\alpha) = V_\Lambda \times X_{A \setminus \Lambda}$$

where $\Lambda \subset\subset A$ and $V_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda$. We will now show that no such V can exist and hence $X_A = C_p$, i.e. X_A is connected.

Define $\phi : X_\Lambda \to X_A$ by $\phi(y) = x$ where

$$x_\alpha = \begin{cases} y_\alpha & \text{if} \quad \alpha \in \Lambda \\ p_\alpha & \text{if} \quad \alpha \notin \Lambda. \end{cases}$$

If $\alpha \in \Lambda$, $\pi_\alpha \circ \phi(y) = y_\alpha = \pi_\alpha(y)$ and if $\alpha \in A \setminus \Lambda$ then $\pi_\alpha \circ \phi(y) = p_\alpha$ so that in every case $\pi_\alpha \circ \phi : X_\Lambda \to X_\alpha$ is continuous and therefore ϕ is continuous.

Since X_Λ is a product of a finite number of connected spaces it is connected by step 1. above. Hence so is the continuous image, $\phi(X_\Lambda) = X_\Lambda \times \{p_\alpha\}_{\alpha \in A \setminus \Lambda}$, of X_Λ. Now $p \in \phi(X_\Lambda)$ and $\phi(X_\Lambda)$ is connected implies that $\phi(X_\Lambda) \subset C$. On the other hand one easily sees that

$$\emptyset \neq V \cap \phi(X_\Lambda) \subset V \cap C$$

contradicting the assumption that $V \subset C^c$.

■

15.3. Tychonoff's Theorem.

The main theorem of this subsection is that the product of compact spaces is compact. Before going to the general case an arbitrary number of factors let us start with only two factors.

Proposition 15.13. *Suppose that X and Y are non-empty compact topological spaces, then $X \times Y$ is compact in the product topology.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. Then for each $(x, y) \in X \times Y$ there exist $U \in \mathcal{U}$ such that $(x, y) \in U$. By definition of the product topology, there also exist $V_x \in \tau_x^X$ and $W_y \in \tau_y^Y$ such that $V_x \times W_y \subset U$. Therefore $\mathcal{V} := \{V_x \times W_y : (x, y) \in X \times Y\}$ is also an open cover of $X \times Y$. We will now show that $\mathcal{V}$ has a finite sub-cover, say $\mathcal{V}_0 \subset\subset \mathcal{V}$. Assuming this is proved for the moment, this implies that $\mathcal{U}$ also has a finite subcover because each $V \in \mathcal{V}_0$ is contained in some $U_V \in \mathcal{U}$. So to complete the proof it suffices to show every cover $\mathcal{V}$ of the form $\mathcal{V} = \{V_\alpha \times W_\alpha : \alpha \in A\}$ where $V_\alpha \subset_o X$ and $W_\alpha \subset_o Y$ has a finite subcover.

Given $x \in X$, let $f_x : Y \to X \times Y$ be the map $f_x(y) = (x, y)$ and notice that f_x is continuous since $\pi_X \circ f_x(y) = x$ and $\pi_Y \circ f_x(y) = y$ are continuous maps. From this we conclude that $\{x\} \times Y = f_x(Y)$ is compact. Similarly, it follows that $X \times \{y\}$ is compact for all $y \in Y$.

Since $\mathcal{V}$ is a cover of $\{x\} \times Y$, there exist $\Gamma_x \subset\subset A$ such that $\{x\} \times Y \subset \bigcup_{\alpha \in \Gamma_x} (V_\alpha \times W_\alpha)$ without loss of generality we may assume that Γ_x is chosen so that $x \in V_\alpha$ for all $\alpha \in \Gamma_x$. Let $U_x \equiv \bigcap_{\alpha \in \Gamma_x} V_\alpha \subset_o X$, see Figure 34 below.

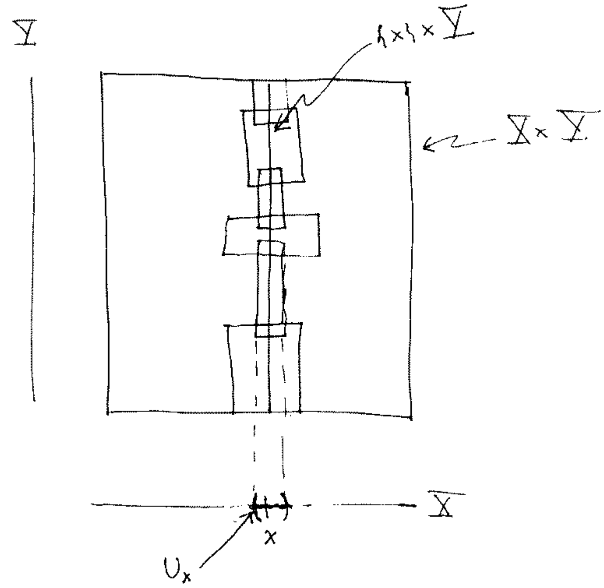

FIGURE 34. Constructing the open set U_x.

Then $\{U_x\}_{x\in X}$ is an open cover of X which is compact, hence there exists $\Lambda \subset\subset X$ such that $X = \cup_{x\in\Lambda} U_x$. The proof is completed by showing that $\mathcal{V}_0 := \cup_{x\in\Lambda} \cup_{\alpha\in\Gamma_x} \{V_\alpha \times W_\alpha\}$ is a cover of $X \times Y$,

$$\cup\mathcal{V}_0 = \cup_{x\in\Lambda} \cup_{\alpha\in\Gamma_x} (V_\alpha \times W_\alpha) \supset \cup_{x\in\Lambda} \cup_{\alpha\in\Gamma_x} (U_x \times W_\alpha) = \cup_{x\in\Lambda} (U_x \times Y) = X \times Y.$$

■

The results of Exercises 3.28 and 4.15 prove Tychonoff's Theorem for a countable product of compact metric spaces. We now state the general version of the theorem.

Theorem 15.14 (Tychonoff's Theorem)**.** *Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of non-empty compact spaces. Then $X := X_A = \prod_{\alpha\in A} X_\alpha$ is compact in the product space topology.*

Proof. The proof requires Zorn's lemma which is equivalent to the axiom of choice, see Theorem B.7 of Appendix B below. For $\alpha \in A$ let π_α denote the projection map from X to X_α. Suppose that $\mathcal{F}$ is a family of closed subsets of X which has the finite intersection property, see Definition 3.25. By Proposition 3.26 the proof will be complete if we can show $\cap\mathcal{F} \neq \emptyset$.

The first step is to apply Zorn's lemma to construct a maximal collection $\mathcal{F}_0$ of (not necessarily closed) subsets of X with the finite intersection property. To do this, let $\Gamma := \left\{\mathcal{G} \subset 2^X : \mathcal{F} \subset \mathcal{G}\right\}$ equipped with the partial order, $\mathcal{G}_1 < \mathcal{G}_2$ if $\mathcal{G}_1 \subset \mathcal{G}_2$. If Φ is a linearly ordered subset of Γ, then $\mathcal{G} := \cup\Phi$ is an upper bound for Γ which still has the finite intersection property as the reader should check. So by Zorn's lemma, Γ has a maximal element $\mathcal{F}_0$.

The maximal $\mathcal{F}_0$ has the following properties.

(1) If $\{F_i\}_{i=1}^n \subset \mathcal{F}_0$ then $\cap_{i=1}^n F_i \in \mathcal{F}_0$ as well. Indeed, if we let $(\mathcal{F}_0)_f$ denote the collection of all finite intersections of elements from $\mathcal{F}_0$, then $(\mathcal{F}_0)_f$ has the finite intersection property and contains $\mathcal{F}_0$. Since $\mathcal{F}_0$ is maximal, this implies $(\mathcal{F}_0)_f = \mathcal{F}_0$.
(2) If $A \subset X$ and $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}_0$ then $A \in \mathcal{F}_0$. For if not $\mathcal{F}_0 \cup \{A\}$ would still satisfy the finite intersection property and would properly contain $\mathcal{F}_0$. this would violate the maximallity of $\mathcal{F}_0$.
(3) For each $\alpha \in A$, $\pi_a(\mathcal{F}_0) := \{\pi_\alpha(F) \subset X_\alpha : F \in \mathcal{F}_0\}$ has the finite intersection property. Indeed, if $\{F_i\}_{i=1}^n \subset \mathcal{F}_0$, then $\cap_{i=1}^n \pi_\alpha(F_i) \supset \pi_\alpha\left(\cap_{i=1}^n F_i\right) \neq \emptyset$.

Since X_α is compact, item 3. above along with Proposition 3.26 implies $\cap_{F\in\mathcal{F}_0}\overline{\pi_\alpha(F)} \neq \emptyset$. Since this true for each $\alpha \in A$, using the axiom of choice, there exists $p \in X$ such that $p_\alpha = \pi_\alpha(p) \in \cap_{F\in\mathcal{F}_0}\overline{\pi_\alpha(F)}$ for all $\alpha \in A$. The proof will be completed by showing $p \in \cap\mathcal{F}$, hence $\cap\mathcal{F}$ is not empty as desired. Since $\cap\left\{\bar{F} : F \in \mathcal{F}_0\right\} \subset \cap\mathcal{F}$, it suffices to show $p \in C := \cap\left\{\bar{F} : F \in \mathcal{F}_0\right\}$. For this suppose that U is an open neighborhood of p in X. By the definition of the product topology, there exists $\Lambda \subset\subset A$ and open sets $U_\alpha \subset X_\alpha$ for all $\alpha \in \Lambda$ such that $p \in \cap_{\alpha\in\Lambda}\pi_\alpha^{-1}(U_\alpha) \subset U$. Since $p_\alpha \in \cap_{F\in\mathcal{F}_0}\overline{\pi_\alpha(F)}$ and $p_\alpha \in U_\alpha$ for all $\alpha \in \Lambda$, it follows that $U_\alpha \cap \pi_\alpha(F) \neq \emptyset$ for all $F \in \mathcal{F}_0$ and all $\alpha \in \Lambda$ and this implies $\pi_\alpha^{-1}(U_\alpha) \cap F \neq \emptyset$ for all $F \in \mathcal{F}_0$ and all $\alpha \in \Lambda$. By item 2. above we concluded that $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{F}_0$ for all $\alpha \in \Lambda$ and by then by item 1., $\cap_{\alpha\in\Lambda}\pi_\alpha^{-1}(U_\alpha) \in \mathcal{F}_0$. In particular $\emptyset \neq F \cap \left(\cap_{\alpha\in\Lambda}\pi_\alpha^{-1}(U_\alpha)\right) \subset F \cap U$ for all $F \in \mathcal{F}_0$ which shows $p \in \bar{F}$ for each $F \in \mathcal{F}_0$. ■

15.4. **Baire Category Theorem.**

Definition 15.15. Let (X,τ) be a topological space. A set $E \subset X$ is said to be **nowhere dense** if $\left(\bar{E}\right)^o = \emptyset$ i.e. $\bar{E}$ has empty interior.

Notice that E is nowhere dense is equivalent to

$$X = \left(\left(\bar{E}\right)^o\right)^c = \overline{\left(\bar{E}\right)^c} = \overline{(E^c)^o}.$$

That is to say E is nowhere dense iff E^c has dense interior.

15.5. **Baire Category Theorem.**

Theorem 15.16 (Baire Category Theorem)**.** *Let (X,ρ) be a complete metric space.*

1. *If $\{V_n\}_{n=1}^\infty$ is a sequence of dense open sets, then $G := \bigcap\limits_{n=1}^\infty V_n$ is dense in X.*
2. *If $\{E_n\}_{n=1}^\infty$ is a sequence of nowhere dense sets, then $X \neq \bigcup_{n=1}^\infty E_n$.*

Proof. 1) We must shows that $\bar{G} = X$ which is equivalent to showing that $W \cap G \neq \emptyset$ for all non-empty open sets $W \subset X$. Since V_1 is dense, $W \cap V_1 \neq \emptyset$ and hence there exists $x_1 \in X$ and $\epsilon_1 > 0$ such that

$$\overline{B(x_1,\epsilon_1)} \subset W \cap V_1.$$

Since V_2 is dense, $B(x_1,\epsilon_1) \cap V_2 \neq \emptyset$ and hence there exists $x_2 \in X$ and $\epsilon_2 > 0$ such that

$$\overline{B(x_2,\epsilon_2)} \subset B(x_1,\epsilon_1) \cap V_2.$$

Continuing this way inductively, we may choose $\{x_n \in X \text{ and } \epsilon_n > 0\}_{n=1}^\infty$ such that

$$\overline{B(x_n,\epsilon_n)} \subset B(x_{n-1},\epsilon_{n-1}) \cap V_n \ \forall n.$$

Furthermore we can clearly do this construction in such a way that $\epsilon_n \downarrow 0$ as $n \uparrow \infty$. Hence $\{x_n\}_{n=1}^\infty$ is Cauchy sequence and $x = \lim\limits_{n\to\infty} x_n$ exists in X since X is complete. Since $\overline{B(x_n,\epsilon_n)}$ is closed, $x \in \overline{B(x_n,\epsilon_n)} \subset V_n$ so that $x \in V_n$ for all n and hence $x \in G$. Moreover, $x \in \overline{B(x_1,\epsilon_1)} \subset W \cap V_1$ implies $x \in W$ and hence $x \in W \cap G$ showing $W \cap G \neq \emptyset$.

2) For the second assertion, since $\bigcup_{n=1}^\infty E_n \subset \bigcup_{n=1}^\infty \bar{E}_n$, it suffices to show $X \neq \bigcup_{n=1}^\infty \bar{E}_n$ or equivalently that $\emptyset \neq \bigcap_{n=1}^\infty \left(\bar{E}_n\right)^c = \bigcap_{n=1}^\infty \left(E_n^c\right)^o$. As we have observed, E_n is nowhere dense is equivalent to $\left(E_n^c\right)^o$ being a dense open set, hence by part 1), $\bigcap_{n=1}^\infty \left(E_n^c\right)^o$ is dense in X and hence not empty. ■

Here is another version of the Baire Category theorem when X is a locally compact Hausdorff space.

Proposition 15.17. *Let X be a locally compact Hausdorff space.*

1. *If $\{V_n\}_{n=1}^\infty$ is a sequence of dense open sets, then $G := \bigcap\limits_{n=1}^\infty V_n$ is dense in X.*
2. *If $\{E_n\}_{n=1}^\infty$ is a sequence of nowhere dense sets, then $X \neq \bigcup_{n=1}^\infty E_n$.*

Proof. As in the previous proof, the second assertion is a consequence of the first. To finish the proof, if suffices to show $G \cap W \neq \emptyset$ for all open sets $W \subset X$. Since V_1 is dense, there exists $x_1 \in V_1 \cap W$ and by Proposition 8.13 there exists $U_1 \subset_o X$ such that $x_1 \in U_1 \subset \bar{U}_1 \subset V_1 \cap W$ with $\bar{U}_1$ being compact. Similarly, there exists a non-empty open set U_2 such that $U_2 \subset \bar{U}_2 \subset U_1 \cap V_2$. Working inductively, we may find non-empty open sets $\{U_k\}_{k=1}^\infty$ such that $U_k \subset \bar{U}_k \subset U_{k-1} \cap V_k$. Since $\cap_{k=1}^n \bar{U}_k = \bar{U}_n \neq \emptyset$ for all n, the finite intersection characterization of $\bar{U}_1$ being compact implies that

$$\emptyset \neq \cap_{k=1}^\infty \bar{U}_k \subset G \cap W.$$

■

Definition 15.18. A subset $E \subset X$ is **meager** or of the **first category** if $E = \bigcup_{n=1}^{\infty} E_n$ where each E_n is nowhere dense. And a set $F \subset X$ is called **residual** if F^c is meager.

Remarks 15.19. The reader should think of meager as being the topological analogue of sets of measure 0 and residual as being the topological analogue of sets of full measure.

(1) F is residual iff F contains a countable intersection of dense open sets. Indeed if F is a residual set, then there exists nowhere dense sets $\{E_n\}$ such that

$$F^c = \cup_{n=1}^\infty E_n \subset \cup_{n=1}^\infty \bar{E}_n.$$

Taking complements of this equation shows that

$$\cap_{n=1}^\infty \bar{E}_n^c \subset F,$$

i.e. F contains a set of the form $\cap_{n=1}^\infty V_n$ with each V_n $(= \bar{E}_n^c)$ being an open dense subset of X.

Conversely, if $\cap_{n=1}^\infty V_n \subset F$ with each V_n being an open dense subset of X, then $F^c \subset \cup_{n=1}^\infty V_n^c$ and hence $F^c = \cup_{n=1}^\infty E_n$ where each $E_n = F^c \cap V_n^c$, is a nowhere dense subset of X.

(2) A countable union of meager sets is meager and any subset of a meager set is meager.

(3) A countable intersection of residual sets is residual.

Remark 15.20. The Baire Category Theorems may now be stated as follows. If X is a complete metric space or X is a locally compact Hausdorff space, then

(1) all residual sets are dense in X and
(2) X is not meager.

Here is an application of Theorem 15.16.

Theorem 15.21. *Let $\mathcal{N} \subset C([0,1],\mathbb{R})$ be the set of nowhere differentiable functions. (Here a function f is said to be differentiable at 0 if $f'(0) := \lim_{t\downarrow 0} \frac{f(t)-f(0)}{t}$ exists and at 1 if $f'(1) := \lim_{t\uparrow 0} \frac{f(1)-f(t)}{1-t}$ exists.) Then $\mathcal{N}$ is a residual set so the "generic" continuous functions is nowhere differentiable.*

Proof. If $f \notin \mathcal{N}$, then $f'(x_0)$ exists for some $x_0 \in [0,1]$ and by the definition of the derivative and compactness of $[0,1]$, there exists $n \in \mathbb{N}$ such that $|f(x) - f(x_0)| \leq n|x - x_0|\ \forall\, x \in [0,1]$. Thus if we define

$$E_n := \{f \in C([0,1]) : \exists\, x_0 \in [0,1] \ni |f(x) - f(x_0)| \leq n|x - x_0|\ \forall\, x \in [0,1]\},$$

then we have just shown $\mathcal{N}^c \subset E := \cup_{n=1}^\infty E_n$. So to finish the proof it suffices to show (for each n) E_n is a closed subset of $C([0,1],\mathbb{R})$ with empty interior.

1) To prove E_n is closed, let $\{f_m\}_{m=1}^\infty \subset E_n$ be a sequence of functions such that there exists $f \in C([0,1],\mathbb{R})$ such that $\|f - f_m\|_u \to 0$ as $m \to \infty$. Since $f_m \in E_n$, there exists $x_m \in [0,1]$ such that

$$|f_m(x) - f_m(x_m)| \le n|x - x_m| \;\forall\; x \in [0,1]. \tag{15.3}$$

Since $[0,1]$ is a compact metric space, by passing to a subsequence if necessary, we may assume $x_0 = \lim_{m\to\infty} x_m \in [0,1]$ exists. Passing to the limit in Eq. (15.3) shows

$$|f(x) - f(x_0)| \le n|x - x_0| \;\forall\; x \in [0,1]$$

and therefore that $f \in E_n$. This shows E_n is a closed subset of $C([0,1],\mathbb{R})$.

2) To finish the proof, we will show $E_n^0 = \emptyset$ by showing for each $f \in E_n$ and $\epsilon > 0$ given, there exits $g \in C([0,1],\mathbb{R}) \setminus E_n$ such that $\|f - g\|_u < \epsilon$. We now construct g.

Since $[0,1]$ is compact and f is continuous there exists $N \in \mathbb{N}$ such that $|f(x) - f(y)| < \epsilon/2$ whenever $|y - x| < 1/N$. Let k denote the piecewise linear function on $[0,1]$ such that $k(\frac{m}{N}) = f(\frac{m}{N})$ for $m = 0, 1, \ldots, N$ and $k''(x) = 0$ for $x \notin P_N := \{m/N : m = 0, 1, \ldots, N\}$. Then it is easily seen that $\|f - k\|_u < \epsilon/2$ and for $x \in (\frac{m}{N}, \frac{m+1}{N})$ that

$$|k'(x)| = \frac{|f(\frac{m+1}{N}) - f(\frac{m}{N})|}{\frac{1}{N}} < N\epsilon/2.$$

We now make k "rougher" by adding a small wiggly function h which we define as follows. Let $M \in \mathbb{N}$ be chosen so that $4\epsilon M > 2n$ and define h uniquely by $h(\frac{m}{M}) = (-1)^m \epsilon/2$ for $m = 0, 1, \ldots, M$ and $h''(x) = 0$ for $x \notin P_M$. Then $\|h\|_u < \epsilon$ and $|h'(x)| = 4\epsilon M > 2n$ for $x \notin P_M$. See Figure 35 below.

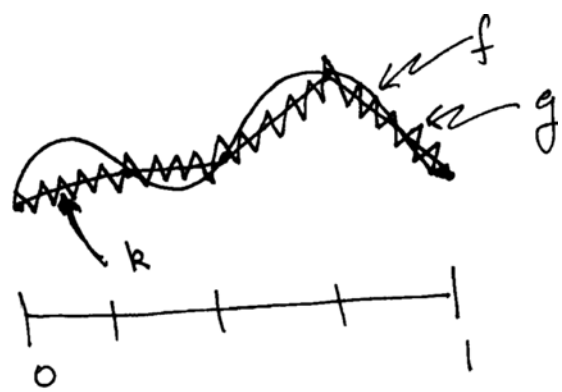

FIGURE 35. Constgructing a rough approximation, g, to a continuous function f.

Finally define $g := k + h$. Then

$$\|f - g\|_u \le \|f - k\|_u + \|h\|_u < \epsilon/2 + \epsilon/2 = \epsilon$$

and

$$|g'(x)| \ge |h'(x)| - |k'(x)| > 2n - n = n \;\forall x \notin P_M \cup P_N.$$

It now follows from this last equation and the mean value theorem that for any $x_0 \in [0,1]$,

$$\left|\frac{g(x) - g(x_0)}{x - x_0}\right| > n$$

for all $x \in [0,1]$ sufficiently close to x_0. This shows $g \notin E_n$ and so the proof is complete. ■

Here is an application of the Baire Category Theorem in Proposition 15.17.

Proposition 15.22. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f'(x)$ exists for all $x \in \mathbb{R}$. Let*

$$U := \cup_{\epsilon>0} \left\{ x \in \mathbb{R} : \sup_{|y|<\epsilon} |f'(x+y)| < \infty \right\}.$$

Then U is a dense open set. (It is not true that $U = \mathbb{R}$ in general, see Example 14.34 above.)

Proof. It is easily seen from the definition of U that U is open. Let $W \subset_o \mathbb{R}$ be an open subset of $\mathbb{R}$. For $k \in \mathbb{N}$, let

$$\begin{aligned} E_k &:= \left\{ x \in W : |f(y) - f(x)| \le k\,|y - x| \text{ when } |y - x| \le \frac{1}{k} \right\} \\ &= \bigcap_{z:|z|\le k^{-1}} \left\{ x \in W : |f(x+z) - f(x)| \le k\,|z| \right\}, \end{aligned}$$

which is a closed subset of $\mathbb{R}$ since f is continuous. Moreover, if $x \in W$ and $M = |f'(x)|$, then

$$\begin{aligned} |f(y) - f(x)| &= |f'(x)\,(y - x) + o\,(y - x)| \\ &\le (M+1)\,|y - x| \end{aligned}$$

for y close to x. (Here $o(y-x)$ denotes a function such that $\lim_{y\to x} o(y-x)/(y-x) = 0$.) In particular, this shows that $x \in E_k$ for all k sufficiently large. Therefore $W = \cup_{k=1}^{\infty} E_k$ and since W is not meager by the Baire category Theorem in Proposition 15.17, some E_k has non-empty interior. That is there exists $x_0 \in E_k \subset W$ and $\epsilon > 0$ such that

$$J := (x_0 - \epsilon, x_0 + \epsilon) \subset E_k \subset W.$$

For $x \in J$, we have $|f(x+z) - f(x)| \le k\,|z|$ provided that $|z| \le k^{-1}$ and therefore that $|f'(x)| \le k$ for $x \in J$. Therefore $x_0 \in U \cap W$ showing U is dense. ■

Remark 15.23. This proposition generalizes to functions $f : \mathbb{R}^n \to \mathbb{R}^m$ in an obvious way.

15.6. **Exercises.**

Exercise 15.1. Prove item 1. of Proposition 15.8. **Hint:** show X is not connected implies X is not path connected.

Exercise 15.2. Prove item 2. of Proposition 15.8. **Hint:** fix $x_0 \in X$ and let W denote the set of $x \in X$ such that there exists $\sigma \in C([0,1], X)$ satisfying $\sigma(0) = x_0$ and $\sigma(1) = x$. Then show W is both open and closed.

Exercise 15.3. Prove item 3. of Proposition 15.8.

Exercise 15.4. Let

$$X := \left\{(x, y) \in \mathbb{R}^2 : y = \sin(x^{-1})\right\} \cup \{(0,0)\}$$

equipped with the relative topology induced from the standard topology on $\mathbb{R}^2$. Show X is connected but not path connected.

Exercise 15.5. Prove the following strong version of item 3. of Proposition 15.8, namely to every pair of points x_0, x_1 in a connected open subset V of $\mathbb{R}^n$ there exists $\sigma \in C^\infty(\mathbb{R}, V)$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. **Hint**: Use a convolution argument.

Exercise 15.6. Folland 5.27. **Hint:** Consider the generalized cantor sets discussed on p. 39 of Folland.

Exercise 15.7. Let $(X, \|\cdot\|)$ be an infinite dimensional normed space and $E \subset X$ be a finite dimensional subspace. Show that $E \subset X$ is nowhere dense.

Exercise 15.8. Now suppose that $(X, \|\cdot\|)$ is an infinite dimensional Banach space. Show that X can not have a countable **algebraic** basis. More explicitly, there is no countable subset $S \subset X$ such that every element $x \in X$ may be written as a **finite** linear combination of elements from S. **Hint:** make use of Exercise 15.7 and the Baire category theorem.