17. Weak and Strong Derivatives and Sobolev Spaces

For this section, let Ω be an open subset of $\mathbb{R}^d$, $p,q,r \in [1,\infty]$, $L^p(\Omega) = L^p(\Omega,\mathcal{B}_\Omega,m)$ and $L^p_{loc}(\Omega) = L^p_{loc}(\Omega,\mathcal{B}_\Omega,m)$, where m is Lebesgue measure on $\mathcal{B}_{\mathbb{R}^d}$ and $\mathcal{B}_\Omega$ is the Borel $\sigma-$algebra on Ω. If $\Omega = \mathbb{R}^d$, we will simply write L^p and L^p_{loc} for $L^p(\mathbb{R}^d)$ and $L^p_{loc}(\mathbb{R}^d)$ respectively. Also let

$$\langle f,g\rangle := \int_\Omega fg dm$$

for any pair of measurable functions $f,g:\Omega\to\mathbb{C}$ such that $fg \in L^1(\Omega)$. For example, by Hölder's inequality, if $\langle f,g\rangle$ is defined for $f\in L^p(\Omega)$ and $g\in L^q(\Omega)$ when $q = \frac{p}{p-1}$. The following simple but useful remark will be used (typically without further comment) in the sequel.

Remark 17.1. Suppose $r,p,q\in[1,\infty]$ are such that $r^{-1} = p^{-1}+q^{-1}$ and $f_t\to f$ in $L^p(\Omega)$ and $g_t\to g$ in $L^q(\Omega)$ as $t\to 0$, then $f_tg_t\to fg$ in $L^r(\Omega)$. Indeed,

$$\begin{aligned}\|f_tg_t - fg\|_r &= \|(f_t-f)\,g_t + f\,(g_t-g)\|_r \\ &\le \|f_t-f\|_p\,\|g_t\|_q + \|f\|_p\,\|g_t-g\|_q \to 0 \text{ as } t\to 0\end{aligned}$$

Definition 17.2 (Weak Differentiability)**.** Let $v\in\mathbb{R}^d$ and $f\in L^p(\Omega)$ ($f\in L^p_{loc}(\Omega)$) then $\partial_v f$ is said to exist **weakly** in $L^p(\Omega)$ ($L^p_{loc}(\Omega)$) if there exists a function $g\in L^p(\Omega)$ ($g\in L^p_{loc}(\Omega)$) such that

$$\langle f,\partial_v\phi\rangle = -\langle g,\phi\rangle \text{ for all } \phi\in C_c^\infty(\Omega). \tag{17.1}$$

The function g if it exists will be denoted by $\partial_v^{(w)}f$. (By Corollary 9.27, there is at most one $g\in L^1_{loc}(\Omega)$ such that Eq. (17.1) holds, so $\partial_v^{(w)}f$ is well defined.)

Lemma 17.3. *Suppose $f\in L^1_{loc}(\Omega)$ and $\partial_v^{(w)}f$ exists weakly in $L^1_{loc}(\Omega)$. Then*

1. $\mathrm{supp}_m(\partial_v^{(w)}f)\subset\mathrm{supp}_m(f)$, *where* $\mathrm{supp}_m(f)$ *is the essential support of f relative to Lebesgue measure, see Definition 9.14.*
2. *If f is continuously differentiable on $U\subset_o\Omega$, then $\partial_v^{(w)}f = \partial_v f$ a.e. on U.*

Proof.

1. Since
$$\langle\partial_v^{(w)}f,\phi\rangle = -\langle f,\partial_v\phi\rangle = 0 \text{ for all } \phi\in C_c^\infty(\Omega\setminus\mathrm{supp}_m(f)),$$
and application of Corollary 9.27 shows $\partial_v^{(w)}f = 0$ a.e. on $\Omega\setminus\mathrm{supp}_m(f)$. So by Lemma 9.15, $\Omega\setminus\mathrm{supp}_m(f)\subset\Omega\setminus\mathrm{supp}_m(\partial_v^{(w)}f)$, i.e. $\mathrm{supp}_m(\partial_v^{(w)}f)\subset\mathrm{supp}_m(f)$.
2. Suppose that $f|_U$ is C^1 and let $\psi\in C_c^\infty(U)$ which we view as a function in $C_c^\infty(\mathbb{R}^d)$ by setting $\psi\equiv 0$ on $\mathbb{R}^d\setminus U$. By Corollary 9.24, there exists $\gamma\in C_c^\infty(\Omega)$ such that $0\le\gamma\le 1$ and $\gamma = 1$ in a neighborhood of $\mathrm{supp}(\psi)$. Then by setting $\gamma f = 0$ on $\mathbb{R}^d\setminus\mathrm{supp}(\gamma)$ we may view $\gamma f\in C_c^1(\mathbb{R}^d)$ and so by standard integration by parts (see Lemma 9.25) and the ordinary product rule,
$$\begin{aligned}\langle\partial_v^{(w)}f,\psi\rangle &= -\langle f,\partial_v\psi\rangle = -\langle\gamma f,\partial_v\psi\rangle \\ &= \langle\partial_v(\gamma f),\psi\rangle = \langle\partial_v\gamma\cdot f + \gamma\partial_v f,\psi\rangle = \langle\partial_v f,\psi\rangle \end{aligned}\tag{17.2}$$

wherein the last equality we have used $\psi\partial_v\gamma = 0$ and $\psi\gamma = \psi$. Since Eq. (17.2) is true for all $\psi \in C_c^\infty(U)$, an application of Corollary 9.27 with $h = \partial_v^{(w)} f(x) - \partial_v f(x)$ and $\mu = m$ shows $\partial_v^{(w)} f(x) = \partial_v f(x)$ for m – a.e. $x \in U$.

■

Lemma 17.4 (Product Rule)**.** *Let $f \in L^1_{loc}(\Omega)$, $v \in \mathbb{R}^d$ and $\phi \in C^\infty(\Omega)$. If $\partial_v^{(w)} f$ exists in $L^1_{loc}(\Omega)$, then $\partial_v^{(w)}(\phi f)$ exists in $L^1_{loc}(\Omega)$ and*

$$\partial_v^{(w)}(\phi f) = \partial_v \phi \cdot f + \phi \partial_v^{(w)} f \text{ a.e.}$$

Moreover if $\phi \in C_c^\infty(\mathbb{R}^d)$ and $F := \phi f \in L^1$ (here we define F on $\mathbb{R}^d$ by setting $F = 0$ on $\mathbb{R}^d \setminus \Omega$), then $\partial^{(w)} F = \partial_v \phi \cdot f + \phi \partial_v^{(w)} f$ exists weakly in $L^1(\mathbb{R}^d)$.

Proof. Let $\psi \in C_c^\infty(\Omega)$, then

$$\begin{aligned} -\langle \phi f, \partial_v \psi \rangle &= -\langle f, \phi \partial_v \psi \rangle = -\langle f, \partial_v(\phi\psi) - \partial_v \phi \cdot \psi \rangle = \langle \partial_v^{(w)} f, \phi\psi \rangle + \langle \partial_v \phi \cdot f, \psi \rangle \\ &= \langle \phi \partial_v^{(w)} f, \psi \rangle + \langle \partial_v \phi \cdot f, \psi \rangle. \end{aligned}$$

This proves the first assertion. To prove the second assertion let $\gamma \in C_c^\infty(\Omega)$ such that $0 \le \gamma \le 1$ and $\gamma = 1$ on a neighborhood of $\text{supp}(\phi)$. So for $\psi \in C_c^\infty(\mathbb{R}^d)$, using $\partial_v \gamma = 0$ on $\text{supp}(\phi)$ and $\gamma\psi \in C_c^\infty(\Omega)$, we find

$$\begin{aligned} \langle F, \partial_v \psi \rangle &= \langle \gamma F, \partial_v \psi \rangle = \langle F, \gamma \partial_v \psi \rangle = \langle (\phi f), \partial_v(\gamma\psi) - \partial_v \gamma \cdot \psi \rangle \\ &= \langle (\phi f), \partial_v(\gamma\psi) \rangle = -\langle \partial_v^{(w)}(\phi f), (\gamma\psi) \rangle \\ &= -\langle \partial_v \phi \cdot f + \phi \partial_v^{(w)} f, \gamma\psi \rangle = -\langle \partial_v \phi \cdot f + \phi \partial_v^{(w)} f, \psi \rangle. \end{aligned}$$

This show $\partial_v^{(w)} F = \partial_v \phi \cdot f + \phi \partial_v^{(w)} f$ as desired. ■

Lemma 17.5. *Suppose $p \in [1, \infty)$, $v \in \mathbb{R}^d$ and $f \in L^p_{loc}(\Omega)$.*

(1) *If there exists $\{f_m\}_{m=1}^\infty \subset L^p_{loc}(\Omega)$ such that $\partial_v^{(w)} f_m$ exists in $L^p_{loc}(\Omega)$ for all m and there exists $g \in L^p_{loc}(\Omega)$ such that for all $\phi \in C_c^\infty(\Omega)$,*

$$\lim_{m\to\infty} \langle f_m, \phi \rangle = \langle f, \phi \rangle \text{ and } \lim_{m\to\infty} \langle \partial_v f_m, \phi \rangle = \langle g, \phi \rangle$$

then $\partial_v^{(w)} f$ exists in $L^p_{loc}(\Omega)$ and $\partial_v f = g$.

(2) *If $\partial_v^{(w)} f$ exists in $L^p_{loc}(\Omega)$ then there exists $f_n \in C_c^\infty(\Omega)$ such that $f_n \to f$ in $L^p(K)$ (i.e. $\lim_{n\to\infty} \|f - f_n\|_{L^p(K)} = 0$) and $\partial_v f_n \to \partial_v^{(w)} f$ in $L^p(K)$ for all $K \sqsubset\sqsubset \Omega$.*

Proof.

(1) Since

$$\langle f, \partial_v \phi \rangle = \lim_{m\to\infty} \langle f_m, \partial_v \phi \rangle = -\lim_{m\to\infty} \langle \partial_v^{(w)} f_m, \phi \rangle = \langle g, \phi \rangle$$

for all $\phi \in C_c^\infty(\Omega)$, $\partial_v^{(w)} f$ exists and is equal to $g \in L^p_{loc}(\Omega)$.

(2) Let $K_0 := \emptyset$ and

$$K_n := \{x \in \Omega : |x| \le n \text{ and } d(x, \Omega^c) \ge 2/n\}$$

(so $K_n \subset K_{n+1}^o \subset K_{n+1}$ for all n and $K_n \uparrow \Omega$ as $n \to \infty$ or see Lemma 8.10) and choose $\psi_n \in C_c^\infty(K_n^o, [0,1])$ using Corollary 9.24 so that $\psi_n = 1$ on a neighborhood of K_{n-1}. Given a compact set $K \subset \Omega$, for all sufficiently large m, $\psi_m f = f$ on K and by Lemma 17.4 and item 1. of Lemma 17.3, we also have

$$\partial_v^{(w)}(\psi_m f) = \partial_v \psi_m \cdot f + \psi_m \partial_v^{(w)} f = \partial_v^{(w)} f \text{ on } K.$$

This argument shows we may assume $\mathrm{supp}_m(f)$ is a compact subset of Ω in which case we extend f to a function F on $\mathbb{R}^d$ by setting $F = f$ on Ω and $F = 0$ on Ω^c. This function F is in $L^p(\mathbb{R}^d)$ and $\partial_v^{(w)} F = \partial_v^{(w)} f$. Indeed, if $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in C_c^\infty(\mathbb{R}^d)$ is chosen so that $\mathrm{supp}(\psi) \subset \Omega$, $0 \le \psi \le 1$ and $\psi = 1$ in a neighborhood of $\mathrm{supp}_m(f)$, then

$$\begin{aligned}\langle F, \partial_v \phi\rangle &= \langle F, \psi \partial_v \phi\rangle = \langle f, \partial_v (\psi\phi)\rangle \\ &= -\langle \partial_v^{(w)} f, \psi\phi\rangle = -\langle \psi \partial_v^{(w)} f, \phi\rangle\end{aligned}$$

which shows $\partial_v^{(w)} F$ exist in $L^p(\mathbb{R}^d)$ and

$$\partial_v^{(w)} F = \psi \partial_v^{(w)} f = 1_\Omega \partial_v^{(w)} f.$$

Let $\chi \in C_c^\infty(B_0(1))$ with $\int_{\mathbb{R}^d} \chi dm = 1$ and set $\delta_k(x) = n^d \chi(nx)$. Then there exists $N \in \mathbb{N}$ and a compact subset $K \subset \Omega$ such that $f_n := F * \delta_n \in C_c^\infty(\Omega)$ and $\mathrm{supp}(f_n) \subset K$ for all $n \ge N$. By Proposition 9.23 and the definition of $\partial_v^{(w)} F$,

$$\begin{aligned}\partial_v f_n(x) = F * \partial_v \delta_n(x) &= \int_{\mathbb{R}^d} F(y) \partial_v \delta_n(x-y) dy \\ &= -\langle F, \partial_v [\delta_n(x - \cdot)]\rangle = \langle \partial_v^{(w)} F, \delta_n(x-\cdot)\rangle = \partial_v^{(w)} F * \delta_n(x).\end{aligned}$$

Hence by Theorem 9.20, $f_n \to F = f$ and $\partial_v f_n \to \partial_v^{(w)} F = \partial_v^{(w)} f$ in $L^p(\Omega)$ as $n \to \infty$.

■

Definition 17.6 (Strong Differentiability)**.** Let $v \in \mathbb{R}^d$ and $f \in L^p$, then $\partial_v f$ is said to exist **strongly** in L^p if the $\lim_{t\to 0} (\tau_{-tv} f - f)/t$ exists in L^p, where as above $\tau_v f(x) := f(x - v)$. We will denote the limit by $\partial_v^{(s)} f$.

It is easily verified that if $f \in L^p$, $v \in \mathbb{R}^d$ and $\partial_v^{(s)} f \in L^p$ exists then $\partial_v^{(w)} f$ exists and $\partial_v^{(w)} f = \partial_v^{(s)} f$. To check this assertion, let $\phi \in C_c^\infty(\mathbb{R}^d)$ and then using Remark 17.1,

$$\partial_v^{(s)} f \cdot \phi = L^1\text{–}\lim_{t\to 0} \frac{\tau_{-tv} f - f}{t} \phi.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^d} \partial_v^{(s)} f \cdot \phi dm &= \lim_{t\to 0} \int_{\mathbb{R}^d} \frac{\tau_{-tv} f - f}{t} \phi dm = \lim_{t\to 0} \int_{\mathbb{R}^d} f \frac{\tau_{tv}\phi - \phi}{t} dm \\ &= \frac{d}{dt}|_0 \int_{\mathbb{R}^d} f \tau_{tv} \phi dm = \int_{\mathbb{R}^d} f \cdot \frac{d}{dt}|_0 \tau_{tv} \phi dm = -\int_{\mathbb{R}^d} f \cdot \partial_v \phi dm,\end{aligned}$$

wherein we have used Corollary 5.43 to differentiate under the integral in the fourth equality. This shows $\partial_v^{(w)} f$ exists and is equal to $\partial_v^{(s)} f$. What is somewhat more surprising is that the converse assertion that if $\partial_v^{(w)} f$ exists then so does $\partial_v^{(s)} f$. The next theorem is a generalization of Theorem 10.36 from L^2 to L^p.

Theorem 17.7 (Weak and Strong Differentiability)**.** *Suppose $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^d)$ and $v \in \mathbb{R}^d \setminus \{0\}$. Then the following are equivalent:*

(1) *There exists* $g \in L^p(\mathbb{R}^d)$ *and* $\{t_n\}_{n=1}^\infty \subset \mathbb{R}\backslash\{0\}$ *such that* $\lim_{n\to\infty} t_n = 0$ *and*

$$\lim_{n\to\infty} \langle \frac{f(\cdot + t_n v) - f(\cdot)}{t_n}, \phi\rangle = \langle g, \phi\rangle \text{ for all } \phi \in C_c^\infty(\mathbb{R}^d).$$

(2) $\partial_v^{(w)} f$ *exists and is equal to* $g \in L^p(\mathbb{R}^d)$, *i.e.* $\langle f, \partial_v \phi\rangle = -\langle g, \phi\rangle$ *for all* $\phi \in C_c^\infty(\mathbb{R}^d)$.

(3) *There exists* $g \in L^p(\mathbb{R}^d)$ *and* $f_n \in C_c^\infty(\mathbb{R}^d)$ *such that* $f_n \overset{L^p}{\to} f$ *and* $\partial_v f_n \overset{L^p}{\to} g$ *as* $n \to \infty$.

(4) $\partial_v^{(s)} f$ *exists and is is equal to* $g \in L^p(\mathbb{R}^d)$, *i.e.*

$$\frac{f(\cdot + tv) - f(\cdot)}{t} \overset{L^p}{\to} g \text{ as } t \to 0.$$

Moreover if $p \in (1,\infty)$ *any one of the equivalent conditions 1. – 4. above are implied by the following condition.*

1′. *There exists* $\{t_n\}_{n=1}^\infty \subset \mathbb{R}\backslash\{0\}$ *such that* $\lim_{n\to\infty} t_n = 0$ *and*

$$\sup_n \left\| \frac{f(\cdot + t_n v) - f(\cdot)}{t_n} \right\|_p < \infty.$$

Proof. 4. $\Longrightarrow$ 1. is simply the assertion that strong convergence implies weak convergence.

1. $\Longrightarrow$ 2. For $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\langle g, \phi\rangle &= \lim_{n\to\infty} \langle \frac{f(\cdot + t_n v) - f(\cdot)}{t_n}, \phi\rangle = \lim_{n\to\infty} \langle f, \frac{\phi(\cdot - t_n v) - \phi(\cdot)}{t_n}\rangle \\ &= \langle f, \lim_{n\to\infty} \frac{\phi(\cdot - t_n v) - \phi(\cdot)}{t_n}\rangle = -\langle f, \partial_v \phi\rangle,\end{aligned}$$

wherein we have used the translation invariance of Lebesgue measure and the dominated convergence theorem.

2. $\Longrightarrow$ 3. Let $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $\int_{\mathbb{R}^d} \phi(x)dx = 1$ and let $\phi_m(x) = m^d \phi(mx)$, then by Proposition 9.23, $h_m := \phi_m * f \in C^\infty(\mathbb{R}^d)$ for all m and

$$\begin{aligned}\partial_v h_m(x) &= \partial_v \phi_m * f(x) = \int_{\mathbb{R}^d} \partial_v \phi_m(x-y) f(y) dy = \langle f, -\partial_v [\phi_m(x - \cdot)]\rangle \\ &= \langle g, \phi_m(x-\cdot)\rangle = \phi_m * g(x).\end{aligned}$$

By Theorem 9.20, $h_m \to f \in L^p(\mathbb{R}^d)$ and $\partial_v h_m = \phi_m * g \to g$ in $L^p(\mathbb{R}^d)$ as $m \to \infty$. This shows 3. holds except for the fact that h_m need not have compact support. To fix this let $\psi \in C_c^\infty(\mathbb{R}^d, [0,1])$ such that $\psi = 1$ in a neighborhood of 0 and let $\psi_\epsilon(x) = \psi(\epsilon x)$ and $(\partial_v \psi)_\epsilon(x) := (\partial_v \psi)(\epsilon x)$. Then

$$\partial_v(\psi_\epsilon h_m) = \partial_v \psi_\epsilon h_m + \psi_\epsilon \partial_v h_m = \epsilon (\partial_v \psi)_\epsilon h_m + \psi_\epsilon \partial_v h_m$$

so that $\psi_\epsilon h_m \to h_m$ in L^p and $\partial_v(\psi_\epsilon h_m) \to \partial_v h_m$ in L^p as $\epsilon \downarrow 0$. Let $f_m = \psi_{\epsilon_m} h_m$ where ϵ_m is chosen to be greater than zero but small enough so that

$$\|\psi_{\epsilon_m} h_m - h_m\|_p + \|\partial_v(\psi_{\epsilon_m} h_m) \to \partial_v h_m\|_p < 1/m.$$

Then $f_m \in C_c^\infty(\mathbb{R}^d)$, $f_m \to f$ and $\partial_v f_m \to g$ in L^p as $m \to \infty$.

3. $\implies$ 4. By the fundamental theorem of calculus

$$\frac{\tau_{-tv}f_m(x)-f_m(x)}{t} = \frac{f_m(x+tv)-f_m(x)}{t}$$

$$= \frac{1}{t}\int_0^1 \frac{d}{ds}f_m(x+stv)ds = \int_0^1 \left(\partial_v f_m\right)(x+stv)ds. \tag{17.3}$$

Let

$$G_t(x) := \int_0^1 \tau_{-stv}g(x)ds = \int_0^1 g(x+stv)ds$$

which is defined for almost every x and is in $L^p(\mathbb{R}^d)$ by Minkowski's inequality for integrals, Theorem 7.27. Therefore

$$\frac{\tau_{-tv}f_m(x)-f_m(x)}{t} - G_t(x) = \int_0^1 \left[\left(\partial_v f_m\right)(x+stv) - g(x+stv)\right]ds$$

and hence again by Minkowski's inequality for integrals,

$$\left\|\frac{\tau_{-tv}f_m - f_m}{t} - G_t\right\|_p \le \int_0^1 \left\|\tau_{-stv}\left(\partial_v f_m\right) - \tau_{-stv}g\right\|_p ds = \int_0^1 \left\|\partial_v f_m - g\right\|_p ds.$$

Letting $m\to\infty$ in this equation implies $\left(\tau_{-tv}f - f\right)/t = G_t$ a.e. Finally one more application of Minkowski's inequality for integrals implies,

$$\left\|\frac{\tau_{-tv}f-f}{t} - g\right\|_p = \left\|G_t - g\right\|_p = \left\|\int_0^1 \left(\tau_{-stv}g - g\right)ds\right\|_p$$

$$\le \int_0^1 \left\|\tau_{-stv}g - g\right\|_p ds.$$

By the dominated convergence theorem and Proposition 9.13, the latter term tends to 0 as $t\to 0$ and this proves 4.

($1'$. $\implies$ 1. when $p>1$) This is a consequence of Theorem 16.27 which asserts, by passing to a subsequence if necessary, that $\frac{f(\cdot+t_n v)-f(\cdot)}{t_n} \overset{w}{\to} g$ for some $g\in L^p(\mathbb{R}^d)$. ■

Example 17.8. The fact that ($1'$) does not imply the equivalent conditions 1 – 4 in Theorem 17.7 when $p=1$ is demonstrated by the following example. Let $f := 1_{[0,1]}$, then

$$\int_{\mathbb{R}}\left|\frac{f(x+t)-f(x)}{t}\right|dx = \frac{1}{|t|}\int_{\mathbb{R}}\left|1_{[-t,1-t]}(x) - 1_{[0,1]}(x)\right|dx = 2$$

for $|t|<1$. For contradiction sake, suppose there exists $g\in L^1(\mathbb{R},dm)$ such that

$$\lim_{n\to\infty}\frac{f(x+t_n)-f(x)}{t_n} = g(x) \text{ in } L^1$$

for some sequence $\{t_n\}_{n=1}^\infty$ as above. Then for $\phi\in C_c^\infty(\mathbb{R})$ we would have on one hand,

$$\int_{\mathbb{R}}\frac{f(x+t_n)-f(x)}{t_n}\phi(x)dx = \int_{\mathbb{R}}\frac{\phi(x-t_n)-\phi(x)}{t_n}f(x)dx \to -\int_0^1 \phi'(x)dx = (\phi(0)-\phi(1)),$$

while on the other hand,

$$\int_{\mathbb{R}}\frac{f(x+t_n)-f(x)}{t_n}\phi(x)dx \to \int_{\mathbb{R}} g(x)\phi(x)dx.$$

These two equations imply

$$\int_{\mathbb{R}} g(x)\phi(x)dx = \phi(0) - \phi(1) \text{ for all } \phi \in C_c^\infty(\mathbb{R}) \tag{17.4}$$

and in particular that $\int_{\mathbb{R}} g(x)\phi(x)dx = 0$ for all $\phi \in C_c(\mathbb{R}\setminus\{0,1\})$. By Corollary 9.27, $g(x) = 0$ for m – a.e. $x \in \mathbb{R}\setminus\{0,1\}$ and hence $g(x) = 0$ for m – a.e. $x \in \mathbb{R}$. But this clearly contradicts Eq. (17.4). This example also shows that the unit ball in $L^1(\mathbb{R}, dm)$ is not sequentially weakly compact.

We will now give a couple of applications of Theorem 17.7.

Proposition 17.9. *Let $\Omega \subset \mathbb{R}$ be an open interval and $f \in L^1_{loc}(\Omega)$. Then $\partial^w f$ exists in $L^1_{loc}(\Omega)$ iff f has a continuous version $\tilde{f}$ which is absolutely continuous on all compact subintervals of Ω. Moreover, $\partial^w f = \tilde{f}'$ a.e., where $\tilde{f}'(x)$ is the usual pointwise derivative.*

Proof. If f is locally absolutely continuous and $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset [a,b] \subset \Omega$, then by Corollary 14.32,

$$\int_\Omega f'\phi dm = \int_a^b f'\phi dm = -\int_a^b f\phi' dm + f\phi|_a^b = -\int_\Omega f\phi' dm.$$

This shows $\partial^w f$ exists and $\partial^w f = f' \in L^1_{loc}(\Omega)$.

Now suppose that $\partial^w f$ exists in $L^1_{loc}(\Omega)$, let $[a,b]$ be a compact subinterval of Ω, $\psi \in C_c^\infty(\Omega)$ such that $\psi = 1$ on a neighborhood of $[a,b]$ and $0 \le \psi \le 1$ on Ω and define $F := \psi f$. By Lemma 17.4, $F \in L^1$ and $\partial^{(w)} F$ exists in L^1 and is given by

$$\partial^{(w)}(\psi f) = \psi' f + \psi \partial^{(w)} f.$$

From Theorem 17.7 there exists $F_n \in C_c^\infty(\mathbb{R})$ such that $F_n \to F$ and $F_n' \to \partial^{(w)} F$ in L^1 as $n \to \infty$. Hence, using the fundamental theorem of calculus,

$$F_n(x) = \int_{-\infty}^x F_n'(y)dy \to \int_{-\infty}^x \partial^{(w)} F(y)dy =: G(x) \tag{17.5}$$

as $n \to \infty$. Since $F_n \to G$ pointwise and $F_n \to F$ in L^1, it follows that $F = \psi f = G$ a.e. In particular, $G = f$ a.e. on $[a,b]$ and we conclude that f has a continuous version on $[a,b]$, this version, $G|_{[a,b]}$, is absolutely continuous on $[a,b]$ and by Eq. (17.5) and fundamental theorem of calculus for absolutely continuous functions (Theorem 14.31),

$$G'(x) = \partial^{(w)} F(x) = \partial^{(w)} f(x) \text{ for } m \text{ a.e. } x \in [a,b].$$

Since $[a,b] \subset \Omega$ was an arbitrary compact subinterval and continuous versions of a measurable functions are unique if they exist, it follows from the above considerations that there exists a unique function $\tilde{f} \in C(\Omega)$ such that $\tilde{f} = G|_{[a,b]}$ for all such pairs $([a,b], G)$ as above. The function $\tilde{f}$ then satisfies all of the desired properties. ■

Because of Lemma 17.3, Theorem 17.7 and Proposition 17.9 it is now safe to simply write $\partial_v f$ for the ordinary partial derivative of f, the weak derivative $\partial_v^{(w)} f$ and the strong derivative $\partial_v^{(s)} f$.

Definition 17.10. Let X and Y be metric spaces. A function $f : X \to Y$ is said to be **Lipschitz** if there exists $C < \infty$ such that

$$d^Y(f(x), f(x')) \le C d^X(x, x') \text{ for all } x, x' \in X$$

and said to be **locally Lipschitz** if for all compact subsets $K \subset X$ there exists $C_K < \infty$ such that

$$d^Y(f(x), f(x')) \le C_K d^X(x, x') \text{ for all } x, x' \in K.$$

Proposition 17.11. *Let $f \in L^1(\mathbb{R}^d)$ such that essential support,* $\operatorname{supp}_m(f)$, *is compact. Then there exists a Lipschitz function $F : \mathbb{R}^d \to \mathbb{C}$ such that $F = f$ a.e. iff $\partial_v^{(w)} f$ exists and is bounded for all $v \in \mathbb{R}^d$.*

Proof. Suppose $f = F$ a.e. and F is Lipschitz and let $p \in (1, \infty)$. Since F is Lipschitz and has compact support, for all $0 < |t| \le 1$,

$$\int_{\mathbb{R}^d} \left| \frac{f(x+tv) - f(x)}{t} \right|^p dx = \int_{\mathbb{R}^d} \left| \frac{F(x+tv) - F(x)}{t} \right|^p dx \le \tilde{C} |v|^p,$$

where $\tilde{C}$ is a constant depending on the size of the support of f and on the Lipschitz constant, C, for F. Therefore Theorem 17.7 may be applied to conclude $\partial_v^{(w)} f$ exists in L^p and moreover,

$$\lim_{t \to 0} \frac{F(x+tv) - F(x)}{t} = \partial_v^{(w)} f(x) \text{ for } m - \text{a.e. } x.$$

Since there exists $\{t_n\}_{n=1}^\infty \subset \mathbb{R} \setminus \{0\}$ such that $\lim_{n\to\infty} t_n = 0$ and

$$\left| \partial_v^{(w)} f(x) \right| = \lim_{n \to \infty} \left| \frac{F(x+t_n v) - F(x)}{t_n} \right| \le C |v| \text{ for a.e. } x \in \mathbb{R}^d,$$

it follows that $\left\| \partial_v^{(w)} f \right\|_\infty \le C |v|$ for all $v \in \mathbb{R}^d$.

Conversely, let ϕ_m be an approximate δ – function sequence as used in the proof of Theorem 17.7 and $f_m := f * \phi_m$. Then $f_m \in C_c^\infty(\mathbb{R}^d)$, $\partial_v f_m = \partial_v f * \phi_m$, $|\partial_v f_m| \le \|\partial_v f\|_\infty < \infty$ and therefore,

$$|f_m(y) - f_m(x)| = \left| \int_0^1 \frac{d}{dt} f_m(x + t(y-x)) dt \right| = \left| \int_0^1 (y-x) \cdot \nabla f_m(x + t(y-x)) dt \right|$$

$$\le \int_0^1 |y-x| \cdot |\nabla f_m(x + t(y-x))| \, dt \le C |y-x| \tag{17.6}$$

where C is a constant independent of m. By passing to a subsequence of the $\{\phi_m\}_{m=1}^\infty$ if necessary, we may assume that $\lim_{m\to\infty} f_m(x) = f(x)$ for m – a.e. $x \in \mathbb{R}^d$. Letting $m \to \infty$ in Eq. (17.6) then implies

$$|f(y) - f(x)| \le C |y-x| \text{ for all } x, y \notin E$$

where $E \subset \mathbb{R}^d$ is a m – null set. It is now easily verified that if $F : \mathbb{R}^d \to \mathbb{C}$ is defined by $F = f$ on E^c and

$$F(x) = \begin{cases} f(x) & \text{if } x \notin E \\ \lim_{\substack{y \to x \\ y \notin E}} f(y) & \text{if } x \in E \end{cases}$$

defines a Lipschitz function F on $\mathbb{R}^d$ such that $F = f$ a.e. ∎

Lemma 17.12. *Let $v \in \mathbb{R}^d$.*

(1) *If $h \in L^1$ and $\partial_v h$ exists in L^1, then $\int_{\mathbb{R}^d} \partial_v h(x) dx = 0$.*

(2) *If $p, q, r \in [1, \infty)$ satisfy $r^{-1} = p^{-1} + q^{-1}$, $f \in L^p$ and $g \in L^q$ are functions such that $\partial_v f$ and $\partial_v g$ exists in L^p and L^q respectively, then $\partial_v(fg)$ exists in L^r and $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$. Moreover if $r = 1$ we have the integration by parts formula,*

$$\langle \partial_v f, g \rangle = -\langle f, \partial_v g \rangle. \tag{17.7}$$

(3) *If $p = 1$, $\partial_v f$ exists in L^1 and $g \in BC^1(\mathbb{R}^d)$ (i.e. $g \in C^1(\mathbb{R}^d)$ with g and its first derivatives being bounded) then $\partial_v(gf)$ exists in L^1 and $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$ and again Eq. (17.7) holds.*

Proof. 1) By item 3. of Theorem 17.7 there exists $h_n \in C_c^\infty(\mathbb{R}^d)$ such that $h_n \to h$ and $\partial_v h_n \to \partial_v h$ in L^1. Then

$$\int_{\mathbb{R}^d} \partial_v h_n(x) dx = \frac{d}{dt}|_0 \int_{\mathbb{R}^d} h_n(x + tv) dx = \frac{d}{dt}|_0 \int_{\mathbb{R}^d} h_n(x) dx = 0$$

and letting $n \to \infty$ proves the first assertion.

2) Similarly there exists $f_n, g_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $\partial_v f_n \to \partial_v f$ in L^p and $g_n \to g$ and $\partial_v g_n \to \partial_v g$ in L^q as $n \to \infty$. So by the standard product rule and Remark 17.1, $f_n g_n \to fg \in L^r$ as $n \to \infty$ and

$$\partial_v(f_n g_n) = \partial_v f_n \cdot g_n + f_n \cdot \partial_v g_n \to \partial_v f \cdot g + f \cdot \partial_v g \text{ in } L^r \text{ as } n \to \infty.$$

It now follows from another application of Theorem 17.7 that $\partial_v(fg)$ exists in L^r and $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$. Eq. (17.7) follows from this product rule and item 1. when $r = 1$.

3) Let $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $\partial_v f_n \to \partial_v f$ in L^1 as $n \to \infty$. Then as above, $gf_n \to gf$ in L^1 and $\partial_v(gf_n) \to \partial_v g \cdot f + g\partial_v f$ in L^1 as $n \to \infty$. In particular if $\phi \in C_c^\infty(\mathbb{R}^d)$, then

$$\begin{aligned}\langle gf, \partial_v \phi \rangle &= \lim_{n\to\infty} \langle gf_n, \partial_v \phi \rangle = -\lim_{n\to\infty} \langle \partial_v (gf_n), \phi \rangle \\ &= -\lim_{n\to\infty} \langle \partial_v g \cdot f_n + g\partial_v f_n, \phi \rangle = -\langle \partial_v g \cdot f + g\partial_v f, \phi \rangle.\end{aligned}$$

This shows $\partial_v(fg)$ exists (weakly) and $\partial_v(fg) = \partial_v f \cdot g + f \cdot \partial_v g$. Again Eq. (17.7) holds in this case by item 1. already proved. ∎

Lemma 17.13. *Let $p, q, r \in [1, \infty]$ satisfy $p^{-1} + q^{-1} = 1 + r^{-1}$, $f \in L^p$, $g \in L^q$ and $v \in \mathbb{R}^d$.*

(1) *If $\partial_v f$ exists strongly in L^r, then $\partial_v(f * g)$ exists strongly in L^p and*

$$\partial_v(f * g) = (\partial_v f) * g.$$

(2) *If $\partial_v g$ exists strongly in L^q, then $\partial_v(f * g)$ exists strongly in L^r and*

$$\partial_v(f * g) = f * \partial_v g.$$

(3) *If $\partial_v f$ exists weakly in L^p and $g \in C_c^\infty(\mathbb{R}^d)$, then $f * g \in C^\infty(\mathbb{R}^d)$, $\partial_v(f * g)$ exists strongly in L^r and*

$$\partial_v(f * g) = f * \partial_v g = (\partial_v f) * g.$$

Proof. Items 1 and 2. By Young's inequality and simple computations:

$$\begin{aligned}\left\|\frac{\tau_{-tv}(f*g)-f*g}{t}-(\partial_v f)*g\right\|_r &= \left\|\frac{\tau_{-tv}f*g-f*g}{t}-(\partial_v f)*g\right\|_r \\ &= \left\|\left[\frac{\tau_{-tv}f-f}{t}-(\partial_v f)\right]*g\right\|_r \\ &\le \left\|\frac{\tau_{-tv}f-f}{t}-(\partial_v f)\right\|_p \|g\|_q\end{aligned}$$

which tends to zero as $t\to 0$. The second item is proved analogously, or just make use of the fact that $f*g=g*f$ and apply Item 1.

Using the fact that $g(x-\cdot)\in C_c^\infty(\mathbb{R}^d)$ and the definition of the weak derivative,

$$\begin{aligned}f*\partial_v g(x) &= \int_{\mathbb{R}^d} f(y)\left(\partial_v g\right)(x-y)dy = -\int_{\mathbb{R}^d} f(y)\left(\partial_v g(x-\cdot)\right)(y)dy \\ &= \int_{\mathbb{R}^d}\partial_v f(y)g(x-y)dy = \partial_v f*g(x).\end{aligned}$$

Item 3. is a consequence of this equality and items 1. and 2. ■

17.1. **Sobolev Spaces.**

Notation 17.14. Let ∂^α be defined as in Notation 9.10 and $f\in L^1_{loc}(\Omega)$. We say $\partial^\alpha f$ exists weakly in $L^1_{loc}(\Omega)$ iff there exists $g\in L^1_{loc}(\Omega)$ such that

$$\langle f,\partial^\alpha\phi\rangle = (-1)^{|\alpha|}\langle g,\phi\rangle \text{ for all } \phi\in C_c^\infty(\Omega).$$

As usual g is unique if it exists and we will denote g by $\partial^\alpha f$.

Definition 17.15. For $p\in[1,\infty]$, $k\in\mathbb{N}$ and Ω an open subset of $\mathbb{R}^d$, let

$$W^{k,p}(\Omega) := \{f\in L^p(\Omega) : \partial^\alpha f\in L^p(\Omega) \text{ (weakly) for all } |\alpha|\le k\}$$

and define

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\le k}\|\partial^\alpha f\|_{L^p(\Omega)}.$$

Theorem 17.16. *The space $W^{k,p}(\Omega)$ with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is a Banach space.*

Proof. Suppose that $\{f_n\}_{n=1}^\infty\subset W^{k,p}(\Omega)$ is a Cauchy sequence, then $\{\partial^\alpha f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(\Omega)$ for all $|\alpha|\le k$. By the completeness of $L^p(\Omega)$, there exists $g_\alpha\in L^p(\Omega)$ such that $g_\alpha = L^p\text{-}\lim_{n\to\infty}\partial^\alpha f_n$ for all $|\alpha|\le k$. Therefore, for all $\phi\in C_c^\infty(\Omega)$,

$$\langle f,\partial^\alpha\phi\rangle = \lim_{n\to\infty}\langle f_n,\partial^\alpha\phi\rangle = (-1)^{|\alpha|}\lim_{n\to\infty}\langle\partial^\alpha f_n,\phi\rangle = (-1)^{|\alpha|}\lim_{n\to\infty}\langle g_\alpha,\phi\rangle.$$

This shows $\partial^\alpha f$ exists weakly and $g_\alpha = \partial^\alpha f$ a.e. ■

Example 17.17. Let Ω be an open subset of $\mathbb{R}^d$, then $H^1(\Omega) := W^{1,2}(\Omega)$ is a Hilbert space with inner product defined by

$$(f,g) = \int_\Omega f\cdot\bar{g}dm + \int_\Omega\nabla f\cdot\nabla\bar{g}dm.$$

Proposition 17.18. *$C_c^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$ for all $1\le p<\infty$.*

Proof. The proof of this proposition is left an exercise to the reader. However, note that the assertion **2.** implies **3.** in Theorem 17.7 essentially proves the statement when $k = 1$ and the same smooth approximations used in the proof of Theorem 17.7 will work here as well. ■

17.2. Hölder Spaces.

Notation 17.19. Let Ω be an open subset of $\mathbb{R}^d$, $BC(\Omega)$ and $BC(\bar{\Omega})$ be the bounded continuous functions on Ω and $\bar{\Omega}$ respectively. By identifying $f \in BC(\bar{\Omega})$ with $f|_\Omega \in BC(\Omega)$, we will consider $BC(\bar{\Omega})$ as a subset of $BC(\Omega)$. For $u \in BC(\Omega)$ and $0 < \beta \leq 1$ let

$$\|u\|_u := \sup_{x \in \Omega} |u(x)| \text{ and } [u]_\beta := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\beta} \right\}.$$

If $[u]_\beta < \infty$, we say u is $\beta -$ **Hölder continuous** with holder exponent[35] β and we let

$$C^{0,\beta}(\Omega) := \{u \in BC(\Omega) : [u]_\beta < \infty\}$$

denote the space of Hölder continuous functions on Ω. For $u \in C^{0,\beta}(\Omega)$ let

$$\|u\|_{C^{0,\beta}(\Omega)} := \|u\|_u + [u]_\beta. \tag{17.8}$$

Remark 17.20. If $u : \Omega \to \mathbb{C}$ and $[u]_\beta < \infty$ for some $\beta > 1$, then u is constant on each connected component of Ω. Indeed, if $x \in \Omega$ and $h \in \mathbb{R}^d$ then

$$\left| \frac{u(x + th) - u(x)}{t} \right| \leq [u]_\beta t^\beta / t \to 0 \text{ as } t \to 0$$

which shows $\partial_h u(x) = 0$ for all $x \in \Omega$. If $y \in \Omega$ is in the same connected component as x, then by Exercise 15.5 there exists as smooth curve $\sigma : [0, 1] \to \Omega$ such that $\sigma(0) = x$ and $\sigma(1) = y$. So by the fundamental theorem of calculus and the chain rule,

$$u(y) - u(x) = \int_0^1 \frac{d}{dt} u(\sigma(t)) dt = \int_0^1 0 \, dt = 0.$$

This is why we do not talk about Hölder spaces with Hölder exponents larger than 1.

Exercise 17.1. Suppose $u \in C^1(\Omega) \cap BC(\Omega)$ and $\partial_i u \in BC(\Omega)$ for $i = 1, 2, \ldots, d$. Show $[u]_1 < \infty$, i.e. $u \in C^{0,1}(\Omega)$.

Theorem 17.21. *Let Ω be an open subset of $\mathbb{R}^d$. Then*

1. *$BC(\bar{\Omega})$ is a closed subspace of $BC(\Omega)$.*
2. *Every element $u \in C^{0,\beta}(\Omega)$ has a unique extension to a continuous function (which we will still denote by u) on $\bar{\Omega}$. Therefore we may identify $C^{0,\beta}(\Omega)$ with a subspace of $BC(\bar{\Omega})$. We may also write $C^{0,\beta}(\Omega)$ as $C^{0,\beta}(\bar{\Omega})$ to emphasize this point.*
3. *The function $u \in C^{0,\beta}(\Omega) \to \|u\|_{C^{0,\beta}(\Omega)} \in [0, \infty)$ is a norm on $C^{0,\beta}(\Omega)$ which make $C^{0,\beta}(\Omega)$ into a Banach space.*

[35] If $\beta = 1$, u is is said to be Lipschitz continuous.

Proof. 1. The first item is trivial since for $u \in BC(\bar{\Omega})$, the sup-norm of u on $\bar{\Omega}$ agrees with the sup-norm on Ω and $BC(\bar{\Omega})$ is complete in this norm.

2. Suppose that $[u]_\beta < \infty$ and $x_0 \in \partial\Omega$. Let $\{x_n\}_{n=1}^\infty \subset \Omega$ be a sequence such that $x_0 = \lim_{n\to\infty} x_n$. Then

$$|u(x_n) - u(x_m)| \le [u]_\beta \left|x_n - x_m\right|^\beta \to 0 \text{ as } m, n \to \infty$$

showing $\{u(x_n)\}_{n=1}^\infty$ is Cauchy so that $\bar{u}(x_0) := \lim_{n\to\infty} u(x_n)$ exists. If $\{y_n\}_{n=1}^\infty \subset \Omega$ is another sequence converging to x_0, then

$$|u(x_n) - u(y_n)| \le [u]_\beta \left|x_n - y_n\right|^\beta \to 0 \text{ as } n \to \infty,$$

showing $\bar{u}(x_0)$ is well defined. In this way we define $\bar{u}(x)$ for all $x \in \partial\Omega$ and let $\bar{u}(x) = u(x)$ for $x \in \Omega$. Since a similar limiting argument shows

$$|\bar{u}(x) - \bar{u}(y)| \le [u]_\beta \left|x - y\right|^\beta \text{ for all } x, y \in \bar{\Omega}$$

it follows that $\bar{u}$ is still continuous. In the sequel we will abuse notation and simply denote $\bar{u}$ by u.

3. For $u, v \in C^{0,\beta}(\Omega)$,

$$\begin{aligned}[v+u]_\beta &= \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|v(y) + u(y) - v(x) - u(x)|}{|x-y|^\beta} \right\} \\ &\le \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|v(y) - v(x)| + |u(y) - u(x)|}{|x-y|^\beta} \right\} \le [v]_\beta + [u]_\beta\end{aligned}$$

and for $\lambda \in \mathbb{C}$ it is easily seen that $[\lambda u]_\beta = \lambda [u]_\beta$. This shows $[\cdot]_\beta$ is a semi-norm on $C^{0,\beta}(\Omega)$ and therefore $\|\cdot\|_{C^{0,\beta}(\Omega)}$ defined in Eq. (17.8) is a norm.

To see that $C^{0,\beta}(\Omega)$ is complete, let $\{u_n\}_{n=1}^\infty$ be a $C^{0,\beta}(\Omega)$–Cauchy sequence. Since $BC(\bar{\Omega})$ is complete, there exists $u \in BC(\bar{\Omega})$ such that $\|u - u_n\|_u \to 0$ as $n \to \infty$. For $x, y \in \Omega$ with $x \neq y$,

$$\frac{|u(x) - u(y)|}{\left|x-y\right|^\beta} = \lim_{n\to\infty} \frac{|u_n(x) - u_n(y)|}{\left|x-y\right|^\beta} \le \limsup_{n\to\infty} [u_n] \le \lim_{n\to\infty} \|u_n\|_{C^{0,\beta}(\Omega)} < \infty,$$

and so we see that $u \in C^{0,\beta}(\Omega)$. Similarly,

$$\begin{aligned}\frac{|u(x) - u_n(x) - (u(y) - u_n(y))|}{\left|x-y\right|^\beta} &= \lim_{m\to\infty} \frac{|(u_m - u_n)(x) - (u_m - u_n)(y)|}{|x-y|^\beta} \\ &\le \limsup_{m\to\infty} [u_m - u_n]_\beta \to 0 \text{ as } n \to \infty,\end{aligned}$$

showing $[u - u_n]_\beta \to 0$ as $n \to \infty$ and therefore $\lim_{n\to\infty} \|u - u_n\|_{C^{0,\beta}(\Omega)} = 0$. ■

Notation 17.22. Since Ω and $\bar{\Omega}$ are locally compact Hausdorff spaces, we may define $C_0(\Omega)$ and $C_0(\bar{\Omega})$ as in Definition 8.29. We will also let

$$C_0^{0,\beta}(\Omega) := C^{0,\beta}(\Omega) \cap C_0(\Omega) \text{ and } C_0^{0,\beta}(\bar{\Omega}) := C^{0,\beta}(\Omega) \cap C_0(\bar{\Omega}).$$

It has already been shown in Proposition 8.30 that $C_0(\Omega)$ and $C_0(\bar{\Omega})$ are closed subspaces of $BC(\Omega)$ and $BC(\bar{\Omega})$ respectively. The next proposition describes the relation between $C_0(\Omega)$ and $C_0(\bar{\Omega})$.

Proposition 17.23. *Each $u \in C_0(\Omega)$ has a unique extension to a continuous function on $\bar{\Omega}$ given by $\bar{u} = u$ on Ω and $\bar{u} = 0$ on $\partial\Omega$ and the extension $\bar{u}$ is in $C_0(\bar{\Omega})$. Conversely if $u \in C_0(\bar{\Omega})$ and $u|_{\partial\Omega} = 0$, then $u|_\Omega \in C_0(\Omega)$. In this way we may identify $C_0(\Omega)$ with those $u \in C_0(\bar{\Omega})$ such that $u|_{\partial\Omega} = 0$.*

Proof. Any extension $u \in C_0(\Omega)$ to an element $\bar{u} \in C(\bar{\Omega})$ is necessarily unique, since Ω is dense inside $\bar{\Omega}$. So define $\bar{u} = u$ on Ω and $\bar{u} = 0$ on $\partial\Omega$. We must show $\bar{u}$ is continuous on $\bar{\Omega}$ and $\bar{u} \in C_0(\bar{\Omega})$.

For the continuity assertion it is enough to show $\bar{u}$ is continuous at all points in $\partial\Omega$. For any $\epsilon > 0$, by assumption, the set $K_\epsilon := \{x \in \Omega : |u(x)| \geq \epsilon\}$ is a compact subset of Ω. Since $\partial\Omega = \bar{\Omega} \setminus \Omega$, $\partial\Omega \cap K_\epsilon = \emptyset$ and therefore the distance, $\delta := d(K_\epsilon, \partial\Omega)$, between K_ϵ and $\partial\Omega$ is positive. So if $x \in \partial\Omega$ and $y \in \bar{\Omega}$ and $|y - x| < \delta$, then $|\bar{u}(x) - \bar{u}(y)| = |u(y)| < \epsilon$ which shows $\bar{u} : \bar{\Omega} \to \mathbb{C}$ is continuous. This also shows $\{|\bar{u}| \geq \epsilon\} = \{|u| \geq \epsilon\} = K_\epsilon$ is compact in Ω and hence also in $\bar{\Omega}$. Since $\epsilon > 0$ was arbitrary, this shows $\bar{u} \in C_0(\bar{\Omega})$.

Conversely if $u \in C_0(\bar{\Omega})$ such that $u|_{\partial\Omega} = 0$ and $\epsilon > 0$, then $K_\epsilon := \left\{x \in \bar{\Omega} : |u(x)| \geq \epsilon\right\}$ is a compact subset of $\bar{\Omega}$ which is contained in Ω since $\partial\Omega \cap K_\epsilon = \emptyset$. Therefore K_ϵ is a compact subset of Ω showing $u|_\Omega \in C_0(\bar{\Omega})$. ■

Definition 17.24. Let Ω be an open subset of $\mathbb{R}^d$, $k \in \mathbb{N} \cup \{0\}$ and $\beta \in (0,1]$. Let $BC^k(\Omega)$ $(BC^k(\bar{\Omega}))$ denote the set of $k-$times continuously differentiable functions u on Ω such that $\partial^\alpha u \in BC(\Omega)$ $(\partial^\alpha u \in BC(\bar{\Omega}))$[36] for all $|\alpha| \leq k$. Similarly, let $BC^{k,\beta}(\Omega)$ denote those $u \in BC^k(\Omega)$ such that $[\partial^\alpha u]_\beta < \infty$ for all $|\alpha| = k$. For $u \in BC^k(\Omega)$ let

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_u \text{ and}$$

$$\|u\|_{C^{k,\beta}(\overline{\Omega})} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_u + \sum_{|\alpha| = k} [\partial^\alpha u]_\beta.$$

Theorem 17.25. *The spaces $BC^k(\Omega)$ and $BC^{k,\beta}(\Omega)$ equipped with $\|\cdot\|_{C^k(\Omega)}$ and $\|\cdot\|_{C^{k,\beta}(\overline{\Omega})}$ respectively are Banach spaces and $BC^k(\bar{\Omega})$ is a closed subspace of $BC^k(\Omega)$ and $BC^{k,\beta}(\Omega) \subset BC^k(\bar{\Omega})$. Also*

$$C_0^{k,\beta}(\Omega) = C_0^{k,\beta}(\bar{\Omega}) = \{u \in BC^{k,\beta}(\Omega) : \partial^\alpha u \in C_0(\Omega) \ \forall \ |\alpha| \leq k\}$$

is a closed subspace of $BC^{k,\beta}(\Omega)$.

Proof. Suppose that $\{u_n\}_{n=1}^\infty \subset BC^k(\Omega)$ is a Cauchy sequence, then $\{\partial^\alpha u_n\}_{n=1}^\infty$ is a Cauchy sequence in $BC(\Omega)$ for $|\alpha| \leq k$. Since $BC(\Omega)$ is complete, there exists $g_\alpha \in BC(\Omega)$ such that $\lim_{n\to\infty} \|\partial^\alpha u_n - g_\alpha\|_u = 0$ for all $|\alpha| \leq k$. Letting $u := g_0$, we must show $u \in C^k(\Omega)$ and $\partial^\alpha u = g_\alpha$ for all $|\alpha| \leq k$. This will be done by induction on $|\alpha|$. If $|\alpha| = 0$ there is nothing to prove. Suppose that we have verified $u \in C^l(\Omega)$ and $\partial^\alpha u = g_\alpha$ for all $|\alpha| \leq l$ for some $l < k$. Then for $x \in \Omega$, $i \in \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}$ sufficiently small,

$$\partial^a u_n(x + te_i) = \partial^a u_n(x) + \int_0^t \partial_i \partial^a u_n(x + \tau e_i) d\tau.$$

Letting $n \to \infty$ in this equation gives

$$\partial^a u(x + te_i) = \partial^a u(x) + \int_0^t g_{\alpha+e_i}(x + \tau e_i) d\tau$$

from which it follows that $\partial_i \partial^\alpha u(x)$ exists for all $x \in \Omega$ and $\partial_i \partial^\alpha u = g_{\alpha+e_i}$. This completes the induction argument and also the proof that $BC^k(\Omega)$ is complete.

[36]To say $\partial^\alpha u \in BC(\bar{\Omega})$ means that $\partial^\alpha u \in BC(\Omega)$ and $\partial^\alpha u$ extends to a continuous function on $\bar{\Omega}$.

It is easy to check that $BC^k(\bar{\Omega})$ is a closed subspace of $BC^k(\Omega)$ and by using Exercise 17.1 and Theorem 17.21 that that $BC^{k,\beta}(\Omega)$ is a subspace of $BC^k(\bar{\Omega})$. The fact that $C_0^{k,\beta}(\Omega)$ is a closed subspace of $BC^{k,\beta}(\Omega)$ is a consequence of Proposition 8.30.

To prove $BC^{k,\beta}(\Omega)$ is complete, let $\{u_n\}_{n=1}^\infty \subset BC^{k,\beta}(\Omega)$ be a $\|\cdot\|_{C^{k,\beta}(\overline{\Omega})}$ – Cauchy sequence. By the completeness of $BC^k(\Omega)$ just proved, there exists $u \in BC^k(\Omega)$ such that $\lim_{n\to\infty}\|u-u_n\|_{C^k(\Omega)} = 0$. An application of Theorem 17.21 then shows $\lim_{n\to\infty}\|\partial^\alpha u_n - \partial^\alpha u\|_{C^{0,\beta}(\Omega)} = 0$ for $|\alpha| = k$ and therefore $\lim_{n\to\infty}\|u - u_n\|_{C^{k,\beta}(\overline{\Omega})} = 0$. ■

17.3. Exercises.

Exercise 17.2. Let $p \in [1,\infty)$, α be a multi index (if $\alpha = 0$ let ∂^0 be the identity operator on L^p),

$$D(\partial^\alpha) := \{f \in L^p(\mathbb{R}^n) : \partial^\alpha f \text{ exists weakly in } L^p(\mathbb{R}^n)\}$$

and for $f \in D(\partial^\alpha)$ (the domain of ∂^α) let $\partial^\alpha f$ denote the α – weak derivative of f. (See Notation 17.14.)

(1) Show ∂^α is a densely defined operator on L^p, i.e. $D(\partial^\alpha)$ is a dense linear subspace of L^p and $\partial^\alpha : D(\partial^\alpha) \to L^p$ is a linear transformation.
(2) Show $\partial^\alpha : D(\partial^\alpha) \to L^p$ is a closed operator, i.e. the graph,
$$\Gamma(\partial^\alpha) := \{(f, \partial^\alpha f) \in L^p \times L^p : f \in D(\partial^\alpha)\},$$
is a closed subspace of $L^p \times L^p$.
(3) Show $\partial^\alpha : D(\partial^\alpha) \subset L^p \to L^p$ is not bounded unless $\alpha = 0$. (The norm on $D(\partial^\alpha)$ is taken to be the L^p – norm.)

Exercise 17.3. Let $p \in [1,\infty)$, $f \in L^p$ and α be a multi index. Show $\partial^\alpha f$ exists weakly (see Notation 17.14) in L^p iff there exists $f_n \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^p$ such that $f_n \to f$ and $\partial^\alpha f_n \to g$ in L^p as $n \to \infty$. **Hint:** One direction follows from Exercise 17.2. For the other direction, see the proof of Theorem 17.7.

Exercise 17.4. Folland 8.8 on p. 246.

Exercise 17.5. Assume $n = 1$ and let $\partial = \partial_{e_1}$ where $e_1 = (1) \in \mathbb{R}^1 = \mathbb{R}$.

(1) Let $f(x) = |x|$, show ∂f exists weakly in $L^1_{loc}(\mathbb{R})$ and $\partial f(x) = \operatorname{sgn}(x)$ for m – a.e. x.
(2) Show $\partial(\partial f)$ does **not** exists weakly in $L^1_{loc}(\mathbb{R})$.
(3) Generalize item 1. as follows. Suppose $f \in C(\mathbb{R},\mathbb{R})$ and there exists a finite set $\Lambda := \{t_1 < t_2 < \cdots < t_N\} \subset \mathbb{R}$ such that $f \in C^1(\mathbb{R}\setminus\Lambda,\mathbb{R})$. Assuming $\partial f \in L^1_{loc}(\mathbb{R})$, show ∂f exists weakly and $\partial^{(w)} f(x) = \partial f(x)$ for m – a.e. x.

Exercise 17.6. Suppose that $f \in L^1_{loc}(\Omega)$ and $v \in \mathbb{R}^d$ and $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^d$. If $\partial_j f := \partial_{e_j} f$ exists weakly in $L^1_{loc}(\Omega)$ for all $j = 1, 2, \dots, n$ then $\partial_v f$ exists weakly in $L^1_{loc}(\Omega)$ and $\partial_v f = \sum_{j=1}^n v_j \partial_j f$.

Exercise 17.7. Show Proposition 17.11 generalizes as follows. Let Ω be an open subset of $\mathbb{R}^d$ and $f \in L^1_{loc}(\Omega)$, then there exists a locally Lipschitz function $F : \Omega \to \mathbb{C}$ such that $F = f$ a.e. iff $\partial_v^{(w)} f$ exists and is locally bounded for all $v \in \mathbb{R}^d$. (Here we say $\partial_v^{(w)} f$ is locally bounded if for all compact subsets $K \subset \Omega$, there exists a constant $M_K < \infty$ such that $\left|\partial_v^{(w)} f(x)\right| \le M_K$ for m – a.e. $x \in K$.

Exercise 17.8. Suppose, $f \in L^1_{loc}(\mathbb{R}^d)$ and $\partial_v f$ exists weakly and $\partial_v f = 0$ in $L^1_{loc}(\mathbb{R}^d)$ for all $v \in \mathbb{R}^d$. Then there exists $\lambda \in \mathbb{C}$ such that $f(x) = \lambda$ for m – a.e. $x \in \mathbb{R}^d$. **Hint:** See steps 1. and 2. in the outline given in Exercise 17.9 below.

Exercise 17.9. (A generalization of Exercise 17.8.) Suppose Ω is a connected open subset of $\mathbb{R}^d$ and $f \in L^1_{loc}(\Omega)$. If $\partial^\alpha f = 0$ weakly for $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| = N+1$, then $f(x) = p(x)$ for m – a.e. x where $p(x)$ is a polynomial of degree at most N. Here is an outline.

(1) Suppose $x_0 \in \Omega$ and $\epsilon > 0$ such that $C := C_{x_0}(\epsilon) \subset \Omega$ and let δ_n be a sequence of approximate δ – functions such $\mathrm{supp}(\delta_n) \subset B_0(1/n)$ for all n. Then for n large enough, $\partial^\alpha(f * \delta_n) = (\partial^\alpha f) * \delta_n$ on C for $|\alpha| = N+1$. Now use Taylor's theorem to conclude there exists a polynomial p_n of degree at most N such that $f_n = p_n$ on C.
(2) Show $p := \lim_{n\to\infty} p_n$ exists on C and then let $n \to \infty$ in step 1. to show there exists a polynomial p of degree at most N such that $f = p$ a.e. on C.
(3) Use Taylor's theorem to show if p and q are two polynomials on $\mathbb{R}^d$ which agree on an open set then $p = q$.
(4) Finish the proof with a connectedness argument using the results of steps 2. and 3. above.

Exercise 17.10. Suppose $\Omega \subset_o \mathbb{R}^d$ and $v, w \in \mathbb{R}^d$. Assume $f \in L^1_{loc}(\Omega)$ and that $\partial_v \partial_w f$ exists weakly in $L^1_{loc}(\Omega)$, show $\partial_w \partial_v f$ also exists weakly and $\partial_w \partial_v f = \partial_v \partial_w f$.

Exercise 17.11. Let $d = 2$ and $f(x, y) = 1_{x \geq 0}$. Show $\partial^{(1,1)} f = 0$ weakly in L^1_{loc} despite the fact that $\partial_1 f$ does not exist weakly in L^1_{loc}!