The results in Eq. (18.8) now follow from Eq. (18.6) and item 5 of Theorem 18.3. For example, since $p_t(x) = t^{-n/2}p_1(x/\sqrt{t})$,

$$(\widehat{p}_t)(\xi) = t^{-n/2}\left(\sqrt{t}\right)^n \hat{p}_1(\sqrt{t}\xi) = e^{-\frac{t}{2}|\xi|^2}.$$

This may also be written as $(\widehat{p}_t)(\xi) = t^{-n/2}p_{\frac{1}{t}}(\xi)$. Using this and the fact that p_t is an even function,

$$(\widehat{p}_t)^\vee(x) = \mathcal{F}\widehat{p}_t(-x) = t^{-n/2}\mathcal{F}p_{\frac{1}{t}}(-x) = t^{-n/2}t^{n/2}p_t(-x) = p_t(x).$$

18.2. **Schwartz Test Functions.**

Definition 18.5. A function $f \in C(\mathbb{R}^n, \mathbb{C})$ is said to have **rapid decay** or **rapid decrease** if

$$\sup_{x\in\mathbb{R}^n} (1 + |x|)^N |f(x)| < \infty \text{ for } N = 1, 2, \ldots.$$

Equivalently, for each $N \in \mathbb{N}$ there exists constants $C_N < \infty$ such that $|f(x)| \leq C_N(1 + |x|)^{-N}$ for all $x \in \mathbb{R}^n$. A function $f \in C(\mathbb{R}^n, \mathbb{C})$ is said to have (at most) **polynomial growth** if there exists $N < \infty$ such

$$\sup (1 + |x|)^{-N} |f(x)| < \infty,$$

i.e. there exists $N \in \mathbb{N}$ and $C < \infty$ such that $|f(x)| \leq C(1 + |x|)^N$ for all $x \in \mathbb{R}^n$.

Definition 18.6 (Schwartz Test Functions)**.** Let $\mathcal{S}$ denote the space of functions $f \in C^\infty(\mathbb{R}^n)$ such that f and all of its partial derivatives have rapid decay and let

$$\|f\|_{N,\alpha} = \sup_{x\in\mathbb{R}^n} \left|(1 + |x|)^N \partial^\alpha f(x)\right|$$

so that

$$\mathcal{S} = \left\{f \in C^\infty(\mathbb{R}^n) : \|f\|_{N,\alpha} < \infty \text{ for all } N \text{ and } \alpha\right\}.$$

Also let $\mathcal{P}$ denote those functions $g \in C^\infty(\mathbb{R}^n)$ such that g and all of its derivatives have at most polynomial growth, i.e. $g \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{P}$ iff for all multi-indices α, there exists $N_\alpha < \infty$ such

$$\sup (1 + |x|)^{-N_\alpha} |\partial^\alpha g(x)| < \infty.$$

(Notice that any polynomial function on $\mathbb{R}^n$ is in $\mathcal{P}$.)

Remark 18.7. Since $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset L^2(\mathbb{R}^n)$, it follows that $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$.

Exercise 18.1. Let

$$L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha \tag{18.10}$$

with $a_\alpha \in \mathcal{P}$. Show $L(\mathcal{S}) \subset \mathcal{S}$ and in particular $\partial^\alpha f$ and $x^\alpha f$ are back in $\mathcal{S}$ for all multi-indices α.

Notation 18.8. Suppose that $p(x, \xi) = \Sigma_{|\alpha|\leq N} a_\alpha(x)\xi^\alpha$ where each function $a_\alpha(x)$ is a smooth function. We then set

$$p(x, D_x) := \Sigma_{|\alpha|\leq N} a_\alpha(x)D_x^\alpha$$

and if each $a_\alpha(x)$ is also a polynomial in x we will let

$$p(-D_\xi, \xi) := \Sigma_{|\alpha|\leq N} a_\alpha(-D_\xi)M_{\xi^\alpha}$$

where M_{ξ^α} is the operation of multiplication by ξ^α.

Proposition 18.9. *Let $p(x,\xi)$ be as above and assume each $a_\alpha(x)$ is a polynomial in x. Then for $f \in \mathcal{S}$,*

$$(p(x,D_x)f)^\wedge(\xi) = p(-D_\xi,\xi)\hat{f}(\xi) \tag{18.11}$$

and

$$p(\xi,D_\xi)\hat{f}(\xi) = [p(D_x,-x)f(x)]^\wedge(\xi). \tag{18.12}$$

Proof. The identities $(-D_\xi)^\alpha e^{-ix\cdot\xi} = x^\alpha e^{-ix\cdot\xi}$ and $D_x^\alpha e^{ix\cdot\xi} = \xi^\alpha e^{ix\cdot\xi}$ imply, for any polynomial function q on $\mathbb{R}^n$,

$$q(-D_\xi)e^{-ix\cdot\xi} = q(x)e^{-ix\cdot\xi} \text{ and } q(D_x)e^{ix\cdot\xi} = q(\xi)e^{ix\cdot\xi}. \tag{18.13}$$

Therefore using Eq. (18.13) repeatedly,

$$\begin{aligned}(p(x,D_x)f)^\wedge(\xi) &= \int_{\mathbb{R}^n} \Sigma_{|\alpha|\le N} a_\alpha(x) D_x^\alpha f(x)\cdot e^{-ix\cdot\xi}\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} \Sigma_{|\alpha|\le N} D_x^\alpha f(x)\cdot a_\alpha(-D_\xi)e^{-ix\cdot\xi}\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} f(x)\Sigma_{|\alpha|\le N}(-D_x)^\alpha\left[a_\alpha(-D_\xi)e^{-ix\cdot\xi}\right]\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} f(x)\Sigma_{|\alpha|\le N} a_\alpha(-D_\xi)\left[\xi^\alpha e^{-ix\cdot\xi}\right]\mathbf{d}\xi = p(-D_\xi,\xi)\hat{f}(\xi)\end{aligned}$$

wherein the third inequality we have used Lemma 9.26 to do repeated integration by parts, the fact that mixed partial derivatives commute in the fourth, and in the last we have repeatedly used Corollary 5.43 to differentiate under the integral. The proof of Eq. (18.12) is similar:

$$\begin{aligned}p(\xi,D_\xi)\hat{f}(\xi) &= p(\xi,D_\xi)\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\mathbf{d}x = \int_{\mathbb{R}^n} f(x)p(\xi,-x)e^{-ix\cdot\xi}\mathbf{d}x \\ &= \sum_\alpha \int_{\mathbb{R}^n} f(x)(-x)^\alpha a_\alpha(\xi)e^{-ix\cdot\xi}\mathbf{d}x = \sum_\alpha \int_{\mathbb{R}^n} f(x)(-x)^\alpha a_\alpha(-D_x)e^{-ix\cdot\xi}\mathbf{d}x \\ &= \sum_\alpha \int_{\mathbb{R}^n} e^{-ix\cdot\xi} a_\alpha(D_x)\left[(-x)^\alpha f(x)\right]\mathbf{d}x = [p(D_x,-x)f(x)]^\wedge(\xi).\end{aligned}$$

■

Corollary 18.10. *The Fourier transform preserves the space $\mathcal{S}$, i.e. $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$.*

Proof. Let $p(x,\xi) = \Sigma_{|\alpha|\le N} a_\alpha(x)\xi^\alpha$ with each $a_\alpha(x)$ being a polynomial function in x. If $f \in \mathcal{S}$ then $p(D_x,-x)f \in \mathcal{S} \subset L^1$ and so by Eq. (18.12), $p(\xi,D_\xi)\hat{f}(\xi)$ is bounded in ξ, i.e.

$$\sup_{\xi\in\mathbb{R}^n} |p(\xi,D_\xi)\hat{f}(\xi)| \le C(p,f) < \infty.$$

Taking $p(x,\xi) = (1+|\xi|^2)^N\xi^\alpha$ with $N \in \mathbb{Z}_+$ in this estimate shows $\hat{f}(\xi)$ and all of its derivatives have rapid decay, i.e. $\hat{f}$ is in $\mathcal{S}$. ■

18.3. **Fourier Inversion Formula .**

Theorem 18.11 (Fourier Inversion Theorem)**.** *Suppose that $f \in L^1$ and $\hat{f} \in L^1$, then*

1. *there exists $f_0 \in C_0(\mathbb{R}^n)$ such that $f = f_0$ a.e.*
2. *$f_0 = \mathcal{F}^{-1}\mathcal{F}f$ and $f_0 = \mathcal{F}\mathcal{F}^{-1}f$,*