3. Metric, Banach and Topological Spaces

3.1. Basic metric space notions.

Definition 3.1. A function $d : X \times X \to [0, \infty)$ is called a metric if

1. (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
2. (Non-degenerate) $d(x, y) = 0$ if and only if $x = y \in X$
3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

As primary examples, any normed space $(X, \|\cdot\|)$ is a metric space with $d(x, y) := \|x - y\|$. Thus the space $\ell^p(\mu)$ is a metric space for all $p \in [1, \infty]$. Also any subset of a metric space is a metric space. For example a surface Σ in $\mathbb{R}^3$ is a metric space with the distance between two points on Σ being the usual distance in $\mathbb{R}^3$.

Definition 3.2. Let (X, d) be a metric space. The **open ball** $B(x, \delta) \subset X$ centered at $x \in X$ with radius $\delta > 0$ is the set

$$B(x, \delta) := \{y \in X : d(x, y) < \delta\}.$$

We will often also write $B(x, \delta)$ as $B_x(\delta)$. We also define the **closed ball** centered at $x \in X$ with radius $\delta > 0$ as the set $C_x(\delta) := \{y \in X : d(x, y) \leq \delta\}$.

Definition 3.3. A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to be convergent if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. In this case we write $\lim_{n\to\infty} x_n = x$ of $x_n \to x$ as $n \to \infty$.

Exercise 3.1. Show that x in Definition 3.3 is necessarily unique.

Definition 3.4. A set $F \subset X$ is closed iff every convergent sequence $\{x_n\}_{n=1}^{\infty}$ which is contained in F has its limit back in F. A set $V \subset X$ is open iff V^c is closed. We will write $F \sqsubset X$ to indicate the F is a closed subset of X and $V \subset_o X$ to indicate the V is an open subset of X. We also let τ_d denote the collection of open subsets of X relative to the metric d.

Exercise 3.2. Let $\mathcal{F}$ be a collection of closed subsets of X, show $\cap\mathcal{F} := \cap_{F\in\mathcal{F}} F$ is closed. Also show that finite unions of closed sets are closed, i.e. if $\{F_k\}_{k=1}^{n}$ are closed sets then $\cup_{k=1}^{n} F_k$ is closed. (By taking complements, this shows that the collection of open sets, τ_d, is closed under finite intersections and arbitrary unions.)

The following "continuity" facts of the metric d will be used frequently in the remainder of this book.

Lemma 3.5. *For any non empty subset $A \subset X$, let $d_A(x) \equiv \inf\{d(x, a) | a \in A\}$, then*

$$|d_A(x) - d_A(y)| \leq d(x, y) \ \forall\, x, y \in X. \tag{3.1}$$

Moreover the set $F_\epsilon \equiv \{x \in X | d_A(x) \geq \epsilon\}$ is closed in X.

Proof. Let $a \in A$ and $x, y \in X$, then

$$d(x, a) \leq d(x, y) + d(y, a).$$

Take the inf over a in the above equation shows that

$$d_A(x) \leq d(x, y) + d_A(y) \quad \forall x, y \in X.$$

Therefore, $d_A(x) - d_A(y) \le d(x,y)$ and by interchanging x and y we also have that $d_A(y) - d_A(x) \le d(x,y)$ which implies Eq. (3.1). Now suppose that $\{x_n\}_{n=1}^\infty \subset F_\epsilon$ is a convergent sequence and $x = \lim_{n\to\infty} x_n \in X$. By Eq. (3.1),

$$\epsilon - d_A(x) \le d_A(x_n) - d_A(x) \le d(x, x_n) \to 0 \text{ as } n \to \infty,$$

so that $\epsilon \le d_A(x)$. This shows that $x \in F_\epsilon$ and hence F_ϵ is closed. ■

Corollary 3.6. *The function d satisfies,*

$$|d(x,y) - d(x',y')| \le d(y,y') + d(x,x')$$

and in particular $d : X \times X \to [0,\infty)$ is continuous.

Proof. By Lemma 3.5 for single point sets and the triangle inequality for the absolute value of real numbers,

$$\begin{aligned} |d(x,y) - d(x',y')| &\le |d(x,y) - d(x,y')| + |d(x,y') - d(x',y')| \\ &\le d(y,y') + d(x,x'). \end{aligned}$$

■

Exercise 3.3. Show that $V \subset X$ is open iff for every $x \in V$ there is a $\delta > 0$ such that $B_x(\delta) \subset V$. In particular show $B_x(\delta)$ is open for all $x \in X$ and $\delta > 0$.

Lemma 3.7. *Let A be a closed subset of X and $F_\epsilon \sqsubset X$ be as defined as in Lemma 3.5. Then $F_\epsilon \uparrow A^c$ as $\epsilon \downarrow 0$.*

Proof. It is clear that $d_A(x) = 0$ for $x \in A$ so that $F_\epsilon \subset A^c$ for each $\epsilon > 0$ and hence $\cup_{\epsilon>0} F_\epsilon \subset A^c$. Now suppose that $x \in A^c \subset_o X$. By Exercise 3.3 there exists an $\epsilon > 0$ such that $B_x(\epsilon) \subset A^c$, i.e. $d(x,y) \ge \epsilon$ for all $y \in A$. Hence $x \in F_\epsilon$ and we have shown that $A^c \subset \cup_{\epsilon>0} F_\epsilon$. Finally it is clear that $F_\epsilon \subset F_{\epsilon'}$ whenever $\epsilon' \le \epsilon$. ■

Definition 3.8. Given a set A contained a metric space X, let $\bar{A} \subset X$ be the **closure of** A defined by

$$\bar{A} := \{x \in X : \exists \{x_n\} \subset A \ni x = \lim_{n\to\infty} x_n\}.$$

That is to say $\bar{A}$ contains all **limit points** of A.

Exercise 3.4. Given $A \subset X$, show $\bar{A}$ is a closed set and in fact

$$\bar{A} = \cap\{F : A \subset F \subset X \text{ with } F \text{ closed}\}. \tag{3.2}$$

That is to say $\bar{A}$ is the smallest closed set containing A.

3.2. **Continuity.** Suppose that (X,d) and (Y,ρ) are two metric spaces and $f : X \to Y$ is a function.

Definition 3.9. A function $f : X \to Y$ is continuous at $x \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$d(f(x), f(x')) < \epsilon \text{ provided that } \rho(x,x') < \delta.$$

The function f is said to be continuous if f is continuous at all points $x \in X$.

The following lemma gives three other ways to characterize continuous functions.

Lemma 3.10 (Continuity Lemma)**.** *Suppose that (X,ρ) and (Y,d) are two metric spaces and $f : X \to Y$ is a function. Then the following are equivalent:*

1. *f is continuous.*

2. *$f^{-1}(V) \in \tau_\rho$ for all $V \in \tau_d$, i.e. $f^{-1}(V)$ is open in X if V is open in Y.*
3. *$f^{-1}(C)$ is closed in X if C is closed in Y.*
4. *For all convergent sequences $\{x_n\} \subset X$, $\{f(x_n)\}$ is convergent in Y and*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Proof. 1. $\Rightarrow$ 2. For all $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ if $\rho(x, x') < \delta$. i.e.

$$B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon))$$

So if $V \subset_o Y$ and $x \in f^{-1}(V)$ we may choose $\epsilon > 0$ such that $B_{f(x)}(\epsilon) \subset V$ then

$$B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon)) \subset f^{-1}(V)$$

showing that $f^{-1}(V)$ is open.

2. $\Rightarrow$ 1. Let $\epsilon > 0$ and $x \in X$, then, since $f^{-1}(B_{f(x)}(\epsilon)) \subset_o X$, there exists $\delta > 0$ such that $B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon))$ i.e. if $\rho(x, x') < \delta$ then $d(f(x'), f(x)) < \epsilon$.

2. $\iff$ 3. If C is closed in Y, then $C^c \subset_o Y$ and hence $f^{-1}(C^c) \subset_o X$. Since $f^{-1}(C^c) = \left(f^{-1}(C)\right)^c$, this shows that $f^{-1}(C)$ is the complement of an open set and hence closed. Similarly one shows that 3. $\Rightarrow$ 2.

1. $\Rightarrow$ 4. If f is continuous and $x_n \to x$ in X, let $\epsilon > 0$ and choose $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ when $\rho(x, x') < \delta$. There exists an $N > 0$ such that $\rho(x, x_n) < \delta$ for all $n \geq N$ and therefore $d(f(x), f(x_n)) < \epsilon$ for all $n \geq N$. That is to say $\lim_{n\to\infty} f(x_n) = f(x)$ as $n \to \infty$.

4. $\Rightarrow$ 1. We will show that not 1. $\Rightarrow$ not 4. Not 1 implies there exists $\epsilon > 0$, a point $x \in X$ and a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $d(f(x), f(x_n)) \geq \epsilon$ while $\rho(x, x_n) < \frac{1}{n}$. Clearly this sequence $\{x_n\}$ violates 4. ∎

There is of course a local version of this lemma. To state this lemma, we will use the following terminology.

Definition 3.11. Let X be metric space and $x \in X$. A subset $A \subset X$ is a **neighborhood** of x if there exists an open set $V \subset_o X$ such that $x \in V \subset A$. We will say that $A \subset X$ is an **open neighborhood** of x if A is open and $x \in A$.

Lemma 3.12 (Local Continuity Lemma)**.** *Suppose that (X, ρ) and (Y, d) are two metric spaces and $f : X \to Y$ is a function. Then following are equivalent:*

1. *f is continuous as $x \in X$.*
2. *For all neighborhoods $A \subset Y$ of $f(x)$, $f^{-1}(A)$ is a neighborhood of $x \in X$.*
3. *For all sequences $\{x_n\} \subset X$ such that $x = \lim_{n\to\infty} x_n$, $\{f(x_n)\}$ is convergent in Y and*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

The proof of this lemma is similar to Lemma 3.10 and so will be omitted.

Example 3.13. The function d_A defined in Lemma 3.5 is continuous for each $A \subset X$. In particular, if $A = \{x\}$, it follows that $y \in X \to d(y, x)$ is continuous for each $x \in X$.

Exercise 3.5. Show the closed ball $C_x(\delta) := \{y \in X : d(x, y) \leq \delta\}$ is a closed subset of X.

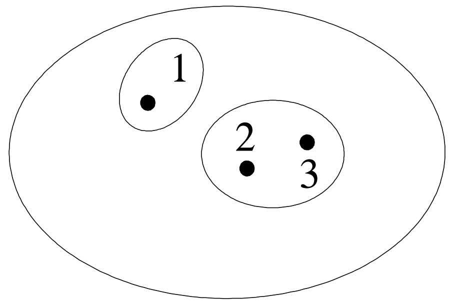

FIGURE 5. A topology

3.3. **Basic Topological Notions.** Using the metric space results above as motivation we will axiomatize the notion of being an open set to more general settings.

Definition 3.14. A collection of subsets τ of X is a **topology** if

1. $\emptyset, X \in \tau$
2. τ is closed under arbitrary unions, i.e. if $V_\alpha \in \tau$, for $\alpha \in I$ then $\bigcup_{\alpha \in I} V_\alpha \in \tau$.
3. τ is closed under finite intersections, i.e. if $V_1, \ldots, V_n \in \tau$ then $V_1 \cap \cdots \cap V_n \in \tau$.

A pair (X, τ) where τ is a topology on X will be called a **topological space**.

Notation 3.15. The subsets $V \subset X$ which are in τ are called open sets and we will abbreviate this by writing $V \subset_o X$ and the those sets $F \subset X$ such that $F^c \in \tau$ are called closed sets. We will write $F \sqsubset X$ if F is a closed subset of X.

Example 3.16.
1. Let (X, d) be a metric space, we write τ_d for the collection of d – open sets in X. We have already seen that τ_d is a topology, see Exercise 3.2.
2. Let X be any set, then $\tau = \mathcal{P}(X)$ is a topology. In this topology all subsets of X are both open and closed. At the opposite extreme we have the **trivial** topology, $\tau = \{\emptyset, X\}$. In this topology only the empty set and X are open (closed).
3. Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, X, \{2, 3\}\}$ is a topology on X which does not come from a metric.
4. Again let $X = \{1, 2, 3\}$. Then $\tau = \{\{1\}, \{2, 3\}, \emptyset, X\}$. is a topology, and the sets X, $\{1\}$, $\{2, 3\}, \phi$ are open and closed. The sets $\{1, 2\}$ and $\{1, 3\}$ are neither open nor closed.

Definition 3.17. Let (X, τ) be a topological space, $A \subset X$ and $i_A : A \to X$ be the inclusion map, i.e. $i_A(a) = a$ for all $a \in A$. Define

$$\tau_A = i_A^{-1}(\tau) = \{A \cap V : V \in \tau\},$$

the so called **relative topology** on A.

Notice that the closed sets in Y relative to τ_Y are precisely those sets of the form $C \cap Y$ where C is close in X. Indeed, $B \subset Y$ is closed iff $Y \setminus B = Y \cap V$ for some $V \in \tau$ which is equivalent to $B = Y \setminus (Y \cap V) = Y \cap V^c$ for some $V \in \tau$.

Exercise 3.6. Show the relative topology is a topology on A. Also show if (X,d) is a metric space and $\tau=\tau_d$ is the topology coming from d, then $(\tau_d)_A$ is the topology induced by making A into a metric space using the metric $d|_{A\times A}$.

Notation 3.18 (Neighborhoods of x)**.** An **open neighborhood** of a point $x\in X$ is an open set $V\subset X$ such that $x\in V$. Let $\tau_x=\{V\in\tau : x\in V\}$ denote the collection of open neighborhoods of x. A collection $\eta\subset\tau_x$ is called a **neighborhood base** at $x\in X$ if for all $V\in\tau_x$ there exists $W\in\eta$ such that $W\subset V$.

The notation τ_x should not be confused with

$$\tau_{\{x\}}:=i_{\{x\}}^{-1}(\tau)=\{\{x\}\cap V : V\in\tau\}=\{\emptyset,\{x\}\}.$$

When (X,d) is a metric space, a typical example of a neighborhood base for x is $\eta=\{B_x(\epsilon):\epsilon\in\mathbb{D}\}$ where $\mathbb{D}$ is any dense subset of $(0,1]$.

Definition 3.19. Let (X,τ) be a topological space and A be a subset of X.

1. The **closure** of A is the smallest closed set $\bar{A}$ containing A, i.e.
$$\bar{A}:=\cap\{F : A\subset F\sqsubset X\}.$$
(Because of Exercise 3.4 this is consistent with Definition 3.8 for the closure of a set in a metric space.)
2. The **interior** of A is the largest open set A^o contained in A, i.e.
$$A^o=\cup\{V\in\tau : V\subset A\}.$$
3. The **accumulation points** of A is the set
$$\operatorname{acc}(A)=\{x\in X : V\cap A\setminus\{x\}\neq\emptyset \text{ for all } V\in\tau_x\}.$$
4. The **boundary** of A is the set $\partial A:=\bar{A}\setminus A^o$.
5. A is a **neighborhood** of a point $x\in X$ if $x\in A^o$. This is equivalent to requiring there to be an open neighborhood of V of $x\in X$ such that $V\subset A$.

Remark 3.20. The relationships between the interior and the closure of a set are:

$$(A^o)^c=\bigcap\{V^c : V\in\tau \text{ and } V\subset A\}=\bigcap\{C : C \text{ is closed } C\supset A^c\}=\overline{A^c}$$

and similarly, $(\bar{A})^c=(A^c)^o$. Hence the boundary of A may be written as

$$\partial A\equiv\bar{A}\setminus A^o=\bar{A}\cap(A^o)^c=\bar{A}\cap\overline{A^c}, \tag{3.3}$$

which is to say ∂A consists of the points in both the closure of A and A^c.

Proposition 3.21. *Let $A\subset X$ and $x\in X$.*

1. *If $V\subset_o X$ and $A\cap V=\emptyset$ then $\bar{A}\cap V=\emptyset$.*
2. *$x\in\bar{A}$ iff $V\cap A\neq\emptyset$ for all $V\in\tau_x$.*
3. *$x\in\partial A$ iff $V\cap A\neq\emptyset$ and $V\cap A^c\neq\emptyset$ for all $V\in\tau_x$.*
4. *$\bar{A}=A\cup\operatorname{acc}(A)$.*

Proof. 1. Since $A\cap V=\emptyset$, $A\subset V^c$ and since V^c is closed, $\bar{A}\subset V^c$. That is to say $\bar{A}\cap V=\emptyset$.

2. By Remark 3.20[3], $\bar{A}=((A^c)^o)^c$ so $x\in\bar{A}$ iff $x\notin(A^c)^o$ which happens iff $V\nsubseteq A^c$ for all $V\in\tau_x$, i.e. iff $V\cap A\neq\emptyset$ for all $V\in\tau_x$.

3. This assertion easily follows from the Item 2. and Eq. (3.3).

[3] Here is another direct proof of item 2. which goes by showing $x\notin\bar{A}$ iff there exists $V\in\tau_x$ such that $V\cap A=\emptyset$. If $x\notin\bar{A}$ then $V=\overline{A^c}\in\tau_x$ and $V\cap A\subset V\cap\bar{A}=\emptyset$. Conversely if there exists $V\in\tau_x$ such that $V\cap A=\emptyset$ then by Item 1. $\bar{A}\cap V=\emptyset$.

4. Item 4. is an easy consequence of the definition of $\text{acc}(A)$ and item 2. ■

Lemma 3.22. *Let $A \subset Y \subset X$, $\bar{A}^Y$ denote the closure of A in Y with its relative topology and $\bar{A} = \bar{A}^X$ be the closure of A in X, then $\bar{A}^Y = \bar{A}^X \cap Y$.*

Proof. Using the comments after Definition 3.17,

$$\begin{aligned}\bar{A}^Y &= \cap\{B \sqsubset Y : A \subset B\} = \cap\{C \cap Y : A \subset C \sqsubset X\} \\ &= Y \cap (\cap\{C : A \subset C \sqsubset X\}) = Y \cap \bar{A}^X.\end{aligned}$$

Alternative proof. Let $x \in Y$ then $x \in \bar{A}^Y$ iff for all $V \in \tau_x^Y$, $V \cap A \neq \emptyset$. This happens iff for all $U \in \tau_x^X$, $U \cap Y \cap A = U \cap A \neq \emptyset$ which happens iff $x \in \bar{A}^X$. That is to say $\bar{A}^Y = \bar{A}^X \cap Y$. ■

Definition 3.23. Let (X, τ) be a topological space and $A \subset X$. We say a subset $\mathcal{U} \subset \tau$ is an **open cover** of A if $A \subset \cup\mathcal{U}$. The set A is said to be **compact** if every open cover of A has finite a sub-cover, i.e. if $\mathcal{U}$ is an open cover of A there exists $\mathcal{U}_0 \subset\subset \mathcal{U}$ such that $\mathcal{U}_0$ is a cover of A. (We will write $A \sqsubset\sqsubset X$ to denote that $A \subset X$ and A is compact.) A subset $A \subset X$ is **precompact** if $\bar{A}$ is compact.

Proposition 3.24. *Suppose that $K \subset X$ is a compact set and $F \subset K$ is a closed subset. Then F is compact. If $\{K_i\}_{i=1}^n$ is a finite collections of compact subsets of X then $K = \cup_{i=1}^n K_i$ is also a compact subset of X.*

Proof. Let $\mathcal{U} \subset \tau$ is an open cover of F, then $\mathcal{U}\cup\{F^c\}$ is an open cover of K. The cover $\mathcal{U}\cup\{F^c\}$ of K has a finite subcover which we denote by $\mathcal{U}_0\cup\{F^c\}$ where $\mathcal{U}_0 \subset\subset \mathcal{U}$. Since $F \cap F^c = \emptyset$, it follows that $\mathcal{U}_0$ is the desired subcover of F.

For the second assertion suppose $\mathcal{U} \subset \tau$ is an open cover of K. Then $\mathcal{U}$ covers each compact set K_i and therefore there exists a finite subset $\mathcal{U}_i \subset\subset \mathcal{U}$ for each i such that $K_i \subset \cup\mathcal{U}_i$. Then $\mathcal{U}_0 := \cup_{i=1}^n \mathcal{U}_i$ is a finite cover of K. ■

Definition 3.25. We say a collection $\mathcal{F}$ of closed subsets of a topological space (X, τ) has the **finite intersection property if** $\cap\mathcal{F}_0 \neq \emptyset$ for all $\mathcal{F}_0 \subset\subset \mathcal{F}$.

The notion of compactness may be expressed in terms of closed sets as follows.

Proposition 3.26. *A topological space X is compact iff every family of closed sets $\mathcal{F} \subset \mathcal{P}(X)$ with the **finite intersection property** satisfies $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. ($\Rightarrow$) Suppose that X is compact and $\mathcal{F} \subset \mathcal{P}(X)$ is a collection of closed sets such that $\bigcap \mathcal{F} = \emptyset$. Let

$$\mathcal{U} = \mathcal{F}^c := \{C^c : C \in \mathcal{F}\} \subset \tau,$$

then $\mathcal{U}$ is a cover of X and hence has a finite subcover, $\mathcal{U}_0$. Let $\mathcal{F}_0 = \mathcal{U}_0^c \subset\subset \mathcal{F}$, then $\cap\mathcal{F}_0 = \emptyset$ so that $\mathcal{F}$ does not have the finite intersection property.

($\Leftarrow$) If X is not compact, there exists an open cover $\mathcal{U}$ of X with no finite subcover. Let $\mathcal{F} = \mathcal{U}^c$, then $\mathcal{F}$ is a collection of closed sets with the finite intersection property while $\bigcap \mathcal{F} = \emptyset$. ■

Exercise 3.7. Let (X, τ) be a topological space. Show that $A \subset X$ is compact iff (A, τ_A) is a compact topological space.

Definition 3.27. Let (X, τ) be a topological space. A sequence $\{x_n\}_{n=1}^\infty \subset X$ **converges** to a point $x \in X$ if for all $V \in \tau_x$, $x_n \in V$ almost always (abbreviated a.a.), i.e. $\#(\{n : x_n \notin V\}) < \infty$. We will write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$ when x_n converges to x.

Example 3.28. Let $Y = \{1, 2, 3\}$ and $\tau = \{Y, \emptyset, \{1, 2\}, \{2, 3\}, \{2\}\}$ and $y_n = 2$ for all n. Then $y_n \to y$ for every $y \in Y$. So limits need not be unique!

Definition 3.29. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(\tau_Y) \subset \tau_X$. We will also say that f is τ_X/τ_Y – continuous or (τ_X, τ_Y) – continuous. We also say that f is continuous at a point $x \in X$ if for every open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $U \subset f^{-1}(V)$. See Figure 6.

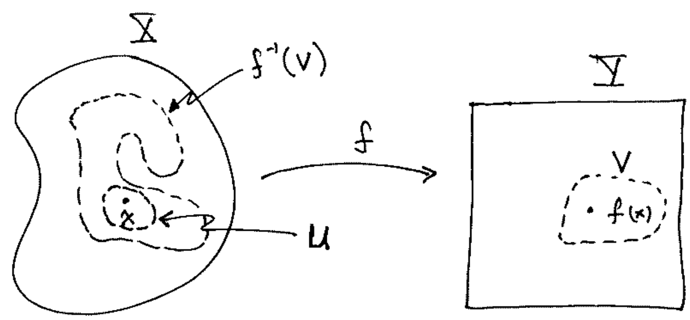

FIGURE 6. Checking that a function is continuous at $x \in X$.

Definition 3.30. A map $f : X \to Y$ between topological spaces is called a **homeomorphism** provided that f is bijective, f is continuous and $f^{-1} : Y \to X$ is continuous. If there exists $f : X \to Y$ which is a homeomorphism, we say that X and Y are homeomorphic. (As topological spaces X and Y are essentially the same.)

Exercise 3.8. Show $f : X \to Y$ is continuous iff f is continuous at all points $x \in X$.

Exercise 3.9. Show $f : X \to Y$ is continuous iff $f^{-1}(C)$ is closed in X for all closed subsets C of Y.

Exercise 3.10. Suppose $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is a compact subset of Y.

Exercise 3.11 (Dini's Theorem)**.** Let X be a compact topological space and $f_n : X \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \downarrow 0$ as $n \to \infty$ for each $x \in X$. Show that in fact $f_n \downarrow 0$ uniformly in x, i.e. $\sup_{x \in X} f_n(x) \downarrow 0$ as $n \to \infty$. **Hint:** Given $\epsilon > 0$, consider the open sets $V_n := \{x \in X : f_n(x) < \epsilon\}$.

Definition 3.31 (First Countable)**.** A topological space, (X, τ), is **first countable** iff every point $x \in X$ has a countable neighborhood base. (All metric space are first countable.)

When τ is first countable, we may formulate many topological notions in terms of sequences.

Proposition 3.32. *If $f : X \to Y$ is continuous at $x \in X$ and $\lim_{n\to\infty} x_n = x \in X$, then $\lim_{n\to\infty} f(x_n) = f(x) \in Y$. Moreover, if there exists a countable neighborhood base η of $x \in X$, then f is continuous at x iff $\lim\limits_{n\to\infty} f(x_n) = f(x)$ for all sequences $\{x_n\}_{n=1}^\infty \subset X$ such that $x_n \to x$ as $n \to \infty$.*

Proof. If $f : X \to Y$ is continuous and $W \in \tau_Y$ is a neighborhood of $f(x) \in Y$, then there exists a neighborhood V of $x \in X$ such that $f(V) \subset W$. Since $x_n \to x$, $x_n \in V$ a.a. and therefore $f(x_n) \in f(V) \subset W$ a.a., i.e. $f(x_n) \to f(x)$ as $n \to \infty$.

Conversely suppose that $\eta \equiv \{W_n\}_{n=1}^{\infty}$ is a countable neighborhood base at x and $\lim_{n\to\infty} f(x_n) = f(x)$ for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$. By replacing W_n by $W_1 \cap \cdots \cap W_n$ if necessary, we may assume that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. If f were **not** continuous at x then there exists $V \in \tau_{f(x)}$ such that $x \notin f^{-1}(V)^0$. Therefore, W_n is not a subset of $f^{-1}(V)$ for all n. Hence for each n, we may choose $x_n \in W_n \setminus f^{-1}(V)$. This sequence then has the property that $x_n \to x$ as $n \to \infty$ while $f(x_n) \notin V$ for all n and hence $\lim_{n\to\infty} f(x_n) \neq f(x)$. ■

Lemma 3.33. *Suppose there exists $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \to x$, then $x \in \bar{A}$. Conversely if (X, τ) is a first countable space (like a metric space) then if $x \in \bar{A}$ there exists $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \to x$.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty} \subset A$ and $x_n \to x \in X$. Since $\bar{A}^c$ is an open set, if $x \in \bar{A}^c$ then $x_n \in \bar{A}^c \subset A^c$ a.a. contradicting the assumption that $\{x_n\}_{n=1}^{\infty} \subset A$. Hence $x \in \bar{A}$.

For the converse we now assume that (X, τ) is first countable and that $\{V_n\}_{n=1}^{\infty}$ is a countable neighborhood base at x such that $V_1 \supset V_2 \supset V_3 \supset \dots$. By Proposition 3.21, $x \in \bar{A}$ iff $V \cap A \neq \emptyset$ for all $V \in \tau_x$. Hence $x \in \bar{A}$ implies there exists $x_n \in V_n \cap A$ for all n. It is now easily seen that $x_n \to x$ as $n \to \infty$. ■

Definition 3.34 (Support)**.** Let $f : X \to Y$ be a function from a topological space (X, τ_X) to a vector space Y. Then we define the support of f by

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}},$$

a closed subset of X.

Example 3.35. For example, let $f(x) = \sin(x)1_{[0,4\pi]}(x) \in \mathbb{R}$, then

$$\{f \neq 0\} = (0, 4\pi) \setminus \{\pi, 2\pi, 3\pi\}$$

and therefore $\operatorname{supp}(f) = [0, 4\pi]$.

Notation 3.36. If X and Y are two topological spaces, let $C(X, Y)$ denote the continuous functions from X to Y. If Y is a Banach space, let

$$BC(X, Y) := \{f \in C(X, Y) : \sup_{x\in X} \|f(x)\|_Y < \infty\}$$

and

$$C_c(X, Y) := \{f \in C(X, Y) : \operatorname{supp}(f) \text{ is compact}\}.$$

If $Y = \mathbb{R}$ or $\mathbb{C}$ we will simply write $C(X)$, $BC(X)$ and $C_c(X)$ for $C(X, Y)$, $BC(X, Y)$ and $C_c(X, Y)$ respectively.

The next result is included for completeness but will not be used in the sequel so may be omitted.

Lemma 3.37. *Suppose that $f : X \to Y$ is a map between topological spaces. Then the following are equivalent:*

1. *f is continuous.*
2. *$f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$*

3. $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ *for all* $B \sqsubset X$.

Proof. If f is continuous, then $f^{-1}\left(\overline{f(A)}\right)$ is closed and since $A \subset f^{-1}(f(A)) \subset f^{-1}\left(\overline{f(A)}\right)$ it follows that $\bar{A} \subset f^{-1}\left(\overline{f(A)}\right)$. From this equation we learn that $f(\bar{A}) \subset \overline{f(A)}$ so that (1) implies (2) Now assume (2), then for $B \subset Y$ (taking $A = f^{-1}(\bar{B})$) we have

$$f(\overline{f^{-1}(B)}) \subset f(\overline{f^{-1}(\bar{B})}) \subset \overline{f(f^{-1}(\bar{B}))} \subset \bar{B}$$

and therefore

$$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}). \tag{3.4}$$

This shows that (2) implies (3) Finally if Eq. (3.4) holds for all B, then when B is closed this shows that

$$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) = f^{-1}(B) \subset \overline{f^{-1}(B)}$$

which shows that

$$f^{-1}(B) = \overline{f^{-1}(B)}.$$

Therefore $f^{-1}(B)$ is closed whenever B is closed which implies that f is continuous. ■

3.4. **Completeness.**

Definition 3.38 (Cauchy sequences)**.** A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is **Cauchy** provided that

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Exercise 3.12. Show that convergent sequences are always Cauchy sequences. The converse is not always true. For example, let $X = \mathbb{Q}$ be the set of rational numbers and $d(x, y) = |x - y|$. Choose a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ which converges to $\sqrt{2} \in \mathbb{R}$, then $\{x_n\}_{n=1}^{\infty}$ is $(\mathbb{Q}, d)$ – Cauchy but not $(\mathbb{Q}, d)$ – convergent. The sequence does converge in $\mathbb{R}$ however.

Definition 3.39. A metric space (X, d) is **complete** if all Cauchy sequences are convergent sequences.

Exercise 3.13. Let (X, d) be a complete metric space. Let $A \subset X$ be a subset of X viewed as a metric space using $d|_{A\times A}$. Show that $(A, d|_{A\times A})$ is complete iff A is a closed subset of X.

Definition 3.40. If $(X, \|\cdot\|)$ is a normed vector space, then we say $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. The normed vector space is a **Banach space** if it is complete, i.e. if every $\{x_n\}_{n=1}^{\infty} \subset X$ which is Cauchy is convergent where $\{x_n\}_{n=1}^{\infty} \subset X$ is convergent iff there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. As usual we will abbreviate this last statement by writing $\lim_{n\to\infty} x_n = x$.

Lemma 3.41. *Suppose that X is a set then the bounded functions $\ell^\infty(X)$ on X is a Banach space with the norm*

$$\|f\| = \|f\|_\infty = \sup_{x\in X} |f(x)|.$$

Moreover if X is a topological space the set $BC(X) \subset \ell^\infty(X) = B(X)$ is closed subspace of $\ell^\infty(X)$ and hence is also a Banach space.

Proof. Let $\{f_n\}_{n=1}^\infty \subset \ell^\infty(X)$ be a Cauchy sequence. Since for any $x \in X$, we have

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \tag{3.5}$$

which shows that $\{f_n(x)\}_{n=1}^\infty \subset \mathbb{F}$ is a Cauchy sequence of numbers. Because $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is complete, $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. Passing to the limit $n \to \infty$ in Eq. (3.5) implies

$$|f(x) - f_m(x)| \le \lim\sup_{n\to\infty} \|f_n - f_m\|_\infty$$

and taking the supremum over $x \in X$ of this inequality implies

$$\|f - f_m\|_\infty \le \lim\sup_{n\to\infty} \|f_n - f_m\|_\infty \to 0 \text{ as } m \to \infty$$

showing $f_m \to f$ in $\ell^\infty(X)$.

For the second assertion, suppose that $\{f_n\}_{n=1}^\infty \subset BC(X) \subset \ell^\infty(X)$ and $f_n \to f \in \ell^\infty(X)$. We must show that $f \in BC(X)$, i.e. that f is continuous. To this end let $x, y \in X$, then

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\le 2\|f - f_n\|_\infty + |f_n(x) - f_n(y)|. \end{aligned}$$

Thus if $\epsilon > 0$, we may choose n large so that $2\|f - f_n\|_\infty < \epsilon/2$ and then for this n there exists an open neighborhood V_x of $x \in X$ such that $|f_n(x) - f_n(y)| < \epsilon/2$ for $y \in V_x$. Thus $|f(x) - f(y)| < \epsilon$ for $y \in V_x$ showing the limiting function f is continuous. ■

Remark 3.42. Let X be a set, Y be a Banach space and $\ell^\infty(X, Y)$ denote the bounded functions $f : X \to Y$ equipped with the norm $\|f\| = \|f\|_\infty = \sup_{x\in X} \|f(x)\|_Y$. If X is a topological space, let $BC(X, Y)$ denote those $f \in \ell^\infty(X, Y)$ which are continuous. The same proof used in Lemma 3.41 shows that $\ell^\infty(X, Y)$ is a Banach space and that $BC(X, Y)$ is a closed subspace of $\ell^\infty(X, Y)$.

Theorem 3.43 (Completeness of $\ell^p(\mu)$)**.** *Let X be a set and $\mu : X \to (0, \infty]$ be a given function. Then for any $p \in [1, \infty]$, $(\ell^p(\mu), \|\cdot\|_p)$ is a Banach space.*

Proof. We have already proved this for $p = \infty$ in Lemma 3.41 so we now assume that $p \in [1, \infty)$. Let $\{f_n\}_{n=1}^\infty \subset \ell^p(\mu)$ be a Cauchy sequence. Since for any $x \in X$,

$$|f_n(x) - f_m(x)| \le \frac{1}{\mu(x)} \|f_n - f_m\|_p \to 0 \text{ as } m, n \to \infty$$

it follows that $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence of numbers and $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. By Fatou's Lemma,

$$\begin{aligned} \|f_n - f\|_p^p &= \sum_X \mu \cdot \lim_{m\to\infty} \inf |f_n - f_m|^p \le \lim_{m\to\infty} \inf \sum_X \mu \cdot |f_n - f_m|^p \\ &= \lim_{m\to\infty} \inf \|f_n - f_m\|_p^p \to 0 \text{ as } n \to \infty. \end{aligned}$$

This then shows that $f = (f - f_n) + f_n \in \ell^p(\mu)$ (being the sum of two ℓ^p – functions) and that $f_n \xrightarrow{\ell^p} f$. ■

Example 3.44. Here are a couple of examples of complete metric spaces.

1. $X = \mathbb{R}$ and $d(x,y) = |x-y|$.
2. $X = \mathbb{R}^n$ and $d(x,y) = \|x-y\|_2 = \sum_{i=1}^n (x_i - y_i)^2$.
3. $X = \ell^p(\mu)$ for $p \in [1,\infty]$ and any weight function μ.
4. $X = C([0,1],\mathbb{R})$ – the space of continuous functions from $[0,1]$ to $\mathbb{R}$ and $d(f,g) := \max_{t\in[0,1]} |f(t) - g(t)|$. This is a special case of Lemma 3.41.
5. Here is a typical example of a non-complete metric space. Let $X = C([0,1],\mathbb{R})$ and
$$d(f,g) := \int_0^1 |f(t) - g(t)|\, dt.$$

3.5. **Compactness in Metric Spaces.** Let (X,ρ) be a metric space and let $B_x'(\epsilon) = B_x(\epsilon) \setminus \{x\}$.

Definition 3.45. A point $x \in X$ is an accumulation point of a subset $E \subset X$ if $\emptyset \neq E \cap V \setminus \{x\}$ for all $V \subset_o X$ containing x.

Let us start with the following elementary lemma which is left as an exercise to the reader.

Lemma 3.46. *Let $E \subset X$ be a subset of a metric space (X,ρ). Then the following are equivalent:*

1. *$x \in X$ is an accumulation point of E.*
2. *$B_x'(\epsilon) \cap E \neq \emptyset$ for all $\epsilon > 0$.*
3. *$B_x(\epsilon) \cap E$ is an infinite set for all $\epsilon > 0$.*
4. *There exists $\{x_n\}_{n=1}^\infty \subset E \setminus \{x\}$ with $\lim_{n\to\infty} x_n = x$.*

Definition 3.47. A metric space (X,ρ) is said to be $\epsilon -$ **bounded** ($\epsilon > 0$) provided there exists a finite cover of X by balls of radius ϵ. The metric space is **totally bounded** if it is $\epsilon -$ bounded for all $\epsilon > 0$.

Theorem 3.48. *Let X be a metric space. The following are equivalent.*

(a) *X is compact.*
(b) *Every infinite subset of X has an accumulation point.*
(c) *X is totally bounded and complete.*

Proof. The proof will consist of showing that $a \Rightarrow b \Rightarrow c \Rightarrow a$.

($a \Rightarrow b$) We will show that **not** $b \Rightarrow$ **not** a. Suppose there exists $E \subset X$, such that $\#(E) = \infty$ and E has no accumulation points. Then for all $x \in X$ there exists $\delta_x > 0$ such that $V_x := B_x(\delta_x)$ satisfies $(V_x \setminus \{x\}) \cap E = \emptyset$. Clearly $\mathcal{V} = \{V_x\}_{x\in X}$ is a cover of X, yet $\mathcal{V}$ has no finite sub cover. Indeed, for each $x \in X$, $V_x \cap E$ consists of at most one point, therefore if $\Lambda \subset\subset X$, $\cup_{x\in\Lambda} V_x$ can only contain a finite number of points from E, in particular $X \neq \cup_{x\in\Lambda} V_x$. (See Figure 7.)

($b \Rightarrow c$) To show X is complete, let $\{x_n\}_{n=1}^\infty \subset X$ be a sequence and $E := \{x_n : n \in \mathbb{N}\}$. If $\#(E) < \infty$, then $\{x_n\}_{n=1}^\infty$ has a subsequence $\{x_{n_k}\}$ which is constant and hence convergent. If E is an infinite set it has an accumulation point by assumption and hence Lemma 3.46 implies that $\{x_n\}$ has a convergence subsequence.

We now show that X is totally bounded. Let $\epsilon > 0$ be given and choose $x_1 \in X$. If possible choose $x_2 \in X$ such that $d(x_2, x_1) \geq \epsilon$, then if possible choose $x_3 \in X$ such that $d(x_3, \{x_1, x_2\}) \geq \epsilon$ and continue inductively choosing points $\{x_j\}_{j=1}^n \subset X$ such that $d(x_n, \{x_1, \ldots, x_{n-1}\}) \geq \epsilon$. This process must terminate, for otherwise we could choose $E = \{x_j\}_{j=1}^\infty$ and infinite number of distinct points such that

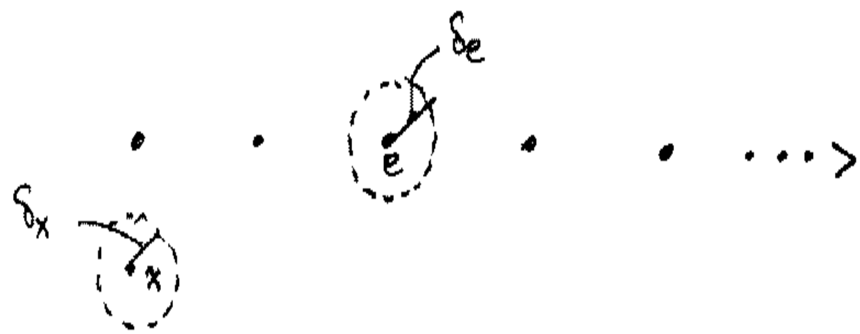

FIGURE 7. The construction of an open cover with no finite sub-cover.

$d(x_j, \{x_1, \ldots, x_{j-1}\}) \geq \epsilon$ for all $j = 2, 3, 4, \ldots$. Since for all $x \in X$ the $B_x(\epsilon/3) \cap E$ can contain at most one point, no point $x \in X$ is an accumulation point of E. (See Figure 8.)

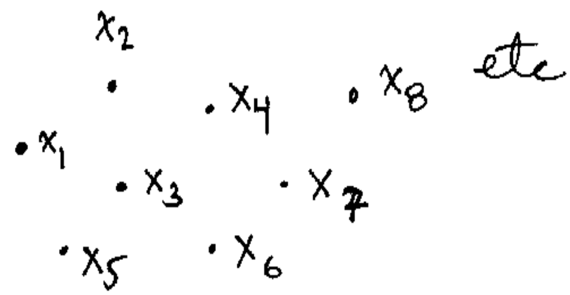

FIGURE 8. Constructing a set with out an accumulation point.

$(c \Rightarrow a)$ For sake of contradiction, assume there exists a cover an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ of X with no finite subcover. Since X is totally bounded for each $n \in \mathbb{N}$ there exists $\Lambda_n \subset\subset X$ such that

$$X = \bigcup_{x \in \Lambda_n} B_x(1/n) \subset \bigcup_{x \in \Lambda_n} C_x(1/n).$$

Choose $x_1 \in \Lambda_1$ such that no finite subset of $\mathcal{V}$ covers $K_1 := C_{x_1}(1)$. Since $K_1 = \cup_{x \in \Lambda_2} K_1 \cap C_x(1/2)$, there exists $x_2 \in \Lambda_2$ such that $K_2 := K_1 \cap C_{x_2}(1/2)$ can not be covered by a finite subset of $\mathcal{V}$. Continuing this way inductively, we construct sets $K_n = K_{n-1} \cap C_{x_n}(1/n)$ with $x_n \in \Lambda_n$ such no K_n can be covered by a finite subset of $\mathcal{V}$. Now choose $y_n \in K_n$ for each n. Since $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of closed sets such that $\operatorname{diam}(K_n) \leq 2/n$, it follows that $\{y_n\}$ is a Cauchy and hence convergent with

$$y = \lim_{n \to \infty} y_n \in \cap_{m=1}^\infty K_m.$$

Since $\mathcal{V}$ is a cover of X, there exists $V \in \mathcal{V}$ such that $x \in V$. Since $K_n \downarrow \{y\}$ and $\operatorname{diam}(K_n) \to 0$, it now follows that $K_n \subset V$ for some n large. But this violates the assertion that K_n can not be covered by a finite subset of $\mathcal{V}$.(See Figure 9.) ■

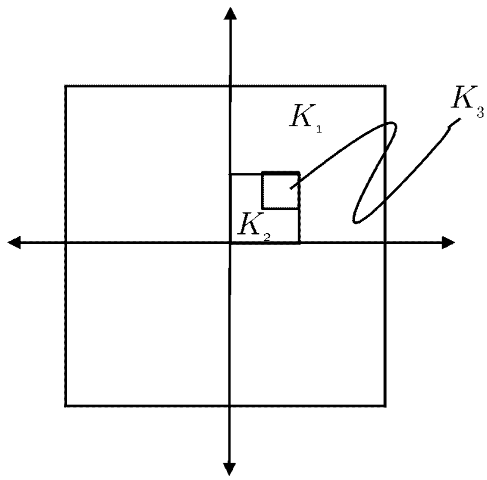

FIGURE 9. Nested Sequence of cubes.

Remark 3.49. Let X be a topological space and Y be a Banach space. By combining Exercise 3.10 and Theorem 3.48 it follows that $C_c(X,Y) \subset BC(X,Y)$.

Corollary 3.50. *Let X be a metric space then X is compact iff* ***all*** *sequences $\{x_n\} \subset X$ have convergent subsequences.*

Proof. Suppose X is compact and $\{x_n\} \subset X$.

1. If $\#(\{x_n : n = 1, 2, \dots\}) < \infty$ then choose $x \in X$ such that $x_n = x$ i.o. and let $\{n_k\} \subset \{n\}$ such that $x_{n_k} = x$ for all k. Then $x_{n_k} \to x$
2. If $\#(\{x_n : n = 1, 2, \dots\}) = \infty$. We know $E = \{x_n\}$ has an accumulation point $\{x\}$, hence there exists $x_{n_k} \to x$.

Conversely if E is an infinite set let $\{x_n\}_{n=1}^{\infty} \subset E$ be a sequence of distinct elements of E. We may, by passing to a subsequence, assume $x_n \to x \in X$ as $n \to \infty$. Now $x \in X$ is an accumulation point of E by Theorem 3.48 and hence X is compact. ■

Corollary 3.51. *Compact subsets of $\mathbb{R}^n$ are the closed and bounded sets.*

Proof. If K is closed and bounded then K is complete (being the closed subset of a complete space) and K is contained in $[-M, M]^n$ for some positive integer M. For $\delta > 0$, let

$$\Lambda_\delta = \delta\mathbb{Z}^n \cap [-M, M]^n := \{\delta x : x \in \mathbb{Z}^n \text{ and } \delta|x_i| \le M \text{ for } i = 1, 2, \dots, n\}.$$

We will show, by choosing $\delta > 0$ sufficiently small, that

$$K \subset [-M, M]^n \subset \cup_{x \in \Lambda_\delta} B(x, \epsilon) \tag{3.6}$$

which shows that K is totally bounded. Hence by Theorem 3.48, K is compact.

Suppose that $y \in [-M, M]^n$, then there exists $x \in \Lambda_\delta$ such that $|y_i - x_i| \le \delta$ for $i = 1, 2, \ldots, n$. Hence

$$d^2(x, y) = \sum_{i=1}^n (y_i - x_i)^2 \le n\delta^2$$

which shows that $d(x, y) \le \sqrt{n}\delta$. Hence if choose $\delta < \epsilon/\sqrt{n}$ we have shows that $d(x, y) < \epsilon$, i.e. Eq. (3.6) holds. ■

Example 3.52. Let $X = \ell^p(\mathbb{N})$ with $p \in [1, \infty)$ and $\rho \in X$ such that $\rho(k) \ge 0$ for all $k \in \mathbb{N}$. The set

$$K := \{x \in X : |x(k)| \le \rho(k) \text{ for all } k \in \mathbb{N}\}$$

is compact. To prove this, let $\{x_n\}_{n=1}^\infty \subset K$ be a sequence. By compactness of closed bounded sets in $\mathbb{C}$, for each $k \in \mathbb{N}$ there is a subsequence of $\{x_n(k)\}_{n=1}^\infty \subset \mathbb{C}$ which is convergent. By Cantor's diagonalization trick, we may choose a subsequence $\{y_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $y(k) := \lim_{n\to\infty} y_n(k)$ exists for all $k \in \mathbb{N}$.[4] Since $|y_n(k)| \le \rho(k)$ for all n it follows that $|y(k)| \le \rho(k)$, i.e. $y \in K$. Finally

$$\lim_{n\to\infty} \|y - y_n\|_p^p = \lim_{n\to\infty} \sum_{k=1}^\infty |y(k) - y_n(k)|^p = \sum_{k=1}^\infty \lim_{n\to\infty} |y(k) - y_n(k)|^p = 0$$

where we have used the Dominated convergence theorem. (Note $|y(k) - y_n(k)|^p \le 2^p \rho^p(k)$ and ρ^p is summable.) Therefore $y_n \to y$ and we are done.

Alternatively, we can prove K is compact by showing that K is closed and totally bounded. It is simple to show K is closed, for if $\{x_n\}_{n=1}^\infty \subset K$ is a convergent sequence in X, $x := \lim_{n\to\infty} x_n$, then $|x(k)| \le \lim_{n\to\infty} |x_n(k)| \le \rho(k)$ for all $k \in \mathbb{N}$. This shows that $x \in K$ and hence K is closed. To see that K is totally bounded, let $\epsilon > 0$ and choose N such that $\left(\sum_{k=N+1}^\infty |\rho(k)|^p\right)^{1/p} < \epsilon$. Since $\prod_{k=1}^N C_{\rho(k)}(0) \subset \mathbb{C}^N$ is closed and bounded, it is compact. Therefore there exists a finite subset $\Lambda \subset \prod_{k=1}^N C_{\rho(k)}(0)$ such that

$$\prod_{k=1}^N C_{\rho(k)}(0) \subset \cup_{z\in\Lambda} B_z^N(\epsilon)$$

where $B_z^N(\epsilon)$ is the open ball centered at $z \in \mathbb{C}^N$ relative to the $\ell^p(\{1, 2, 3, \ldots, N\})$ – norm. For each $z \in \Lambda$, let $\tilde{z} \in X$ be defined by $\tilde{z}(k) = z(k)$ if $k \le N$ and $\tilde{z}(k) = 0$ for $k \ge N + 1$. I now claim that

$$K \subset \cup_{z\in\Lambda} B_{\tilde{z}}(2\epsilon) \tag{3.7}$$

which, when verified, shows K is totally bounced. To verify Eq. (3.7), let $x \in K$ and write $x = u + v$ where $u(k) = x(k)$ for $k \le N$ and $u(k) = 0$ for $k < N$. Then

[4] The argument is as follows. Let $\{n_j^1\}_{j=1}^\infty$ be a subsequence of $\mathbb{N} = \{n\}_{n=1}^\infty$ such that $\lim_{j\to\infty} x_{n_j^1}(1)$ exists. Now choose a subsequence $\{n_j^2\}_{j=1}^\infty$ of $\{n_j^1\}_{j=1}^\infty$ such that $\lim_{j\to\infty} x_{n_j^2}(2)$ exists and similalry $\{n_j^3\}_{j=1}^\infty$ of $\{n_j^2\}_{j=1}^\infty$ such that $\lim_{j\to\infty} x_{n_j^3}(3)$ exists. Continue on this way inductively to get

$$\{n\}_{n=1}^\infty \supset \{n_j^1\}_{j=1}^\infty \supset \{n_j^2\}_{j=1}^\infty \supset \{n_j^3\}_{j=1}^\infty \supset \ldots$$

such that $\lim_{j\to\infty} x_{n_j^k}(k)$ exists for all $k \in \mathbb{N}$. Let $m_j := n_j^j$ so that eventually $\{m_j\}_{j=1}^\infty$ is a subsequnce of $\{n_j^k\}_{j=1}^\infty$ for all k. Therefore, we may take $y_j := x_{m_j}$.

by construction $u \in B_{\tilde{z}}(\epsilon)$ for some $\tilde{z} \in \Lambda$ and

$$\|v\|_p \leq \left(\sum_{k=N+1}^{\infty} |\rho(k)|^p\right)^{1/p} < \epsilon.$$

So we have

$$\|x - \tilde{z}\|_p = \|u + v - \tilde{z}\|_p \leq \|u - \tilde{z}\|_p + \|v\|_p < 2\epsilon.$$

Exercise 3.14 (Extreme value theorem)**.** Let (X, τ) be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Show $-\infty < \inf f \leq \sup f < \infty$ and there exists $a, b \in X$ such that $f(a) = \inf f$ and $f(b) = \sup f$. [5] **Hint:** use Exercise 3.10 and Corollary 3.51.

Exercise 3.15 (Uniform Continuity)**.** Let (X, d) be a compact metric space, $f : X \to \mathbb{R}$ be a continuous function. Show that f is uniformly continuous, i.e. if $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ if $x, y \in X$ with $d(x, y) < \delta$. **Hint:** I think the easiest proof is by using a sequence argument.

Definition 3.53. Let L be a vector space. We say that two norms, $|\cdot|$ and $\|\cdot\|$, on L are equivalent if there exists constants $\alpha, \beta \in (0, \infty)$ such that

$$\|f\| \leq \alpha |f| \text{ and } |f| \leq \beta \|f\| \text{ for all } f \in L.$$

Lemma 3.54. *Let L be a finite dimensional vector space. Then any two norms $|\cdot|$ and $\|\cdot\|$ on L are equivalent. (This is typically not true for norms on infinite dimensional spaces.)*

Proof. Let $\{f_i\}_{i=1}^n$ be a basis for L and define a new norm on L by

$$\left\|\sum_{i=1}^n a_i f_i\right\|_1 \equiv \sum_{i=1}^n |a_i| \text{ for } a_i \in \mathbb{F}.$$

By the triangle inequality of the norm $|\cdot|$, we find

$$\left|\sum_{i=1}^n a_i f_i\right| \leq \sum_{i=1}^n |a_i| \, |f_i| \leq M \sum_{i=1}^n |a_i| = M \left\|\sum_{i=1}^n a_i f_i\right\|_1$$

where $M = \max_i |f_i|$. Thus we have

$$|f| \leq M \|f\|_1$$

for all $f \in L$. This inequality shows that $|\cdot|$ is continuous relative to $\|\cdot\|_1$. Now let $S := \{f \in L : \|f\|_1 = 1\}$, a compact subset of L relative to $\|\cdot\|_1$. Therefore by Exercise 3.14 there exists $f_0 \in S$ such that

$$m = \inf\{|f| : f \in S\} = |f_0| > 0.$$

Hence given $0 \neq f \in L$, then $\frac{f}{\|f\|_1} \in S$ so that

$$m \leq \left|\frac{f}{\|f\|_1}\right| = |f| \frac{1}{\|f\|_1}$$

[5] Here is a proof if X is a metric space. Let $\{x_n\}_{n=1}^\infty \subset X$ be a sequence such that $f(x_n) \uparrow \sup f$. By compactness of X we may assume, by passing to a subsequence if necessary that $x_n \to b \in X$ as $n \to \infty$. By continuity of f, $f(b) = \sup f$.

or equivalently

$$\|f\|_1 \leq \frac{1}{m}|f|.$$

This shows that $|\cdot|$ and $\|\cdot\|_1$ are equivalent norms. Similarly one shows that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent and hence so are $|\cdot|$ and $\|\cdot\|$. ■

Definition 3.55. A subset D of a topological space X is **dense** if $\bar{D} = X$. A topological space is said to be **separable** if it contains a countable dense subset, D.

Example 3.56. The following are examples of countable dense sets.

1. The rational number $\mathbb{Q}$ are dense in $\mathbb{R}$ equipped with the usual topology.
2. More generally, $\mathbb{Q}^d$ is a countable dense subset of $\mathbb{R}^d$ for any $d \in \mathbb{N}$.
3. Even more generally, for any function $\mu : \mathbb{N} \to (0, \infty)$, $\ell^p(\mu)$ is separable for all $1 \leq p < \infty$. For example, let $\Gamma \subset \mathbb{F}$ be a countable dense set, then
$$D := \{x \in \ell^p(\mu) : x_i \in \Gamma \text{ for all } i \text{ and } \#\{j : x_j \neq 0\} < \infty\}.$$
The set Γ can be taken to be $\mathbb{Q}$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q} + i\mathbb{Q}$ if $\mathbb{F} = \mathbb{C}$.
4. If (X, ρ) is a metric space which is separable then every subset $Y \subset X$ is also separable in the induced topology.

To prove 4. above, let $A = \{x_n\}_{n=1}^\infty \subset X$ be a countable dense subset of X. Let $\rho(x, Y) = \inf\{\rho(x, y) : y \in Y\}$ be the distance from x to Y. Recall that $\rho(\cdot, Y) : X \to [0, \infty)$ is continuous. Let $\epsilon_n = \rho(x_n, Y) \geq 0$ and for each n let $y_n \in B_{x_n}(\frac{1}{n}) \cap Y$ if $\epsilon_n = 0$ otherwise choose $y_n \in B_{x_n}(2\epsilon_n) \cap Y$. Then if $y \in Y$ and $\epsilon > 0$ we may choose $n \in \mathbb{N}$ such that $\rho(y, x_n) \leq \epsilon_n < \epsilon/3$ and $\frac{1}{n} < \epsilon/3$. If $\epsilon_n > 0$, $\rho(y_n, x_n) \leq 2\epsilon_n < 2\epsilon/3$ and if $\epsilon_n = 0$, $\rho(y_n, x_n) < \epsilon/3$ and therefore

$$\rho(y, y_n) \leq \rho(y, x_n) + \rho(x_n, y_n) < \epsilon.$$

This shows that $B \equiv \{y_n\}_{n=1}^\infty$ is a countable dense subset of Y.

Lemma 3.57. *Any compact metric space (X, d) is separable.*

Proof. To each integer n, there exists $\Lambda_n \subset\subset X$ such that $X = \cup_{x\in\Lambda_n} B(x, 1/n)$. Let $D := \cup_{n=1}^\infty \Lambda_n$ – a countable subset of X. Moreover, it is clear by construction that $\bar{D} = X$. ■

3.6. Compactness in Function Spaces.

In this section, let (X, τ) be a topological space.

Definition 3.58. Let $\mathcal{F} \subset C(X)$.

1. $\mathcal{F}$ is equicontinuous at $x \in X$ iff for all $\epsilon > 0$ there exists $U \in \tau_x$ such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and $f \in \mathcal{F}$.
2. $\mathcal{F}$ is equicontinuous if $\mathcal{F}$ is equicontinuous at all points $x \in X$.
3. $\mathcal{F}$ is pointwise bounded if $\sup\{|f(x)| : |f \in \mathcal{F}\} < \infty$ for all $x \in X$.

Theorem 3.59 (Ascoli-Arzela Theorem)**.** *Let (X, τ) be a compact topological space and $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is precompact in $C(X)$ iff $\mathcal{F}$ is equicontinuous and pointwise bounded.*

Proof. ($\Leftarrow$) Since $B(X)$ is a complete metric space, we must show $\mathcal{F}$ is totally bounded. Let $\epsilon > 0$ be given. By equicontinuity there exists $V_x \in \tau_x$ for all $x \in X$ such that $|f(y) - f(x)| < \epsilon/2$ if $y \in V_x$ and $f \in \mathcal{F}$. Since X is compact we may choose $\Lambda \subset\subset X$ such that $X = \cup_{x\in\Lambda} V_x$. We have now decomposed X into "blocks" $\{V_x\}_{x\in\Lambda}$ such that each $f \in \mathcal{F}$ is constant to within ϵ on V_x. Since $\sup\{|f(x)| : x \in \Lambda \text{ and } f \in \mathcal{F}\} < \infty$, it is now evident that

$$M \equiv \sup\{|f(x)| : x \in X \text{ and } f \in \mathcal{F}\} \leq \sup\{|f(x)| : x \in \Lambda \text{ and } f \in \mathcal{F}\} + \epsilon < \infty.$$

Let $\mathbb{D} \equiv \{k\epsilon/2 : k \in \mathbb{Z}\} \cap [-M, M]$. If $f \in \mathcal{F}$ and $\phi \in \mathbb{D}^\Lambda$ (i.e. $\phi : \Lambda \to \mathbb{D}$ is a function) is chosen so that $|\phi(x) - f(x)| \leq \epsilon/2$ for all $x \in \Lambda$, then

$$|f(y) - \phi(x)| \leq |f(y) - f(x)| + |f(x) - \phi(x)| < \epsilon \ \forall\ x \in \Lambda \text{ and } y \in V_x.$$

From this it follows that $\mathcal{F} = \bigcup\left\{\mathcal{F}_\phi : \phi \in \mathbb{D}^\Lambda\right\}$ where, for $\phi \in \mathbb{D}^\Lambda$,

$$\mathcal{F}_\phi \equiv \{f \in \mathcal{F} : |f(y) - \phi(x)| < \epsilon \text{ for } y \in V_x \text{ and } x \in \Lambda\}.$$

Let $\Gamma := \left\{\phi \in \mathbb{D}^\Lambda : \mathcal{F}_\phi \neq \emptyset\right\}$ and for each $\phi \in \Gamma$ choose $f_\phi \in \mathcal{F}_\phi \cap \mathcal{F}$. For $f \in \mathcal{F}_\phi$, $x \in \Lambda$ and $y \in V_x$ we have

$$|f(y) - f_\phi(y)| \leq |f(y) - \phi(x))| + |\phi(x) - f_\phi(y)| < 2\epsilon.$$

So $\|f - f_\phi\| < 2\epsilon$ for all $f \in \mathcal{F}_\phi$ showing that $\mathcal{F}_\phi \subset B_{f_\phi}(2\epsilon)$. Therefore,

$$\mathcal{F} = \cup_{\phi\in\Gamma}\mathcal{F}_\phi \subset \cup_{\phi\in\Gamma} B_{f_\phi}(2\epsilon)$$

and because $\epsilon > 0$ was arbitrary we have shown that $\mathcal{F}$ is totally bounded.

($\Rightarrow$) Since $\|\cdot\| : C(X) \to [0, \infty)$ is a continuous function on $C(X)$ it is bounded on any compact subset $\mathcal{F} \subset C(X)$. This shows that $\sup\{\|f\| : f \in \mathcal{F}\} < \infty$ which clearly implies that $\mathcal{F}$ is pointwise bounded.[6] Suppose $\mathcal{F}$ were **not** equicontinuous at some point $x \in X$ that is to say there exists $\epsilon > 0$ such that for all $V \in \tau_x$, $\sup_{y\in V} \sup_{f\in\mathcal{F}} |f(y) - f(x)| > \epsilon$.[7] Equivalently said, to each $V \in \tau_x$ we may choose

$$f_V \in \mathcal{F} \text{ and } x_V \in V \text{ such that } |f_V(x) - f_V(x_V)| \geq \epsilon. \tag{3.8}$$

Set $\mathcal{C}_V = \overline{\{f_W : W \in \tau_x \text{ and } W \subset V\}}^{\|\cdot\|_\infty} \subset \mathcal{F}$ and notice for any $\mathcal{V} \subset\subset \tau_x$ that

$$\cap_{V\in\mathcal{V}}\mathcal{C}_V \supseteq \mathcal{C}_{\cap\mathcal{V}} \neq \emptyset,$$

[6] One could also prove that $\mathcal{F}$ is pointwise bounded by considering the continuous evaluation maps $e_x : C(X) \to \mathbb{R}$ given by $e_x(f) = f(x)$ for all $x \in X$.

[7] If X is first countable we could finish the proof with the following argument. Let $\{V_n\}_{n=1}^\infty$ be a neighborhood base at x such that $V_1 \supset V_2 \supset V_3 \supset \ldots$. By the assumption that $\mathcal{F}$ is not equicontinuous at x, there exist $f_n \in \mathcal{F}$ and $x_n \in V_n$ such that $|f_n(x) - f_n(x_n)| \geq \epsilon\ \forall\ n$. Since $\mathcal{F}$ is a compact metric space by passing to a subsequence if necessary we may assume that f_n converges uniformly to some $f \in \mathcal{F}$. Because $x_n \to x$ as $n \to \infty$ we learn that

$$\begin{aligned}\epsilon &\leq |f_n(x) - f_n(x_n)| \leq |f_n(x) - f(x)| + |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| \\ &\leq 2\|f_n - f\| + |f(x) - f(x_n)| \to 0 \text{ as } n \to \infty\end{aligned}$$

which is a contradiction.

so that $\{\mathcal{C}_V\}_V \in \tau_x \subset \mathcal{F}$ has the finite intersection property.[8] Since $\mathcal{F}$ is compact, it follows that there exists some

$$f \in \bigcap_{V \in \tau_x} \mathcal{C}_V \neq \emptyset.$$

Since f is continuous, there exists $V \in \tau_x$ such that $|f(x) - f(y)| < \epsilon/3$ for all $y \in V$. Because $f \in \mathcal{C}_V$, there exists $W \subset V$ such that $\|f - f_W\| < \epsilon/3$. We now arrive at a contradiction;

$$\begin{aligned}\epsilon &\leq |f_W(x) - f_W(x_W)| \leq |f_W(x) - f(x)| + |f(x) - f(x_W)| + |f(x_W) - f_W(x_W)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

■

3.7. Bounded Linear Operators Basics.

Definition 3.60. Let X and Y be normed spaces and $T : X \to Y$ be a linear map. Then T is said to be bounded provided there exists $C < \infty$ such that $\|T(x)\| \leq C\|x\|_X$ for all $x \in X$. We denote the best constant by $\|T\|$, i.e.

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{x \neq 0} \left\{ \|T(x)\| : \|x\| = 1 \right\}.$$

The number $\|T\|$ is called the operator norm of T.

Proposition 3.61. *Suppose that X and Y are normed spaces and $T : X \to Y$ is a linear map. The the following are equivalent:*

(a) *T is continuous.*
(b) *T is continuous at 0.*
(c) *T is bounded.*

Proof. (a) $\Rightarrow$ (b) trivial. (b) $\Rightarrow$ (c) If T continuous at 0 then there exist $\delta > 0$ such that $\|T(x)\| \leq 1$ if $\|x\| \leq \delta$. Therefore for any $x \in X$, $\|T(\delta x/\|x\|)\| \leq 1$ which implies that $\|T(x)\| \leq \frac{1}{\delta}\|x\|$ and hence $\|T\| \leq \frac{1}{\delta} < \infty$. (c) $\Rightarrow$ (a) Let $x \in X$ and $\epsilon > 0$ be given. Then

$$\|T(y) - T(x)\| = \|T(y - x)\| \leq \|T\| \; \|y - x\| < \epsilon$$

provided $\|y - x\| < \epsilon/\|T\| \equiv \delta$. ■

Example 3.62. Suppose that $K : [0, 1] \times [0, 1] \to \mathbb{C}$ is a continuous function. For $f \in C([0, 1])$, let

$$Tf(x) = \int_0^1 K(x, y) f(y) dy.$$

[8] If we are willing to use Net's described in Appendix D below we could finish the proof as follows. Since $\mathcal{F}$ is compact, the net $\{f_V\}_{V \in \tau_x} \subset \mathcal{F}$ has a cluster point $f \in \mathcal{F} \subset C(X)$. Choose a subnet $\{g_\alpha\}_{\alpha \in A}$ of $\{f_V\}_{V \in \tau_X}$ such that $g_\alpha \to f$ uniformly. Then, since $x_V \to x$ implies $x_{V_\alpha} \to x$, we may conclude from Eq. (3.8) that

$$\epsilon \leq |g_\alpha(x) - g_\alpha(x_{V_\alpha})| \to |g(x) - g(x)| = 0$$

which is a contradiction.

Since

$$|Tf(x) - Tf(z)| \leq \int_0^1 |K(x,y) - K(z,y)| \, |f(y)| \, dy$$
$$\leq \|f\|_\infty \max_y |K(x,y) - K(z,y)| \tag{3.9}$$

and the latter expression tends to 0 as $x \to z$ by uniform continuity of K. Therefore $Tf \in C([0,1])$ and by the linearity of the Riemann integral, $T : C([0,1]) \to C([0,1])$ is a linear map. Moreover,

$$|Tf(x)| \leq \int_0^1 |K(x,y)| \, |f(y)| \, dy \leq \int_0^1 |K(x,y)| \, dy \cdot \|f\|_\infty \leq A \|f\|_\infty$$

where

$$A := \sup_{x\in[0,1]} \int_0^1 |K(x,y)| \, dy < \infty. \tag{3.10}$$

This shows $\|T\| \leq A < \infty$ and therefore T is bounded. We may in fact show $\|T\| = A$. To do this let $x_0 \in [0,1]$ be such that

$$\sup_{x\in[0,1]} \int_0^1 |K(x,y)| \, dy = \int_0^1 |K(x_0,y)| \, dy.$$

Such an x_0 can be found since, using a similar argument to that in Eq. (3.9), $x \to \int_0^1 |K(x,y)| \, dy$ is continuous. Given $\epsilon > 0$, let

$$f_\epsilon(y) := \frac{\overline{K(x_0,y)}}{\sqrt{\epsilon + |K(x_0,y)|^2}}$$

and notice that $\lim_{\epsilon\downarrow 0} \|f_\epsilon\|_\infty = 1$ and

$$\|Tf_\epsilon\|_\infty \geq |Tf_\epsilon(x_0)| = Tf_\epsilon(x_0) = \int_0^1 \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} dy.$$

Therefore,

$$\|T\| \geq \lim_{\epsilon\downarrow 0} \frac{1}{\|f_\epsilon\|_\infty} \int_0^1 \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} dy$$
$$= \lim_{\epsilon\downarrow 0} \int_0^1 \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} dy = A$$

since

$$0 \leq |K(x_0,y)| - \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} = \frac{|K(x_0,y)|}{\sqrt{\epsilon + |K(x_0,y)|^2}} \left[\sqrt{\epsilon + |K(x_0,y)|^2} - |K(x_0,y)|\right]$$
$$\leq \sqrt{\epsilon + |K(x_0,y)|^2} - |K(x_0,y)|$$

and the latter expression tends to zero uniformly in y as $\epsilon \downarrow 0$.

We may also consider other norms on $C([0,1])$. Let (for now) $L^1([0,1])$ denote $C([0,1])$ with the norm

$$\|f\|_1 = \int_0^1 |f(x)| \, dx,$$

then $T : L^1([0,1], dm) \to C([0,1])$ is bounded as well. Indeed, let $M = \sup\{|K(x,y)| : x,y \in [0,1]\}$, then

$$|(Tf)(x)| \leq \int_0^1 |K(x,y)f(y)|\,dy \leq M\,\|f\|_1$$

which shows $\|Tf\|_\infty \leq M\,\|f\|_1$ and hence,

$$\|T\|_{L^1 \to C} \leq \max\{|K(x,y)| : x,y \in [0,1]\} < \infty.$$

We can in fact show that $\|T\| = M$ as follows. Let $(x_0,y_0) \in [0,1]^2$ satisfying $|K(x_0,y_0)| = M.$ Then given $\epsilon > 0$, there exists a neighborhood $U = I \times J$ of (x_0,y_0) such that $|K(x,y) - K(x_0,y_0)| < \epsilon$ for all $(x,y) \in U$. Let $f \in C_c(I,[0,\infty))$ such that $\int_0^1 f(x)dx = 1$. Choose $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha K(x_0,y_0) = M$, then

$$\begin{aligned}|(T\alpha f)(x_0)| &= \left|\int_0^1 K(x_0,y)\alpha f(y)dy\right| = \left|\int_I K(x_0,y)\alpha f(y)dy\right| \\ &\geq \operatorname{Re}\int_I \alpha K(x_0,y)f(y)dy \geq \int_I (M-\epsilon)\,f(y)dy = (M-\epsilon)\,\|\alpha f\|_{L^1}\end{aligned}$$

and hence

$$\|T\alpha f\|_C \geq (M-\epsilon)\,\|\alpha f\|_{L^1}$$

showing that $\|T\| \geq M - \epsilon$. Since $\epsilon > 0$ is arbitrary, we learn that $\|T\| \geq M$ and hence $\|T\| = M$.

One may also view T as a map from $T : C([0,1]) \to L^1([0,1])$ in which case one may show

$$\|T\|_{L^1 \to C} \leq \int_0^1 \max_y |K(x,y)|\,dx < \infty.$$

For the next three exercises, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and $T : X \to Y$ be a linear transformation so that T is given by matrix multiplication by an $m \times n$ matrix. Let us identify the linear transformation T with this matrix.

Exercise 3.16. Assume the norms on X and Y are the ℓ^1 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \sum_{j=1}^n |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\leq j\leq n} \sum_{i=1}^m |T_{ij}|.$$

Exercise 3.17. Assume the norms on X and Y are the ℓ^∞ – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \max_{1\leq j\leq n} |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\leq i\leq m} \sum_{j=1}^n |T_{ij}|.$$

Exercise 3.18. Assume the norms on X and Y are the ℓ^2 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\|^2 = \sum_{j=1}^n x_j^2$. Show $\|T\|^2$ is the largest eigenvalue of the matrix $T^{tr}T : \mathbb{R}^n \to \mathbb{R}^n$.

Exercise 3.19. If X is finite dimensional normed space then all linear maps are bounded.

Notation 3.63. Let $L(X,Y)$ denote the bounded linear operators from X to Y. If $Y = \mathbb{F}$ we write X^* for $L(X,\mathbb{F})$ and call X^* the (continuous) **dual space** to X.

Lemma 3.64. *Let X, Y be normed spaces, then the operator norm $\|\cdot\|$ on $L(X,Y)$ is a norm. Moreover if Z is another normed space and $T : X \to Y$ and $S : Y \to Z$ are linear maps, then $\|ST\| \le \|S\|\|T\|$, where $ST := S \circ T$.*

Proof. As usual, the main point in checking the operator norm is a norm is to verify the triangle inequality, the other axioms being easy to check. If $A, B \in L(X,Y)$ then the triangle inequality is verified as follows:

$$\begin{aligned}\|A+B\| &= \sup_{x\neq 0} \frac{\|Ax+Bx\|}{\|x\|} \le \sup_{x\neq 0} \frac{\|Ax\|+\|Bx\|}{\|x\|} \\ &\le \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x\neq 0} \frac{\|Bx\|}{\|x\|} = \|A\| + \|B\|.\end{aligned}$$

For the second assertion, we have for $x \in X$, that

$$\|STx\| \le \|S\|\|Tx\| \le \|S\|\|T\|\|x\|.$$

From this inequality and the definition of $\|ST\|$, it follows that $\|ST\| \le \|S\|\|T\|$. ∎

Proposition 3.65. *Suppose that X is a normed vector space and Y is a Banach space. Then $(L(X,Y), \|\cdot\|_{op})$ is a Banach space. In particular the dual space X^* is always a Banach space.*

We will use the following characterization of a Banach space in the proof of this proposition.

Theorem 3.66. *A normed space $(X, \|\cdot\|)$ is a Banach space iff for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then $\lim_{N\to\infty} \sum_{n=1}^{N} x_n = S$ exists in X (that is to say every absolutely convergent series is a convergent series in X). As usual we will denote S by $\sum_{n=1}^{\infty} x_n$.*

Proof. ($\Rightarrow$)If X is complete and $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then sequence $S_N \equiv \sum_{n=1}^{N} x_n$ for $N \in \mathbb{N}$ is Cauchy because (for $N > M$)

$$\|S_N - S_M\| \le \sum_{n=M+1}^{N} \|x_n\| \to 0 \text{ as } M, N \to \infty.$$

Therefore $S = \sum_{n=1}^{\infty} x_n := \lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists in X.

($\Longleftarrow$) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and let $\{y_k = x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|y_{n+1} - y_n\| < \infty$. By assumption

$$y_{N+1} - y_1 = \sum_{n=1}^{N} (y_{n+1} - y_n) \to S = \sum_{n=1}^{\infty} (y_{n+1} - y_n) \in X \text{ as } N \to \infty.$$

This shows that $\lim_{N\to\infty} y_N$ exists and is equal to $x := y_1 + S$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy,

$$\|x - x_n\| \le \|x - y_k\| + \|y_k - x_n\| \to 0 \text{ as } k, n \to \infty$$

showing that $\lim_{n\to\infty} x_n$ exists and is equal to x. ∎

Proof. (Proof of Proposition 3.65.) We must show $(L(X,Y),\|\cdot\|_{op})$ is complete. Suppose that $T_n \in L(X,Y)$ is a sequence of operators such that $\sum_{n=1}^{\infty} \|T_n\| < \infty$. Then

$$\sum_{n=1}^{\infty} \|T_n x\| \le \sum_{n=1}^{\infty} \|T_n\| \, \|x\| < \infty$$

and therefore by the completeness of Y, $Sx := \sum_{n=1}^{\infty} T_n x = \lim_{N\to\infty} S_N x$ exists in Y, where $S_N := \sum_{n=1}^{N} T_n$. The reader should check that $S : X \to Y$ so defined in linear. Since,

$$\|Sx\| = \lim_{N\to\infty} \|S_N x\| \le \lim_{N\to\infty} \sum_{n=1}^{N} \|T_n x\| \le \sum_{n=1}^{\infty} \|T_n\| \, \|x\|,$$

S is bounded and

$$\|S\| \le \sum_{n=1}^{\infty} \|T_n\|. \tag{3.11}$$

Similarly,

$$\|Sx - S_M x\| = \lim_{N\to\infty} \|S_N x - S_M x\| \le \lim_{N\to\infty} \sum_{n=M+1}^{N} \|T_n\| \, \|x\| = \sum_{n=M+1}^{\infty} \|T_n\| \, \|x\|$$

and therefore,

$$\|S - S_M\| \le \sum_{n=M}^{\infty} \|T_n\| \to 0 \text{ as } M \to \infty.$$

■

Of course we did not actually need to use Theorem 3.66 in the proof. Here is another proof. Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L(X,Y)$. Then for each $x \in X$,

$$\|T_n x - T_m x\| \le \|T_n - T_m\| \, \|x\| \to 0 \text{ as } m, n \to \infty$$

showing $\{T_n x\}_{n=1}^{\infty}$ is Cauchy in Y. Using the completeness of Y, there exists an element $Tx \in Y$ such that

$$\lim_{n\to\infty} \|T_n x - Tx\| = 0.$$

It is a simple matter to show $T : X \to Y$ is a linear map. Moreover,

$$\|Tx - T_n x\| \le \|Tx - T_m x\| + \|T_m x - T_n x\| \le \|Tx - T_m x\| + \|T_m - T_n\| \, \|x\|$$

and therefore

$$\|Tx - T_n x\| \le \lim\sup_{m\to\infty} \left(\|Tx - T_m x\| + \|T_m - T_n\| \, \|x\|\right) = \|x\| \cdot \lim\sup_{m\to\infty} \|T_m - T_n\|.$$

Hence

$$\|T - T_n\| \le \lim\sup_{m\to\infty} \|T_m - T_n\| \to 0 \text{ as } n \to \infty.$$

Thus we have shown that $T_n \to T$ in $L(X,Y)$ as desired.

3.8. **Inverting Elements in $L(X)$ and Linear ODE.**

Definition 3.67. A linear map $T : X \to Y$ is an **isometry** if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. T is said to be **invertible** if T is a bijection and T^{-1} is bounded.

Notation 3.68. We will write $L^{\times}(X,Y)$ for those $T \in L(X,Y)$ which are invertible. If $X = Y$ we simply write $L(X)$ and $L^{\times}(X)$ for $L(X,X)$ and $L^{\times}(X,X)$ respectively.

Proposition 3.69. *Suppose X is a Banach space and $\Lambda \in L(X) \equiv L(X,X)$ satisfies $\sum_{n=0}^{\infty} \|\Lambda^n\| < \infty$. Then $I - \Lambda$ is invertible and*

$$(I-\Lambda)^{-1} = \text{“}\frac{1}{I-\Lambda}\text{”} = \sum_{n=0}^{\infty} \Lambda^n \text{ and } \left\|(I-\Lambda)^{-1}\right\| \le \sum_{n=0}^{\infty} \|\Lambda^n\|.$$

In particular if $\|\Lambda\| < 1$ then the above formula holds and

$$\left\|(I-\Lambda)^{-1}\right\| \le \frac{1}{1-\|\Lambda\|}.$$

Proof. Since $L(X)$ is a Banach space and $\sum_{n=0}^{\infty} \|\Lambda^n\| < \infty$, it follows from Theorem 3.66 that

$$S := \lim_{N\to\infty} S_N := \lim_{N\to\infty} \sum_{n=0}^{N} \Lambda^n$$

exists in $L(X)$. Moreover, by Exercise 3.38 below,

$$\begin{aligned}(I-\Lambda)\,S &= (I-\Lambda)\lim_{N\to\infty} S_N = \lim_{N\to\infty} (I-\Lambda)\,S_N \\ &= \lim_{N\to\infty} (I-\Lambda)\sum_{n=0}^{N} \Lambda^n = \lim_{N\to\infty} (I-\Lambda^{N+1}) = I\end{aligned}$$

and similarly $S\,(I-\Lambda) = I$. This shows that $(I-\Lambda)^{-1}$ exists and is equal to S. Moreover, $(I-\Lambda)^{-1}$ is bounded because

$$\left\|(I-\Lambda)^{-1}\right\| = \|S\| \le \sum_{n=0}^{\infty} \|\Lambda^n\|.$$

If we further assume $\|\Lambda\| < 1$, then $\|\Lambda^n\| \le \|\Lambda\|^n$ and

$$\sum_{n=0}^{\infty} \|\Lambda^n\| \le \sum_{n=0}^{\infty} \|\Lambda\|^n \le \frac{1}{1-\|\Lambda\|} < \infty.$$

■

Corollary 3.70. *Let X and Y be Banach spaces. Then $L^{\times}(X,Y)$ is an open (possibly empty) subset of $L(X,Y)$. More specifically, if $A \in L^{\times}(X,Y)$ and $B \in L(X,Y)$ satisfies*

$$\|B - A\| < \|A^{-1}\|^{-1} \tag{3.12}$$

then $B \in L^{\times}(X,Y)$

$$B^{-1} = \sum_{n=0}^{\infty} \left[I_X - A^{-1}B\right]^n A^{-1} \in L(Y,X) \tag{3.13}$$

and

$$\left\|B^{-1}\right\| \leq \|A^{-1}\| \frac{1}{1-\|A^{-1}\|\,\|A-B\|}.$$

Proof. Let A and B be as above, then

$$B = A-(A-B) = A\left[I_X - A^{-1}(A-B))\right] = A(I_X-\Lambda)$$

where $\Lambda : X \to X$ is given by

$$\Lambda := A^{-1}(A-B) = I_X - A^{-1}B.$$

Now

$$\|\Lambda\| = \left\|A^{-1}(A-B))\right\| \leq \|A^{-1}\|\,\|A-B\| < \|A^{-1}\|\|A^{-1}\|^{-1} = 1.$$

Therefore $I-\Lambda$ is invertible and hence so is B (being the product of invertible elements) with

$$B^{-1} = (I-\Lambda)^{-1}A^{-1} = \left[I_X - A^{-1}(A-B))\right]^{-1}A^{-1}.$$

For the last assertion we have,

$$\left\|B^{-1}\right\| \leq \left\|(I_X-\Lambda)^{-1}\right\|\|A^{-1}\| \leq \|A^{-1}\|\frac{1}{1-\|\Lambda\|} \leq \|A^{-1}\|\frac{1}{1-\|A^{-1}\|\,\|A-B\|}.$$

■

3.8.1. *An Application to Linear Ordinary Differential Equations.* Consider the linear differential equation

$$\dot{x}(t) = A(t)x(t) \text{ where } x(0) = x_0 \in \mathbb{R}^n. \tag{3.14}$$

Here $A \in C(\mathbb{R} \to L(\mathbb{R}^n))$ and $x \in C^1(\mathbb{R} \to \mathbb{R}^n)$. As usual this equation may be written in its equivalent integral form, i.e. we are looking for $x \in C(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = x_0 + \int_0^t A(\tau)x(\tau)d\tau. \tag{3.15}$$

In what follows, we will let $\|\cdot\|$ denote some norm on $\mathbb{R}^n$ – for example the sup-norm. By abuse of notation, also let $\|\cdot\|$ denote the corresponding operator norm on $L(\mathbb{R}^n)$. We will also fix $T \in (0,\infty)$ and let $\|\phi\|_\infty := \max_{0\leq t\leq T}\|\phi(t)\|$ for $\phi \in C([0,T],\mathbb{R}^n)$ or $C([0,T], L(\mathbb{R}^n))$.

Theorem 3.71. *Let $\phi \in C([0,T],\mathbb{R}^n)$, then the integral equation*

$$x(t) = \phi(t) + \int_0^t A(\tau)x(\tau)d\tau \tag{3.16}$$

has a unique solution given by

$$x(t) = \phi(t) + \sum_{n=1}^{\infty}\int_{\Delta_n(t)} A(\tau_n)\dots A(\tau_1)\phi(\tau_1)d\tau_1\dots d\tau_n$$

where

$$\Delta_n(t) = \{0 \leq \tau_1 \leq \cdots \leq \tau_n \leq t\}.$$

Moreover,

$$\|x(t)\| \leq \|\phi\|_\infty e^{\int_0^T \|A(\tau)\|d\tau}.$$

Proof. Define $\Lambda : C([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R}^n)$ by

$$(\Lambda x)(t) = \int_0^t A(\tau)x(\tau)d\tau.$$

Then x solves Eq. (3.15) iff $x = \phi + \Lambda x$ or equivalently iff $(I - \Lambda)x = \phi$. The theorem will be proved by showing $(I-\Lambda)^{-1}$ exists via Proposition 3.69. To apply this proposition it suffices to show $\sum_{n=0}^{\infty} \|\Lambda^n\|_{op} < \infty$, where $\|\cdot\|_{op}$ denotes the operator norm on $L\left(C([0,T],\mathbb{R}^n)\right)$.

An induction argument shows

$$\begin{aligned}
(\Lambda^n \phi)(t) &= \int_0^t d\tau_n A(\tau_n)(\Lambda^{n-1}\phi)(\tau_n) \\
&= \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} A(\tau_n)A(\tau_{n-1})(\Lambda^{n-2}\phi)(\tau_{n-1}) \\
&\vdots \\
&= \int\limits_{0\le\tau_1\le\cdots\le\tau_n\le t} A(\tau_n)\dots A(\tau_1)\phi(\tau_1)d\tau_1\dots d\tau_n \\
&= \int_{\Delta_n(t)} A(\tau_n)\dots A(\tau_1)\phi(\tau_1)d\tau_1\dots d\tau_n.
\end{aligned}$$

Hence

$$\|(\Lambda^n\phi)(t)\| \le \left\{ \int\limits_{0\le\tau_1\le\cdots\le\tau_n\le t} \|A(\tau_n)\|\dots\|A(\tau_1)\|d\tau_1\dots d\tau_n \right\} \|\phi\|_\infty.$$

Therefore

$$\begin{aligned}
\|\Lambda^n\|_{op} &\le \int\limits_{0\le\tau_1\le\cdots\le\tau_n\le T} \|A(\tau_n)\|\dots\|A(\tau_1)\|d\tau_1\dots d\tau_n \\
&= \frac{1}{n!}\int\limits_{[0,T]^n} \|A(\tau_n)\|\dots\|A(\tau_1)\|d\tau_1\dots d\tau_n \\
&= \frac{1}{n!}\left(\int_0^T \|A(\tau)\|d\tau\right)^n. \qquad (3.17)
\end{aligned}$$

Alternatively, one can prove this last equality by induction on n. Namely let

$$F(t) = \int_0^t \|A(\tau)\|d\tau$$

then by induction one shows that

$$I_n(t) := \int\limits_{0\le\tau_1\le\cdots\le\tau_n\le T} \|A(\tau_n)\|\dots\|A(\tau_1)\|d\tau_1\dots d\tau_n = \frac{1}{n!}F^n(t).$$

Indeed,

$$I_{n+1}(t) = \int_0^t \frac{1}{n!}F^n(\tau)\dot{F}(\tau)d\tau = \int_0^t \frac{1}{(n+1)!}\frac{d}{d\tau}F^{n+1}(\tau)d\tau = \frac{1}{(n+1)!}F^{n+1}(t)$$

proving Eq. (3.17) again. Using this estimate we then have

$$\sum_{n=0}^{\infty}\|\Lambda^n\|_{op}\le e^{\int_0^T\|A(\tau)\|d\tau}<\infty.$$

Therefore $(I-\Lambda)^{-1}$ exists and $(I-\Lambda)^{-1}=\sum_{n=0}^{\infty}\Lambda^n$ and

$$\left\|(I-\Lambda)^{-1}\right\|_{op}\le e^{\int_0^T\|A(\tau)\|d\tau}.$$

■

3.9. **Appendix: Sums in Banach Spaces.**

Definition 3.72. Suppose that X is a normed space and $\{v_\alpha\in X:\alpha\in A\}$ is a given collection of vectors in X. We say that $s=\sum_{\alpha\in A}v_\alpha\in X$ if for all $\epsilon>0$ there exists a finite set $\Gamma_\epsilon\subset A$ such that $\left\|s-\sum_{\alpha\in\Lambda}v_\alpha\right\|<\epsilon$ for all $\Lambda\subset\subset A$ such that $\Gamma_\epsilon\subset\Lambda$. (Unlike the case of real valued sums, this does not imply that $\sum_{\alpha\in\Lambda}\|v_\alpha\|<\infty$. See Proposition 13.20 below, from which one may manufacture counter-examples to this false premise.)

Lemma 3.73. *(1) When X is a Banach space, $\sum_{\alpha\in A}v_\alpha$ exists in X iff for all $\epsilon>0$ there exists $\Gamma_\epsilon\subset\subset A$ such that $\left\|\sum_{\alpha\in\Lambda}v_\alpha\right\|<\epsilon$ for all $\Lambda\subset\subset A\setminus\Gamma_\epsilon$. Also if $\sum_{\alpha\in A}v_\alpha$ exists in X then $\{\alpha\in A:v_\alpha\neq 0\}$ is at most countable. (2) If $s=\sum_{\alpha\in A}v_\alpha\in X$ exists and $T:X\to Y$ is a bounded linear map between normed spaces, then $\sum_{\alpha\in A}Tv_\alpha$ exists in Y and*

$$Ts=T\sum_{\alpha\in A}v_\alpha=\sum_{\alpha\in A}Tv_\alpha.$$

Proof. (1) Suppose that $s=\sum_{\alpha\in A}v_\alpha$ exists and $\epsilon>0$. Let $\Gamma_\epsilon\subset\subset A$ be as in Definition 3.72. Then for $\Lambda\subset\subset A\setminus\Gamma_\epsilon$,

$$\begin{aligned}\left\|\sum_{\alpha\in\Lambda}v_\alpha\right\|&\le\left\|\sum_{\alpha\in\Lambda}v_\alpha+\sum_{\alpha\in\Gamma_\epsilon}v_\alpha-s\right\|+\left\|\sum_{\alpha\in\Gamma_\epsilon}v_\alpha-s\right\|\\&=\left\|\sum_{\alpha\in\Gamma_\epsilon\cup\Lambda}v_\alpha-s\right\|+\epsilon<2\epsilon.\end{aligned}$$

Conversely, suppose for all $\epsilon>0$ there exists $\Gamma_\epsilon\subset\subset A$ such that $\left\|\sum_{\alpha\in\Lambda}v_\alpha\right\|<\epsilon$ for all $\Lambda\subset\subset A\setminus\Gamma_\epsilon$. Let $\gamma_n:=\cup_{k=1}^n\Gamma_{1/k}\subset A$ and set $s_n:=\sum_{\alpha\in\gamma_n}v_\alpha$. Then for $m>n$,

$$\|s_m-s_n\|=\left\|\sum_{\alpha\in\gamma_m\setminus\gamma_n}v_\alpha\right\|\le 1/n\to 0\text{ as }m,n\to\infty.$$

Therefore $\{s_n\}_{n=1}^\infty$ is Cauchy and hence convergent in X. Let $s:=\lim_{n\to\infty}s_n$, then for $\Lambda\subset\subset A$ such that $\gamma_n\subset\Lambda$, we have

$$\left\|s-\sum_{\alpha\in\Lambda}v_\alpha\right\|\le\|s-s_n\|+\left\|\sum_{\alpha\in\Lambda\setminus\gamma_n}v_\alpha\right\|\le\|s-s_n\|+\frac{1}{n}.$$

Since the right member of this equation goes to zero as $n\to\infty$, it follows that $\sum_{\alpha\in A}v_\alpha$ exists and is equal to s.

Let $\gamma := \cup_{n=1}^{\infty}\gamma_n$ – a countable subset of A. Then for $\alpha \notin \gamma$, $\{\alpha\} \subset A \setminus \gamma_n$ for all n and hence

$$\|v_\alpha\| = \left\| \sum_{\beta \in \{\alpha\}} v_\beta \right\| \leq 1/n \to 0 \text{ as } n \to \infty.$$

Therefore $v_\alpha = 0$ for all $\alpha \in A \setminus \gamma$.

(2) Let Γ_ϵ be as in Definition 3.72 and $\Lambda \subset\subset A$ such that $\Gamma_\epsilon \subset \Lambda$. Then

$$\left\| Ts - \sum_{\alpha \in \Lambda} Tv_\alpha \right\| \leq \|T\| \left\| s - \sum_{\alpha \in \Lambda} v_\alpha \right\| < \|T\| \, \epsilon$$

which shows that $\sum_{\alpha \in \Lambda} Tv_\alpha$ exists and is equal to Ts. ■

3.10. Aside: Word of Caution.

Example 3.74. Let (X, d) be a metric space. It is always true that $\overline{B_x(\epsilon)} \subset C_x(\epsilon)$ since $C_x(\epsilon)$ is a closed set containing $B_x(\epsilon)$. However, it is not always true that $\overline{B_x(\epsilon)} = C_x(\epsilon)$. For example let $X = \{1, 2\}$ and $d(1, 2) = 1$, then $B_1(1) = \{1\}$, $\overline{B_1(1)} = \{1\}$ while $C_1(1) = X$. For another counter example, take

$$X = \left\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } x = 1\right\}$$

with the usually Euclidean metric coming from the plane. Then

$$\begin{aligned} B_{(0,0)}(1) &= \left\{(0, y) \in \mathbb{R}^2 : |y| < 1\right\}, \\ \overline{B_{(0,0)}(1)} &= \left\{(0, y) \in \mathbb{R}^2 : |y| \leq 1\right\}, \text{ while} \\ C_{(0,0)}(1) &= \overline{B_{(0,0)}(1)} \cup \{(0, 1)\}. \end{aligned}$$

In spite of the above examples, Lemmas 3.75 and 3.76 below shows that for certain metric spaces of interest it is true that $\overline{B_x(\epsilon)} = C_x(\epsilon)$.

Lemma 3.75. *Suppose that $(X, |\cdot|)$ is a normed vector space and d is the metric on X defined by $d(x, y) = |x - y|$. Then*

$$\begin{aligned} \overline{B_x(\epsilon)} &= C_x(\epsilon) \text{ and} \\ \partial B_x(\epsilon) &= \{y \in X : d(x, y) = \epsilon\}. \end{aligned}$$

Proof. We must show that $C := C_x(\epsilon) \subset \overline{B_x(\epsilon)} =: \bar{B}$. For $y \in C$, let $v = y - x$, then

$$|v| = |y - x| = d(x, y) \leq \epsilon.$$

Let $\alpha_n = 1 - 1/n$ so that $\alpha_n \uparrow 1$ as $n \to \infty$. Let $y_n = x + \alpha_n v$, then $d(x, y_n) = \alpha_n d(x, y) < \epsilon$, so that $y_n \in B_x(\epsilon)$ and $d(y, y_n) = 1 - \alpha_n \to 0$ as $n \to \infty$. This shows that $y_n \to y$ as $n \to \infty$ and hence that $y \in \bar{B}$. ■

3.10.1. *Riemannian Metrics.*

This subsection is not completely self contained and may safely be skipped.

Lemma 3.76. *Suppose that X is a Riemannian (or sub-Riemannian) manifold and d is the metric on X defined by*

$$d(x, y) = \inf \{\ell(\sigma) : \sigma(0) = x \text{ and } \sigma(1) = y\}$$

where $\ell(\sigma)$ is the length of the curve σ. We define $\ell(\sigma) = \infty$ if σ is not piecewise smooth.

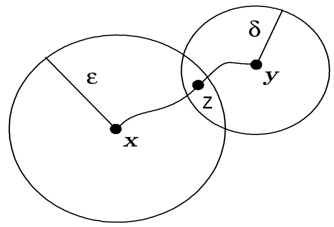

FIGURE 10. An almost length minimizing curve joining x to y.

Then

$$\overline{B_x(\epsilon)} = C_x(\epsilon) \text{ and}$$
$$\partial B_x(\epsilon) = \{y \in X : d(x,y) = \epsilon\}.$$

Proof. Let $C := C_x(\epsilon) \subset \overline{B_x(\epsilon)} =: \bar{B}$. We will show that $C \subset \bar{B}$ by showing $\bar{B}^c \subset C^c$. Suppose that $y \in \bar{B}^c$ and choose $\delta > 0$ such that $B_y(\delta) \cap \bar{B} = \emptyset$. In particular this implies that

$$B_y(\delta) \cap B_x(\epsilon) = \emptyset.$$

We will finish the proof by showing that $d(x,y) \ge \epsilon + \delta > \epsilon$ and hence that $y \in C^c$. This will be accomplished by showing: if $d(x,y) < \epsilon + \delta$ then $B_y(\delta) \cap B_x(\epsilon) \neq \emptyset$.

If $d(x,y) < \max(\epsilon, \delta)$ then either $x \in B_y(\delta)$ or $y \in B_x(\epsilon)$. In either case $B_y(\delta) \cap B_x(\epsilon) \neq \emptyset$. Hence we may assume that $\max(\epsilon,\delta) \le d(x,y) < \epsilon + \delta$. Let $\alpha > 0$ be a number such that

$$\max(\epsilon,\delta) \le d(x,y) < \alpha < \epsilon + \delta$$

and choose a curve σ from x to y such that $\ell(\sigma) < \alpha$. Also choose $0 < \delta' < \delta$ such that $0 < \alpha - \delta' < \epsilon$ which can be done since $\alpha - \delta < \epsilon$. Let $k(t) = d(y, \sigma(t))$ a continuous function on $[0,1]$ and therefore $k([0,1]) \subset \mathbb{R}$ is a connected set which contains 0 and $d(x,y)$. Therefore there exists $t_0 \in [0,1]$ such that $d(y,\sigma(t_0)) = k(t_0) = \delta'$. Let $z = \sigma(t_0) \in B_y(\delta)$ then

$$d(x,z) \le \ell(\sigma|_{[0,t_0]}) = \ell(\sigma) - \ell(\sigma|_{[t_0,1]}) < \alpha - d(z,y) = \alpha - \delta' < \epsilon$$

and therefore $z \in B_x(\epsilon) \cap B_x(\delta) \neq \emptyset$. ■

Remark 3.77. Suppose again that X is a Riemannian (or sub-Riemannian) manifold and

$$d(x,y) = \inf\{\ell(\sigma) : \sigma(0) = x \text{ and } \sigma(1) = y\}.$$

Let σ be a curve from x to y and let $\epsilon = \ell(\sigma) - d(x,y)$. Then for all $0 \le u < v \le 1$,

$$d(\sigma(u), \sigma(v)) \le \ell(\sigma|_{[u,v]}) + \epsilon.$$

So if σ is within ϵ of a length minimizing curve from x to y that $\sigma|_{[u,v]}$ is within ϵ of a length minimizing curve from $\sigma(u)$ to $\sigma(v)$. In particular if $d(x,y) = \ell(\sigma)$ then $d(\sigma(u),\sigma(v)) = \ell(\sigma|_{[u,v]})$ for all $0 \le u < v \le 1$, i.e. if σ is a length minimizing curve from x to y that $\sigma|_{[u,v]}$ is a length minimizing curve from $\sigma(u)$ to $\sigma(v)$.

To prove these assertions notice that

$$\begin{aligned} d(x,y)+\epsilon = \ell(\sigma) &= \ell(\sigma|_{[0,u]}) + \ell(\sigma|_{[u,v]}) + \ell(\sigma|_{[v,1]}) \\ &\geq d(x,\sigma(u)) + \ell(\sigma|_{[u,v]}) + d(\sigma(v),y) \end{aligned}$$

and therefore

$$\begin{aligned} \ell(\sigma|_{[u,v]}) &\leq d(x,y)+\epsilon - d(x,\sigma(u)) - d(\sigma(v),y) \\ &\leq d(\sigma(u),\sigma(v)) + \epsilon. \end{aligned}$$

3.11. **Exercises.**

Exercise 3.20. Prove Lemma 3.46.

Exercise 3.21. Let $X = C([0,1],\mathbb{R})$ and for $f \in X$, let

$$\|f\|_1 := \int_0^1 |f(t)|\, dt.$$

Show that $(X, \|\cdot\|_1)$ is normed space and show by example that this space is **not** complete.

Exercise 3.22. Let (X,d) be a metric space. Suppose that $\{x_n\}_{n=1}^\infty \subset X$ is a sequence and set $\epsilon_n := d(x_n, x_{n+1})$. Show that for $m > n$ that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} \epsilon_k \leq \sum_{k=n}^{\infty} \epsilon_k.$$

Conclude from this that if

$$\sum_{k=1}^{\infty} \epsilon_k = \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

then $\{x_n\}_{n=1}^\infty$ is Cauchy. Moreover, show that if $\{x_n\}_{n=1}^\infty$ is a convergent sequence and $x = \lim_{n\to\infty} x_n$ then

$$d(x, x_n) \leq \sum_{k=n}^{\infty} \epsilon_k.$$

Exercise 3.23. Show that (X,d) is a complete metric space iff every sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$ is a convergent sequence in X. You may find it useful to prove the following statements in the course of the proof.

1. If $\{x_n\}$ is Cauchy sequence, then there is a subsequence $y_j \equiv x_{n_j}$ such that $\sum_{j=1}^\infty d(y_{j+1}, y_j) < \infty$.
2. If $\{x_n\}_{n=1}^\infty$ is Cauchy and there exists a subsequence $y_j \equiv x_{n_j}$ of $\{x_n\}$ such that $x = \lim_{j\to\infty} y_j$ exists, then $\lim_{n\to\infty} x_n$ also exists and is equal to x.

Exercise 3.24. Suppose that $f : [0,\infty) \to [0,\infty)$ is a C^2 – function such that $f(0) = 0$, $f' > 0$ and $f'' \leq 0$ and (X,ρ) is a metric space. Show that $d(x,y) = f(\rho(x,y))$ is a metric on X. In particular show that

$$d(x,y) \equiv \frac{\rho(x,y)}{1+\rho(x,y)}$$

is a metric on X. (Hint: use calculus to verify that $f(a+b) \leq f(a) + f(b)$ for all $a, b \in [0,\infty)$.)

Exercise 3.25. Let $d : C(\mathbb{R}) \times C(\mathbb{R}) \to [0, \infty)$ be defined by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n},$$

where $\|f\|_n \equiv \sup\{|f(x)| : |x| \leq n\} = \max\{|f(x)| : |x| \leq n\}$.

1. Show that d is a metric on $C(\mathbb{R})$.
2. Show that a sequence $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{R})$ converges to $f \in C(\mathbb{R})$ as $n \to \infty$ iff f_n converges to f uniformly on compact subsets of $\mathbb{R}$.
3. Show that $(C(\mathbb{R}), d)$ is a complete metric space.

Exercise 3.26. Let $\{(X_n, d_n)\}_{n=1}^{\infty}$ be a sequence of metric spaces, $X := \prod_{n=1}^{\infty} X_n$, and for $x = (x(n))_{n=1}^{\infty}$ and $y = (y(n))_{n=1}^{\infty}$ in X let

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}.$$

Show: 1) (X, d) is a metric space, 2) a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converges to $x \in X$ iff $x_k(n) \to x(n) \in X_n$ as $k \to \infty$ for every $n = 1, 2, \ldots$, and 3) X is complete if X_n is complete for all n.

Exercise 3.27 (Tychonoff's Theorem)**.** Let us continue the notation of the previous problem. Further assume that the spaces X_n are compact for all n. Show (X, d) is compact. **Hint:** Either use Cantor's method to show every sequence $\{x_m\}_{m=1}^{\infty} \subset X$ has a convergent subsequence or alternatively show (X, d) is complete and totally bounded.

Exercise 3.28. Let (X_i, d_i) for $i = 1, \ldots, n$ be a finite collection of metric spaces and for $1 \leq p \leq \infty$ and $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $X := \prod_{i=1}^{n} X_i$, let

$$\rho_p(x, y) = \begin{cases} \left(\sum_{i=1}^{n} [d_i(x_i, y_i)]^p\right)^{1/p} & \text{if} \quad p \neq \infty \\ \max_i d_i(x_i, y_i) & \text{if} \quad p = \infty \end{cases}.$$

1. Show (X, ρ_p) is a metric space for $p \in [1, \infty]$. **Hint**: Minkowski's inequality.
2. Show that all of the metric $\{\rho_p : 1 \leq p \leq \infty\}$ are equivalent, i.e. for any $p, q \in [1, \infty]$ there exists constants $c, C < \infty$ such that

   $$\rho_p(x, y) \leq C\rho_q(x, y) \text{ and } \rho_q(x, y) \leq c\rho_p(x, y) \text{ for all } x, y \in X.$$

   Hint: This can be done with explicit estimates or more simply using Lemma 3.54.
3. Show that the topologies associated to the metrics ρ_p are the same for all $p \in [1, \infty]$.

Exercise 3.29. Let C be a closed proper subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus C$. Show there exists a $y \in C$ such that $d(x, y) = d_C(x)$.

Exercise 3.30. Let $\mathbb{F} = \mathbb{R}$ in this problem and $A \subset \ell^2(\mathbb{N})$ be defined by

$$\begin{aligned} A &= \{x \in \ell^2(\mathbb{N}) : x(n) \geq 1 + 1/n \text{ for some } n \in \mathbb{N}\} \\ &= \cup_{n=1}^{\infty} \{x \in \ell^2(\mathbb{N}) : x(n) \geq 1 + 1/n\}. \end{aligned}$$

Show A is a closed subset of $\ell^2(\mathbb{N})$ with the property that $d_A(0) = 1$ while there is no $y \in A$ such that $d_A(y) = 1$. (Remember that in general an infinite union of closed sets need not be closed.)

3.11.1. *Banach Space Problems.*

Exercise 3.31. Show that all finite dimensional normed vector spaces $(L, \|\cdot\|)$ are necessarily complete. Also show that closed and bounded sets (relative to the given norm) are compact.

Exercise 3.32. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Show the map

$$(\lambda, x, y) \in \mathbb{F} \times X \times X \to x + \lambda y \in X$$

is continuous relative to the topology on $\mathbb{F} \times X \times X$ defined by the norm

$$\|(\lambda, x, y)\|_{\mathbb{F}\times X\times X} := |\lambda| + \|x\| + \|y\|.$$

(See Exercise 3.28 for more on the metric associated to this norm.) Also show that $\|\cdot\| : X \to [0, \infty)$ is continuous.

Exercise 3.33. Let $p \in [1, \infty]$ and X be an infinite set. Show the closed unit ball in $\ell^p(X)$ is not compact.

Exercise 3.34. Let $X = \mathbb{N}$ and for $p, q \in [1, \infty)$ let $\|\cdot\|_p$ denote the $\ell^p(\mathbb{N})$ – norm. Show $\|\cdot\|_p$ and $\|\cdot\|_q$ are inequivalent norms for $p \neq q$ by showing

$$\sup_{f\neq 0} \frac{\|f\|_p}{\|f\|_q} = \infty \text{ if } p < q.$$

Exercise 3.35. Folland Problem 5.5. Closure of subspaces are subspaces.

Exercise 3.36. Folland Problem 5.9. Showing $C^k([0,1])$ is a Banach space.

Exercise 3.37. Folland Problem 5.11. Showing Holder spaces are Banach spaces.

Exercise 3.38. Let X, Y and Z be normed spaces. Prove the maps

$$(S, x) \in L(X, Y) \times X \longrightarrow Sx \in Y$$

and

$$(S, T) \in L(X, Y) \times L(Y, Z) \longrightarrow ST \in L(X, Z)$$

are continuous relative to the norms

$$\|(S, x)\|_{L(X,Y)\times X} := \|S\|_{L(X,Y)} + \|x\|_X \text{ and}$$
$$\|(S, T)\|_{L(X,Y)\times L(Y,Z)} := \|S\|_{L(X,Y)} + \|T\|_{L(Y,Z)}$$

on $L(X, Y) \times X$ and $L(X, Y) \times L(Y, Z)$ respectively.

3.11.2. *Ascoli-Arzela Theorem Problems.*

Exercise 3.39. Let $T \in (0, \infty)$ and $\mathcal{F} \subset C([0, T])$ be a family of functions such that:

1. $\dot{f}(t)$ exists for all $t \in (0, T)$ and $f \in \mathcal{F}$.
2. $\sup_{f\in\mathcal{F}} |f(0)| < \infty$ and
3. $M := \sup_{f\in\mathcal{F}} \sup_{t\in(0,T)} \left|\dot{f}(t)\right| < \infty$.

Show $\mathcal{F}$ is precompact in the Banach space $C([0, T])$ equipped with the norm $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)|$.

Exercise 3.40. Folland Problem 4.63.

Exercise 3.41. Folland Problem 4.64.

3.11.3. *General Topological Space Problems.*

Exercise 3.42. Give an example of continuous map, $f : X \to Y$, and a compact subset K of Y such that $f^{-1}(K)$ is not compact.