8. Locally Compact Hausdorff Spaces

In this section X will always be a topological space with topology τ. We are now interested in restrictions on τ in order to insure there are "plenty" of continuous functions. One such restriction is to assume $\tau = \tau_d$ – is the topology induced from a metric on X. The following two results shows that (X, τ_d) has lots of continuous functions. Recall for $A \subset X$, $d_A(x) = \inf\{d(x,y) : y \in A\}$.

Lemma 8.1 (Urysohn's Lemma for Metric Spaces)**.** *Let (X,d) be a metric space, $V \subset_o X$ and $F \sqsubset X$ such that $F \subset V$. Then*

$$f(x) = \frac{d_{V^c}(x)}{d_F(x) + d_{V^c}(x)} \text{ for } x \in X \tag{8.1}$$

defines a continuous function, $f : X \to [0,1]$, such that $f(x) = 1$ for $x \in F$ and $f(x) = 0$ if $x \notin V$. (This may also be stated as follows. Let A $(A = F)$ and B $(B = V^c)$ be two disjoint closed subsets of X, then there exists $f \in C(X,[0,1])$ such that $f = 1$ on A and $f = 0$ on B.)

Proof. By Lemma 3.5, d_F and d_{V^c} are continuous functions on X. Since F and V^c are closed, $d_F(x) > 0$ if $x \notin F$ and $d_{V^c}(x) > 0$ if $x \in V$. Since $F \cap V^c = \emptyset$, $d_F(x) + d_{V^c}(x) > 0$ for all x and $(d_F + d_{V^c})^{-1}$ is continuous as well. The remaining assertions about f are all easy to verify. ■

Theorem 8.2 (Metric Space Tietze Extension Theorem)**.** *Let (X,d) be a metric space, D be a closed subset of X, $-\infty < a < b < \infty$ and $f \in C(D,[a,b])$. (Here we are viewing D as a topological space with the relative topology, τ_D, see Definition 3.17.) Then there exists $F \in C(X,[a,b])$ such that $F|_D = f$.*

Proof.

1. By scaling and translation (i.e. by replacing f by $\frac{f-a}{b-a}$), it suffices to prove Theorem 8.2 with $a = 0$ and $b = 1$.
2. Suppose $\alpha \in (0,1]$ and $f : D \to [0,\alpha]$ is continuous function. Let $A := f^{-1}([0, \frac{1}{3}\alpha])$ and $B := f^{-1}([\frac{2}{3}\alpha, 1])$. By Lemma 8.1 there exists a function $\tilde{g} \in C(X,[0,\alpha/3])$ such that $\tilde{g} = 0$ on A and $\tilde{g} = 1$ on B. Letting $g := \frac{\alpha}{3}\tilde{g}$, we have $g \in C(X,[0,\alpha/3])$ such that $g = 0$ on A and $g = \alpha/3$ on B. Further notice that
$$0 \le f(x) - g(x) \le \frac{2}{3}\alpha \text{ for all } x \in D.$$
3. Now suppose $f : D \to [0,1]$ is a continuous function as in step 1. Let $g_1 \in C(X,[0,1/3])$ be as in step 2. with $\alpha = 1$ and let $f_1 := f - g_1|_D \in C(D,[0,2/3])$. Apply step 2. with $\alpha = 2/3$ and $f = f_1$ to find $g_2 \in C(X,[0,\frac{1}{3}\frac{2}{3}])$ such that $f_2 := f - (g_1 + g_2)|_D \in C(D,[0,\left(\frac{2}{3}\right)^2])$. Continue this way inductively to find $g_n \in C(X,[0,\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}])$ such that
$$f - \sum_{n=1}^N g_n|_D =: f_N \in C(D,[0,\left(\frac{2}{3}\right)^N]). \tag{8.2}$$
4. Define $F := \sum_{n=1}^\infty g_n$. Since
$$\sum_{n=1}^\infty \|g_n\|_u \le \sum_{n=1}^\infty \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1,$$

the series defining F is uniformly convergent so $F \in C(X, [0,1])$. Passing to the limit in Eq. (8.2) shows $f = F|_D$.

■

The main thrust of this section is to study locally compact (and σ – compact) Hausdorff spaces as defined below. We will see again that this class of topological spaces have an ample supply of continuous functions. We will start out with the notion of a Hausdorff topology. The following example shows a pathology which occurs when there are not enough open sets in a topology.

Example 8.3. Let $X = \{1,2,3\}$ and $\tau = \{X, \emptyset, \{1,2\}, \{2,3\}, \{2\}\}$ and $x_n = 2$ for all n. Then $x_n \to x$ for every $x \in X$!

Definition 8.4 (Hausdorff Topology)**.** A topological space, (X, τ), is **Hausdorff** if for each pair of distinct points, $x, y \in X$, there exists disjoint open neighborhoods, U and V of x and y respectively. (Metric spaces are typical examples of Hausdorff spaces.)

Remark 8.5. When τ is Hausdorff the "pathologies" appearing in Example 8.3 do not occur. Indeed if $x_n \to x \in X$ and $y \in X \setminus \{x\}$ we may choose $V \in \tau_x$ and $W \in \tau_y$ such that $V \cap W = \emptyset$. Then $x_n \in V$ a.a. implies $x_n \notin W$ for all but a finite number of n and hence $x_n \nrightarrow y$, so limits are unique.

Proposition 8.6. *Suppose that (X, τ) is a Hausdorff space, $K \sqsubset\sqsubset X$ and $x \in K^c$. Then there exists $U, V \in \tau$ such that $U \cap V = \emptyset$, $x \in U$ and $K \subset V$. In particular K is closed. (So compact subsets of Hausdorff topological spaces are closed.) More generally if K and F are two disjoint compact subsets of X, there exist disjoint open sets $U, V \in \tau$ such that $K \subset V$ and $F \subset U$.*

Proof. Because X is Hausdorff, for all $y \in K$ there exists $V_y \in \tau_y$ and $U_y \in \tau_x$ such that $V_y \cap U_y = \emptyset$. The cover $\{V_y\}_{y \in K}$ of K has a finite subcover, $\{V_y\}_{y \in \Lambda}$ for some $\Lambda \subset\subset K$. Let $V = \cup_{y \in \Lambda} V_y$ and $U = \cap_{y \in \Lambda} U_y$, then $U, V \in \tau$ satisfy $x \in U$, $K \subset V$ and $U \cap V = \emptyset$. This shows that K^c is open and hence that K is closed.

Suppose that K and F are two disjoint compact subsets of X. For each $x \in F$ there exists disjoint open sets U_x and V_x such that $K \subset V_x$ and $x \in U_x$. Since $\{U_x\}_{x \in F}$ is an open cover of F, there exists a finite subset Λ of F such that $F \subset U := \cup_{x \in \Lambda} U_x$. The proof is completed by defining $V := \cap_{x \in \Lambda} V_x$. ■

Exercise 8.1. Show any finite set X admits exactly one Hausdorff topology τ.

Exercise 8.2. Given an example of a topological space which has a non-closed compact subset.

Proposition 8.7. *Suppose that X is a compact topological space, Y is a Hausdorff topological space, and $f : X \to Y$ is a continuous bijection then f is a homeomorphism, i.e. $f^{-1} : Y \to X$ is continuous as well.*

Proof. Since closed subsets of compact sets are compact, continuous images of compact subsets are compact and compact subsets of Hausdorff spaces are closed, it follows that $\left(f^{-1}\right)^{-1}(C) = f(C)$ is closed in X for all closed subsets C of X. Thus f^{-1} is continuous. ■

Definition 8.8 (Local and σ – compactness)**.** Let (X, τ) be a topological space.

1. (X,τ) is **locally compact** if for all $x\in X$ there exists an open neighborhood $V\subset X$ of x such that $\bar{V}$ is compact. (Alternatively, in light of Definition 3.19, this is equivalent to requiring that to each $x\in X$ there exists a compact neighborhood N_x of x.)
2. (X,τ) is $\sigma-$ **compact** if there exists compact sets $K_n\subset X$ such that $X=\cup_{n=1}^\infty K_n$. (Notice that we may assume, by replacing K_n by $K_1\cup K_2\cup\dots\cup K_n$ if necessary, that $K_n\uparrow X$.)

Example 8.9. Any open subset of $X\subset\mathbb{R}^n$ is a locally compact and $\sigma-$compact metric space (and hence Hausdorff). The proof of local compactness is easy and is left to the reader. To see that X is $\sigma-$compact, for $k\in\mathbb{N}$, let

$$K_k:=\{x\in X:|x|\le k\text{ and }d_{X^c}(x)\ge 1/k\}.$$

Then K_k is a closed and bounded subset of $\mathbb{R}^n$ and hence compact. Moreover $K_k^o\uparrow X$ as $k\to\infty$ since[16]

$$K_k^o\supset\{x\in X:|x|<k\text{ and }d_{X^c}(x)>1/k\}\uparrow X\text{ as }k\to\infty.$$

Exercise 8.3. Suppose that (X,d) is a metric space and $U\subset X$ is an open subset.

1. If X is locally compact then (U,d) is locally compact.
2. If X is $\sigma-$compact then (U,d) is $\sigma-$compact.

Exercise 8.4. Every separable locally compact metric space is $\sigma-$compact. **Hint:** Let $\{x_n\}_{n=1}^\infty\subset X$ be a countable dense subset of X and define

$$\epsilon_n=\frac{1}{2}\sup\{\epsilon>0:C_{x_n}(\epsilon)\text{ is compact}\}\wedge 1.$$

Exercise 8.5. Every $\sigma-$compact metric space is separable. Therefore a locally compact metric space is separable iff it is $\sigma-$compact.

Lemma 8.10. *Let (X,τ) be a locally compact and $\sigma-$compact topological space. Then there exists compact sets $K_n\uparrow X$ such that $K_n\subset K_{n+1}^o\subset K_{n+1}$ for all n.*

Proof. Suppose that $C\subset X$ is a compact set. For each $x\in C$ let $V_x\subset_o X$ be an open neighborhood of x such that $\bar{V}_x$ is compact. Then $C\subset\cup_{x\in C}V_x$ so there exists $\Lambda\subset\subset C$ such that

$$C\subset\cup_{x\in\Lambda}V_x\subset\cup_{x\in\Lambda}\bar{V}_x=:K.$$

Then K is a compact set, being a finite union of compact subsets of X, and $C\subset\cup_{x\in\Lambda}V_x\subset K^o$.

Now let $C_n\subset X$ be compact sets such that $C_n\uparrow X$ as $n\to\infty$. Let $K_1=C_1$ and then choose a compact set K_2 such that $C_2\subset K_2^o$. Similarly, choose a compact set K_3 such that $K_2\cup C_3\subset K_3^o$ and continue inductively to find compact sets K_n such that $K_n\cup C_{n+1}\subset K_{n+1}^o$ for all n. Then $\{K_n\}_{n=1}^\infty$ is the desired sequence. ■

Remark 8.11. Lemma 8.10 may also be stated as saying there exists precompact open sets $\{G_n\}_{n=1}^\infty$ such that $G_n\subset\bar{G}_n\subset G_{n+1}$ for all n and $G_n\uparrow X$ as $n\to\infty$. Indeed if $\{G_n\}_{n=1}^\infty$ are as above, let $K_n:=\bar{G}_n$ and if $\{K_n\}_{n=1}^\infty$ are as in Lemma 8.10, let $G_n:=K_n^o$.

The following result is a Corollary of Lemma 8.10 and Theorem 3.59.

[16] In fact this is an equality, but we will not need this here.

Corollary 8.12 (Locally compact form of Ascoli-Arzela Theorem). *Let (X,τ) be a locally compact and σ – compact topological space and $\{f_m\}\subset C(X)$ be a pointwise bounded sequence of functions such that $\{f_m|_K\}$ is equicontinuous for any compact subset $K\subset X$. Then there exists a subsequence $\{m_n\}\subset\{m\}$ such that $\{g_n:=f_{m_n}\}_{n=1}^\infty\subset C(X)$ is a sequence which is uniformly convergent on compact subsets of X.*

Proof. Let $\{K_n\}_{n=1}^\infty$ be the compact subsets of X constructed in Lemma 8.10. We may now apply Theorem 3.59 repeatedly to find a nested family of subsequences

$$\{f_m\}\supset\{g_m^1\}\supset\{g_m^2\}\supset\{g_m^3\}\supset\dots$$

such that the sequence $\{g_m^n\}_{m=1}^\infty\subset C(X)$ is uniformly convergent on K_n. Using Cantor's trick, define the subsequence $\{h_n\}$ of $\{f_m\}$ by $h_n\equiv g_n^n$. Then $\{h_n\}$ is uniformly convergent on K_l for each $l\in\mathbb{N}$. Now if $K\subset X$ is an arbitrary compact set, there exists $l<\infty$ such that $K\subset K_l^o\subset K_l$ and therefore $\{h_n\}$ is uniformly convergent on K as well. ■

The next two results shows that locally compact Hausdorff spaces have plenty of open sets and plenty of continuous functions.

Proposition 8.13. *Suppose X is a locally compact Hausdorff space and $U\subset_o X$ and $K\sqsubset\sqsubset U$. Then there exists $V\subset_o X$ such that $K\subset V\subset\overline{V}\subset U\subset X$ and $\bar{V}$ is compact.*

Proof. By local compactness, for all $x\in K$, there exists $U_x\in\tau_x$ such that $\bar{U}_x$ is compact. Since K is compact, there exists $\Lambda\subset\subset K$ such that $\{U_x\}_{x\in\Lambda}$ is a cover of K. The set $O=U\cap(\cup_{x\in\Lambda}U_x)$ is an open set such that $K\subset O\subset U$ and O is precompact since $\bar{O}$ is a closed subset of the compact set $\cup_{x\in\Lambda}\bar{U}_x$. ($\cup_{x\in\Lambda}\bar{U}_x$. is compact because it is a finite union of compact sets.) So by replacing U by O if necessary, we may assume that $\bar{U}$ is compact.

Since $\bar{U}$ is compact and $\partial U=\bar{U}\cap U^c$ is a closed subset of $\bar{U}$, ∂U is compact. Because $\partial U\subset U^c$, it follows that $\partial U\cap K=\emptyset$, so by Proposition 8.6, there exists disjoint open sets V and W such that $K\subset V$ and $\partial U\subset W$. By replacing V by $V\cap U$ if necessary we may further assume that $K\subset V\subset U$, see Figure 17.

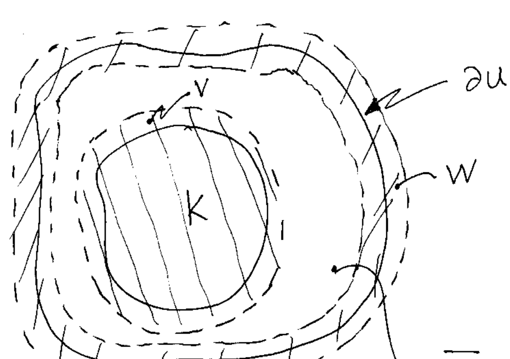

FIGURE 17. The construction of V.

Because $\bar{U}\cap W^c$ is a closed set containing V and $U^c\cap\bar{U}\cap W^c=\partial U\cap W^c=\emptyset$,

$$\bar{V}\subset\bar{U}\cap W^c=U\cap W^c\subset U\subset\bar{U}.$$

Since $\bar{U}$ is compact it follows that $\bar{V}$ is compact and the proof is complete. ■

Exercise 8.6. Give a "simpler" proof of Proposition 8.13 under the additional assumption that X is a metric space. **Hint:** show for each $x \in K$ there exists $V_x := B_x(\epsilon_x)$ with $\epsilon_x > 0$ such that $\overline{B_x(\epsilon_x)} \subset C_x(\epsilon_x) \subset U$ with $C_x(\epsilon_x)$ being compact. Recall that $C_x(\epsilon)$ is the closed ball of radius ϵ about x.

Definition 8.14. Let U be an open subset of a topological space (X, τ). We will write $f \prec U$ to mean a function $f \in C_c(X, [0,1])$ such that $\text{supp}(f) := \overline{\{f \neq 0\}} \subset U$.

Lemma 8.15 (Locally Compact Version of Urysohn's Lemma)**.** *Let X be a locally compact Hausdorff space and $K \sqsubset\sqsubset U \subset_o X$. Then there exists $f \prec U$ such that $f = 1$ on K. In particular, if K is compact and C is closed in X such that $K \cap C = \emptyset$, there exists $f \in C_c(X, [0,1])$ such that $f = 1$ on K and $f = 0$ on C.*

Proof. For notational ease later it is more convenient to construct $g := 1 - f$ rather than f. To motivate the proof, suppose $g \in C(X, [0,1])$ such that $g = 0$ on K and $g = 1$ on U^c. For $r > 0$, let $U_r = \{g < r\}$. Then for $0 < r < s \leq 1$, $U_r \subset \{g \leq r\} \subset U_s$ and since $\{g \leq r\}$ is closed this implies

$$K \subset U_r \subset \bar{U}_r \subset \{g \leq r\} \subset U_s \subset U.$$

Therefore associated to the function g is the collection open sets $\{U_r\}_{r>0} \subset \tau$ with the property that $K \subset U_r \subset \bar{U}_r \subset U_s \subset U$ for all $0 < r < s \leq 1$ and $U_r = X$ if $r > 1$. Finally let us notice that we may recover the function g from the sequence $\{U_r\}_{r>0}$ by the formula

$$g(x) = \inf\{r > 0 : x \in U_r\}. \tag{8.3}$$

The idea of the proof to follow is to turn these remarks around and define g by Eq. (8.3).

Step 1. (Construction of the U_r.) Let

$$\mathbb{D} \equiv \left\{k2^{-n} : k = 1, 2, \ldots, 2^{-1}, n = 1, 2, \ldots\right\}$$

be the dyadic rationales in $(0, 1]$. Use Proposition 8.13 to find a precompact open set U_1 such that $K \subset U_1 \subset \bar{U}_1 \subset U$. Apply Proposition 8.13 again to construct an open set $U_{1/2}$ such that

$$K \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_1$$

and similarly use Proposition 8.13 to find open sets $U_{1/2}, U_{3/4} \subset_o X$ such that

$$K \subset U_{1/4} \subset \bar{U}_{1/4} \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_{3/4} \subset \bar{U}_{3/4} \subset U_1.$$

Likewise there exists open set $U_{1/8}, U_{3/8}, U_{5/8}, U_{7/8}$ such that

$$\begin{aligned} K &\subset U_{1/8} \subset \bar{U}_{1/8} \subset U_{1/4} \subset \bar{U}_{1/4} \subset U_{3/8} \subset \bar{U}_{3/8} \subset U_{1/2} \\ &\subset \bar{U}_{1/2} \subset U_{5/8} \subset \bar{U}_{5/8} \subset U_{3/4} \subset \bar{U}_{3/4} \subset U_{7/8} \subset \bar{U}_{7/8} \subset U_1. \end{aligned}$$

Continuing this way inductively, one shows there exists precompact open sets $\{U_r\}_{r \in \mathbb{D}} \subset \tau$ such that

$$K \subset U_r \subset \overline{U}_r \subset U_s \subset U_1 \subset \bar{U}_1 \subset U$$

for all $r, s \in \mathbb{D}$ with $0 < r < s \leq 1$.

Step 2. Let $U_r \equiv X$ if $r > 1$ and define

$$g(x) = \inf\{r \in \mathbb{D} \cup (1, \infty) : x \in U_r\},$$

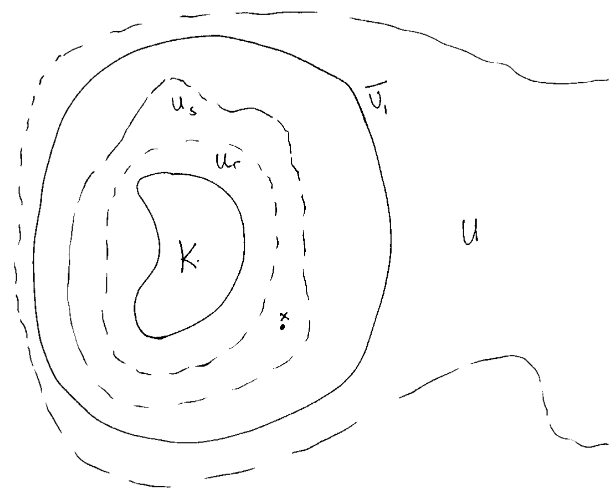

FIGURE 18. Determining g from $\{U_r\}$.

see Figure 18. Then $g(x) \in [0,1]$ for all $x \in X$, $g(x) = 0$ for $x \in K$ since $x \in K \subset U_r$ for all $r \in \mathbb{D}$. If $x \in U_1^c$, then $x \notin U_r$ for all $r \in \mathbb{D}$ and hence $g(x) = 1$. Therefore $f := 1-g$ is a function such that $f = 1$ on K and $\{f \neq 0\} = \{g \neq 1\} \subset U_1 \subset \bar{U}_1 \subset U$ so that $\operatorname{supp}(f) = \overline{\{f \neq 0\}} \subset \bar{U}_1 \subset U$ is a compact subset of U. Thus it only remains to show f, or equivalently g, is continuous.

Since $\mathcal{E} = \{(\alpha,\infty), (-\infty,\alpha) : \alpha \in \mathbb{R}\}$ generates the standard topology on $\mathbb{R}$, to prove g is continuous it suffices to show $\{g < \alpha\}$ and $\{g > \alpha\}$ are open sets for all $\alpha \in \mathbb{R}$. But $g(x) < \alpha$ iff there exists $r \in \mathbb{D} \cup (1,\infty)$ with $r < \alpha$ such that $x \in U_r$. Therefore

$$\{g < \alpha\} = \bigcup \{U_r : r \in \mathbb{D} \cup (1,\infty) \ni r < \alpha\}$$

which is open in X. If $\alpha \geq 1$, $\{g > \alpha\} = \emptyset$ and if $\alpha < 0$, $\{g > \alpha\} = X$. If $\alpha \in (0,1)$, then $g(x) > \alpha$ iff there exists $r \in \mathbb{D}$ such that $r > \alpha$ and $x \notin U_r$. Now if $r > \alpha$ and $x \notin U_r$ then for $s \in \mathbb{D} \cap (\alpha, r)$, $x \notin \bar{U}_s \subset U_r$. Thus we have shown that

$$\{g > \alpha\} = \bigcup \left\{ \left(\overline{U}_s\right)^c : s \in \mathbb{D} \ni s > \alpha \right\}$$

which is again an open subset of X. ■

Exercise 8.7. Give a simpler proof of Lemma 8.15 under the additional assumption that X is a metric space.

Theorem 8.16 (Locally Compact Tietz Extension Theorem)**.** *Let (X,τ) be a locally compact Hausdorff space, $K \sqsubset\sqsubset U \subset_o X$, $f \in C(K,\mathbb{R})$, $a = \min f(K)$ and $b = \max f(K)$. Then there exists $F \in C(X,[a,b])$ such that $F|_K = f$. Moreover given $c \in [a,b]$, F can be chosen so that* $\operatorname{supp}(F-c) = \overline{\{F \neq c\}} \subset U$.

The proof of this theorem is similar to Theorem 8.2 and will be left to the reader, see Exercise 8.10.

Lemma 8.17. *Suppose that (X,τ) is a locally compact second countable Hausdorff space. (For example any separable locally compact metric space and in particular any open subsets of $\mathbb{R}^n$.) Then:*

1. *every open subset* $U \subset X$ *is* σ *– compact.*
2. *If* $F \subset X$ *is a closed set, there exist open sets* $V_n \subset X$ *such that* $V_n \downarrow F$ *as* $n \to \infty$.
3. *To each open set* $U \subset X$ *there exists* $f_n \prec U$ *such that* $\lim_{n\to\infty} f_n = 1_U$.
4. *The* σ *– algebra generated by* $C_c(X)$ *is the Borel* σ *– algebra,* $\mathcal{B}_X$.

Proof.

1. Let U be an open subset of X, $\mathcal{V}$ be a countable base for τ and
$$\mathcal{V}^U := \{W \in \mathcal{V} : \bar{W} \subset U \text{ and } \bar{W} \text{ is compact}\}.$$
For each $x \in U$, by Proposition 8.13, there exists an open neighborhood V of x such that $\bar{V} \subset U$ and $\bar{V}$ is compact. Since $\mathcal{V}$ is a base for the topology τ, there exists $W \in \mathcal{V}$ such that $x \in W \subset V$. Because $\bar{W} \subset \bar{V}$, it follows that $\bar{W}$ is compact and hence $W \in \mathcal{V}^U$. As $x \in U$ was arbitrary, $U = \cup\mathcal{V}^U$.
Let $\{W_n\}_{n=1}^\infty$ be an enumeration of $\mathcal{V}^U$ and set $K_n := \cup_{k=1}^n \bar{W}_k$. Then $K_n \uparrow U$ as $n \to \infty$ and K_n is compact for each n.
2. Let $\{K_n\}_{n=1}^\infty$ be compact subsets of F^c such that $K_n \uparrow F^c$ as $n \to \infty$ and set $V_n := K_n^c = X \setminus K_n$. Then $V_n \downarrow F$ and by Proposition 8.6, V_n is open for each n.
3. Let $U \subset X$ be an open set and $\{K_n\}_{n=1}^\infty$ be compact subsets of U such that $K_n \uparrow U$. By Lemma 8.15, there exist $f_n \prec U$ such that $f_n = 1$ on K_n. These functions satisfy, $1_U = \lim_{n\to\infty} f_n$.
4. By Item 3., 1_U is $\sigma(C_c(X,\mathbb{R}))$ – measurable for all $U \in \tau$. Hence $\tau \subset \sigma(C_c(X,\mathbb{R}))$ and therefore $\mathcal{B}_X = \sigma(\tau) \subset \sigma(C_c(X,\mathbb{R}))$. The converse inclusion always holds since continuous functions are always Borel measurable.

■

Corollary 8.18. *Suppose that* (X,τ) *is a second countable locally compact Hausdorff space,* $\mathcal{B}_X = \sigma(\tau)$ *is the Borel* σ *– algebra on* X *and* $\mathcal{H}$ *is a subspace of* $B(X,\mathbb{R})$ *which is closed under bounded convergence and contains* $C_c(X,\mathbb{R})$. *Then* $\mathcal{H}$ *contains all bounded* $\mathcal{B}_X$ *– measurable real valued functions on* X.

Proof. Since $\mathcal{H}$ is closed under bounded convergence and $C_c(X,\mathbb{R}) \subset \mathcal{H}$, it follows by Item 3. of Lemma 8.17 that $1_U \in \mathcal{H}$ for all $U \in \tau$. Since τ is a π – class the corollary follows by an application of Theorem 6.12. ■

8.1. **Partitions of Unity.**

Definition 8.19. Let (X,τ) be a topological space and $X_0 \subset X$ be a set. A collection of sets $\{B_\alpha\}_{\alpha\in A} \subset 2^X$ is **locally finite** on X_0 if for all $x \in X_0$, there is an open neighborhood $N_x \in \tau$ of x such that $\#\{\alpha \in A : B_\alpha \cap N_x \neq \emptyset\} < \infty$.

Lemma 8.20. *Let* (X,τ) *be a locally compact Hausdorff space.*

1. *A subset* $E \subset X$ *is closed iff* $E \cap K$ *is closed for all* $K \sqsubset\sqsubset X$.
2. *Let* $\{C_\alpha\}_{\alpha\in A}$ *be a locally finite collection of closed subsets of* X, *then* $C = \cup_{\alpha\in A} C_\alpha$ *is closed in* X. *(Recall that in general closed sets are only closed under finite unions.)*

Proof. Item 1. Since compact subsets of Hausdorff spaces are closed, $E \cap K$ is closed if E is closed and K is compact. Now suppose that $E \cap K$ is closed for all compact subsets $K \subset X$ and let $x \in E^c$. Since X is locally compact, there exists a precompact open neighborhood, V, of x.[17] By assumption $E \cap \bar{V}$ is closed so $x \in \left(E \cap \bar{V}\right)^c$ – an open subset of X. By Proposition 8.13 there exists an open set U such that $x \in U \subset \bar{U} \subset \left(E \cap \bar{V}\right)^c$, see Figure 19. Let $W := U \cap V$. Since

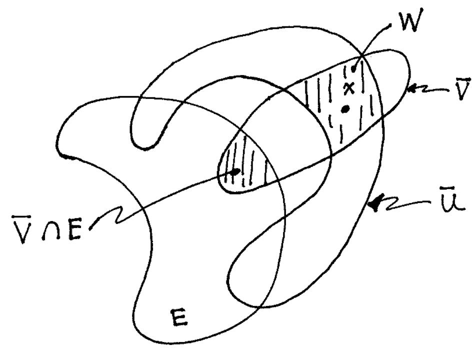

FIGURE 19. Showing E^c is open.

$$W \cap E = U \cap V \cap E \subset U \cap \bar{V} \cap E = \emptyset,$$

and W is an open neighborhood of x and $x \in E^c$ was arbitrary, we have shown E^c is open hence E is closed.

Item 2. Let K be a compact subset of X and for each $x \in K$ let N_x be an open neighborhood of x such that $\#\{\alpha \in A : C_\alpha \cap N_x \neq \emptyset\} < \infty$. Since K is compact, there exists a finite subset $\Lambda \subset K$ such that $K \subset \cup_{x\in\Lambda} N_x$. Letting $\Lambda_0 := \{\alpha \in A : C_\alpha \cap K \neq \emptyset\}$, then

$$\#\left(\Lambda_0\right) \le \sum_{x\in\Lambda} \#\{\alpha \in A : C_\alpha \cap N_x \neq \emptyset\} < \infty$$

and hence $K \cap (\cup_{\alpha\in A} C_\alpha) = K \cap (\cup_{\alpha\in\Lambda_0} C_\alpha)$. The set $(\cup_{\alpha\in\Lambda_0} C_\alpha)$ is a finite union of closed sets and hence closed. Therefore, $K \cap (\cup_{\alpha\in A} C_\alpha)$ is closed and by Item (1) it follows that $\cup_{\alpha\in A} C_\alpha$ is closed as well. ∎

Definition 8.21. Suppose that $\mathcal{U}$ is an open cover of $X_0 \subset X$. A collection $\{\phi_i\}_{i=1}^N \subset C(X,[0,1])$ ($N = \infty$ is allowed here) is a **partition of unity** on X_0 subordinate to the cover $\mathcal{U}$ if:

1. for all i there is a $U \in \mathcal{U}$ such that $\mathrm{supp}(\phi_i) \subset U$,
2. the collection of sets, $\{\mathrm{supp}(\phi_i)\}_{i=1}^N$, is locally finite on X_0, and
3. $\sum_{i=1}^N \phi_i = 1$ on X_0. (Notice by (2), that for each $x \in X_0$ there is a neighborhood N_x such that $\phi_i|_{N_x}$ is not identically zero for only a finite number of terms. So the sum is well defined and we say the sum is **locally finite**.)

Proposition 8.22 (Partitions of Unity: The Compact Case)**.** *Suppose that X is a locally compact Hausdorff space, $K \subset X$ is a compact set and $\mathcal{U} = \{U_j\}_{j=1}^n$ is an open cover of K. Then there exists a partition of unity $\{h_j\}_{j=1}^n$ of K such that $h_j \prec U_j$ for all $j = 1, 2, \ldots, n$.*

[17]If X were a metric space we could finish the proof as follows. If there does not exist an open neighborhood of x which is disjoint from E, then there would exists $x_n \in E$ such that $x_n \to x$. Since $E \cap \bar{V}$ is closed and $x_n \in E \cap \bar{V}$ for all large n, it follows (see Exercise 3.4) that $x \in E \cap \bar{V}$ and in particular that $x \in E$. But we chose $x \in E^c$.

Proof. For all $x \in K$ choose a precompact open neighborhood, V_x, of x such that $\overline{V}_x \subset U_j$. Since K is compact, there exists a finite subset, Λ, of K such that $K \subset \bigcup_{x\in\Lambda} V_x$. Let

$$F_j = \cup\left\{\bar{V}_x : x \in \Lambda \text{ and } \overline{V}_x \subset U_j\right\}.$$

Then F_j is compact, $F_j \subset U_j$ for all j, and $K \subset \cup_{j=1}^n F_j$. By Urysohn's Lemma 8.15 there exists $f_j \prec U_j$ such that $f_j = 1$ on F_j. We will now give two methods to finish the proof.

Method 1. Let $h_1 = f_1$, $h_2 = f_2(1 - h_1) = f_2(1 - f_1)$,

$$h_3 = f_3(1 - h_1 - h_2) = f_3(1 - f_1 - (1 - f_1)f_2) = f_3(1 - f_1)(1 - f_2)$$

and continue on inductively to define

$$h_k = (1 - h_1 - \cdots - h_{k-1})f_k = f_k \cdot \prod_{j=1}^{k-1}(1 - f_j)\,\forall\, k = 2, 3, \ldots, n \tag{8.4}$$

and to show

$$(1 - h_1 - \cdots - h_n) = \prod_{j=1}^{n}(1 - f_j). \tag{8.5}$$

From these equations it clearly follows that $h_j \in C_c(X, [0, 1])$ and that $\mathrm{supp}(h_j) \subset \mathrm{supp}(f_j) \subset U_j$, i.e. $h_j \prec U_j$. Since $\prod_{j=1}^n(1 - f_j) = 0$ on K, $\sum_{j=1}^n h_j = 1$ on K and $\{h_j\}_{j=1}^n$ is the desired partition of unity.

Method 2. Let $g := \sum_{j=1}^{n} f_j \in C_c(X)$. Then $g \geq 1$ on K and hence $K \subset \{g > \frac{1}{2}\}$. Choose $\phi \in C_c(X, [0, 1])$ such that $\phi = 1$ on K and $\mathrm{supp}(\phi) \subset \{g > \frac{1}{2}\}$ and define $f_0 \equiv 1 - \phi$. Then $f_0 = 0$ on K, $f_0 = 1$ if $g \leq \frac{1}{2}$ and therefore,

$$f_0 + f_1 + \cdots + f_n = f_0 + g > 0$$

on X. The desired partition of unity may be constructed as

$$h_j(x) = \frac{f_j(x)}{f_0(x) + \cdots + f_n(x)}.$$

Indeed $\mathrm{supp}\,(h_j) = \mathrm{supp}\,(f_j) \subset U_j$, $h_j \in C_c(X, [0, 1])$ and on K,

$$h_1 + \cdots + h_n = \frac{f_1 + \cdots + f_n}{f_0 + f_1 + \cdots + f_n} = \frac{f_1 + \cdots + f_n}{f_1 + \cdots + f_n} = 1.$$

■

Proposition 8.23. *Let (X, τ) be a locally compact and σ – compact Hausdorff space. Suppose that $\mathcal{U} \subset \tau$ is an open cover of X. Then we may construct two locally finite open covers $\mathcal{V} = \{V_i\}_{i=1}^N$ and $\mathcal{W} = \{W_i\}_{i=1}^N$ of X ($N = \infty$ is allowed here) such that:*

1. *$W_i \subset \bar{W}_i \subset V_i \subset \bar{V}_i$ and $\bar{V}_i$ is compact for all i.*
2. *For each i there exist $U \in \mathcal{U}$ such that $\bar{V}_i \subset U$.*

Proof. By Remark 8.11, there exists an open cover of $\mathcal{G} = \{G_n\}_{n=1}^\infty$ of X such that $G_n \subset \bar{G}_n \subset G_{n+1}$. Then $X = \cup_{k=1}^\infty(\bar{G}_k \setminus \bar{G}_{k-1})$, where by convention $G_{-1} = G_0 = \emptyset$. For the moment fix $k \geq 1$. For each $x \in \bar{G}_k \setminus G_{k-1}$, let $U_x \in \mathcal{U}$ be chosen so that $x \in U_x$ and by Proposition 8.13 choose an open neighborhood N_x of x such that $\bar{N}_x \subset U_x \cap (G_{k+1} \setminus \bar{G}_{k-2})$, see Figure 20 below. Since $\{N_x\}_{x\in\bar{G}_k\setminus G_{k-1}}$ is an open

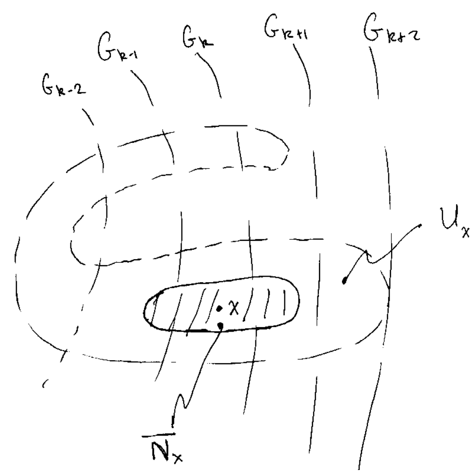

FIGURE 20. Constructing the $\{W_i\}_{i=1}^N$.

cover of the compact set $\bar{G}_k \setminus G_{k-1}$, there exist a finite subset $\Gamma_k \subset \{N_x\}_{x \in \bar{G}_k \setminus G_{k-1}}$ which also covers $\bar{G}_k \setminus G_{k-1}$. By construction, for each $W \in \Gamma_k$, there is a $U \in \mathcal{U}$ such that $\bar{W} \subset U \cap (G_{k+1} \setminus \bar{G}_{k-2})$. Apply Proposition 8.13 one more time to find, for each $W \in \Gamma_k$, an open set V_W such that $\bar{W} \subset V_W \subset \bar{V}_W \subset U \cap (G_{k+1} \setminus \bar{G}_{k-2})$.

We now choose and enumeration $\{W_i\}_{i=1}^N$ of the countable open cover $\cup_{k=1}^\infty \Gamma_k$ of X and define $V_i = V_{W_i}$. Then the collection $\{W_i\}_{i=1}^N$ and $\{V_i\}_{i=1}^N$ are easily checked to satisfy all the conclusions of the proposition. In particular notice that for each k that the set of i's such that $V_i \cap G_k \neq \emptyset$ is finite. ■

Theorem 8.24 (Locally Compact Partitions of Unity)**.** *Let (X, τ) be a locally compact and σ – compact Hausdorff space and $\mathcal{U} \subset \tau$ be an open cover of X. Then there exists a partition of unity of $\{h_i\}_{i=1}^N$ ($N = \infty$ is allowed here) subordinate to the cover $\mathcal{U}$ such that* supp(h_i) *is compact for all i.*

Proof. Let $\mathcal{V} = \{V_i\}_{i=1}^N$ and $\mathcal{W} = \{W_i\}_{i=1}^N$ be open covers of X with the properties described in Proposition 8.23. By Urysohn's Lemma 8.15, there exists $f_i \prec V_i$ such that $f_i = 1$ on $\bar{W}_i$ for each i.

As in the proof of Proposition 8.22 there are two methods to finish the proof.

Method 1. Define $h_1 = f_1$, h_j by Eq. (8.4) for all other j. Then as in Eq. (8.5)

$$1 - \sum_{j=1}^N h_j = \prod_{j=1}^N (1 - f_j) = 0$$

since for $x \in X$, $f_j(x) = 1$ for some j. As in the proof of Proposition 8.22, it is easily checked that $\{h_i\}_{i=1}^N$ is the desired partition of unity.

Method 2. Let $f \equiv \sum_{i=1}^N f_i$, a locally finite sum, so that $f \in C(X)$. Since $\{W_i\}_{i=1}^\infty$ is a cover of X, $f \geq 1$ on X so that $1/f \in C(X))$ as well. The functions $h_i \equiv f_i/f$ for $i = 1, 2, \ldots, N$ give the desired partition of unity. ■

Corollary 8.25. *Let (X, τ) be a locally compact and σ – compact Hausdorff space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A} \subset \tau$ be an open cover of X. Then there exists a partition of unity of $\{h_\alpha\}_{\alpha \in A}$ subordinate to the cover $\mathcal{U}$ such that* supp$(h_\alpha) \subset U_\alpha$ *for all*

$\alpha \in A$. *(Notice that we do not assert that h_α has compact support. However if $\bar{U}_\alpha$ is compact then* $\mathrm{supp}(h_\alpha)$ *will be compact.)*

Proof. By the $\sigma-$compactness of X, we may choose a countable subset, $\{\alpha_i\}_{i<N}$ ($N = \infty$ allowed here), of A such that $\{U_i \equiv U_{\alpha_i}\}_{i<N}$ is still an open cover of X. Let $\{g_j\}_{j<N}$ be a partition of unity subordinate to the cover $\{U_i\}_{i<N}$ as in Theorem 8.24. Define $\tilde{\Gamma}_k \equiv \{j : \mathrm{supp}(g_j) \subset U_k\}$ and $\Gamma_k := \tilde{\Gamma}_k \setminus \left(\cup_{j=1}^{k-1}\tilde{\Gamma}_k\right)$, where by convention $\tilde{\Gamma}_0 = \emptyset$. Then

$$\{i \in \mathbb{N} : i < N\} = \bigcup_{k=1}^{\infty} \tilde{\Gamma}_k = \coprod_{k=1}^{\infty} \Gamma_k.$$

If $\Gamma_k = \emptyset$ let $h_k \equiv 0$ otherwise let $h_k := \sum_{j\in\Gamma_k} g_j$, a locally finite sum. Then $\sum_{k=1}^{\infty} h_k = \sum_{j=1}^{N} g_j = 1$ and the sum $\sum_{k=1}^{\infty} h_k$ is still locally finite. (Why?) Now for $\alpha = \alpha_k \in \{\alpha_i\}_{i=1}^N$, let $h_\alpha := h_k$ and for $\alpha \notin \{\alpha_i\}_{i=1}^N$ let $h_\alpha \equiv 0$. Since

$$\{h_k \neq 0\} = \cup_{j\in\Gamma_k} \{g_j \neq 0\} \subset \cup_{j\in\Gamma_k} \mathrm{supp}(g_j) \subset U_k$$

and, by Item 2. of Lemma 8.20, $\cup_{j\in\Gamma_k} \mathrm{supp}(g_j)$ is closed, we see that

$$\mathrm{supp}(h_k) = \overline{\{h_k \neq 0\}} \subset \cup_{j\in\Gamma_k} \mathrm{supp}(g_j) \subset U_k.$$

Therefore $\{h_\alpha\}_{\alpha\in A}$ is the desired partition of unity. ■

Corollary 8.26. *Let (X, τ) be a locally compact and σ – compact Hausdorff space and A, B be disjoint closed subsets of X. Then there exists $f \in C(X, [0, 1])$ such that $f = 1$ on A and $f = 0$ on B. In fact f can be chosen so that* $\mathrm{supp}(f) \subset B^c$.

Proof. Let $U_1 = A^c$ and $U_2 = B^c$, then $\{U_1, U_2\}$ is an open cover of X. By Corollary 8.25 there exists $h_1, h_2 \in C(X, [0, 1])$ such that $\mathrm{supp}(h_i) \subset U_i$ for $i = 1, 2$ and $h_1 + h_2 = 1$ on X. The function $f = h_2$ satisfies the desired properties. ■

8.2. $C_0(X)$ and the Alexanderov Compactification.

Definition 8.27. Let (X, τ) be a topological space. A continuous function $f : X \to \mathbb{C}$ is said to **vanish at infinity** if $\{|f| \geq \epsilon\}$ is compact in X for all $\epsilon > 0$. The functions, $f \in C(X)$, vanishing at infinity will be denoted by $C_0(X)$.

Proposition 8.28. *Let X be a topological space, $BC(X)$ be the space of bounded continuous functions on X with the supremum norm topology. Then*

1. *$C_0(X)$ is a closed subspace of $BC(X)$.*
2. *If we further assume that X is a locally compact Hausdorff space, then $C_0(X) = \overline{C_c(X)}$.*

Proof.

1. If $f \in C_0(X)$, $K_1 := \{|f| \geq 1\}$ is a compact subset of X and therefore $f(K_1)$ is a compact and hence bounded subset of $\mathbb{C}$ and so $M := \sup_{x\in K_1} |f(x)| < \infty$. Therefore $\|f\|_u \leq M \vee 1 < \infty$ showing $f \in BC(X)$.

   Now suppose $f_n \in C_0(X)$ and $f_n \to f$ in $BC(X)$. Let $\epsilon > 0$ be given and choose n sufficiently large so that $\|f - f_n\|_u \leq \epsilon/2$. Since

   $$|f| \leq |f_n| + |f - f_n| \leq |f_n| + \|f - f_n\|_u \leq |f_n| + \epsilon/2,$$

   $$\{|f| \geq \epsilon\} \subset \{|f_n| + \epsilon/2 \geq \epsilon\} = \{|f_n| \geq \epsilon/2\}.$$

   Because $\{|f| \geq \epsilon\}$ is a closed subset of the compact set $\{|f_n| \geq \epsilon/2\}$, $\{|f| \geq \epsilon\}$ is compact and we have shown $f \in C_0(X)$.

2. Since $C_0(X)$ is a closed subspace of $BC(X)$ and $C_c(X) \subset C_0(X)$, we always have $\overline{C_c(X)} \subset C_0(X)$. Now suppose that $f \in C_0(X)$ and let $K_n \equiv \{|f| \geq \frac{1}{n}\} \sqsubset\sqsubset X$. By Lemma 8.15 we may choose $\phi_n \in C_c(X, [0,1])$ such that $\phi_n \equiv 1$ on K_n. Define $f_n \equiv \phi_n f \in C_c(X)$. Then
$$\|f - f_n\|_u = \|(1-\phi_n)f\|_u \leq \frac{1}{n} \to 0 \text{ as } n \to \infty.$$
This shows that $f \in \overline{C_c(X)}$.

■

Proposition 8.29 (Alexanderov Compactification)**.** *Suppose that (X, τ) is a non-compact locally compact Hausdorff space. Let $X^* = X \cup \{\infty\}$, where $\{\infty\}$ is a new symbol not in X. The collection of sets,*
$$\tau^* = \tau \cup \{X^* \setminus K : K \sqsubset\sqsubset X\} \subset \mathcal{P}(X^*),$$
is a topology on X^* and (X^*, τ^*) is a compact Hausdorff space. Moreover $f \in C(X)$ extends continuously to X^* iff $f = g + c$ with $g \in C_0(X)$ and $c \in \mathbb{C}$ in which case the extension is given by $f(\infty) = c$.

Proof. Let $\mathcal{F} := \{F \subset X^* : X^* \setminus F \in \tau^*\}$, i.e. $F \in \mathcal{F}$ iff F is a compact subset of X or $F = F_0 \cup \{\infty\}$ with F_0 being a closed subset of X. Since the finite union of compact (closed) subsets is compact (closed), it is easily seen that $\mathcal{F}$ is closed under finite unions. Because arbitrary intersections of closed subsets of X are closed and closed subsets of compact subsets of X are compact, it is also easily checked that $\mathcal{F}$ is closed under arbitrary intersections. Therefore $\mathcal{F}$ satisfies the axioms of the closed subsets associated to a topology and hence τ^* is a topology.

Let $i : X \to X^*$ be the inclusion map. Then i is continuous and open, i.e. $i(V)$ is open in X^* for all V open in X. If $f \in C(X^*)$, then $g = f|_X - f(\infty) = f \circ i - f(\infty)$ is continuous on X. Moreover, for all $\epsilon > 0$ there exists an open neighborhood $V \in \tau^*$ of ∞ such that
$$|g(x)| = |f(x) - f(\infty)| < \epsilon \text{ for all } x \in V.$$
Since V is an open neighborhood of ∞, there exists a compact subset, $K \subset X$, such that $V = X^* \setminus K$. By the previous equation we see that $\{x \in X : |g(x)| \geq \epsilon\} \subset K$, so $\{|g| \geq \epsilon\}$ is compact and we have shown g vanishes at ∞.

Conversely if $g \in C_0(X)$, extend g to X^* by setting $g(\infty) = 0$. Given $\epsilon > 0$, the set $K = \{|g| \geq \epsilon\}$ is compact, hence $X^* \setminus K$ is open in X^*. Since $g(X^* \setminus K) \subset (-\epsilon, \epsilon)$ we have shown that g is continuous at ∞. Since g is also continuous at all points in X it follows that g is continuous on X^*. Now it $f = g + c$ with $c \in \mathbb{C}$ and $g \in C_0(X)$, it follows by what we just proved that defining $f(\infty) = c$ extends f to a continuous function on X^*. ■

8.3. **More on Separation Axioms: Normal Spaces.** (The reader may skip to Definition 8.32 if he/she wishes. The following material will not be used in the rest of the book.)

Definition 8.30 ($T_0 - T_2$ Separation Axioms)**.** Let (X, τ) be a topological space. The topology τ is said to be:

1. T_0 if for $x \neq y$ in X there exists $V \in \tau$ such that $x \in V$ and $y \notin V$ or V such that $y \in V$ but $x \notin V$.

2. T_1 if for every $x, y \in X$ with $x \neq y$ there exists $V \in \tau$ such that $x \in V$ and $y \notin V$. Equivalently, τ is T_1 iff all one point subsets of X are closed.[18]
3. T_2 if it is Hausdorff.

Note T_2 implies T_1 which implies T_0. The topology in Example 8.3 is T_0 but not T_1. If X is a finite set and τ is a T_1 – topology on X then $\tau = 2^X$. To prove this let $x \in X$ be fixed. Then for every $y \neq x$ in X there exists $V_y \in \tau$ such that $x \in V_y$ while $y \notin V_y$. Thus $\{x\} = \cap_{y \neq x} V_y \in \tau$ showing τ contains all one point subsets of X and therefore all subsets of X. So we have to look to infinite sets for an example of T_1 topology which is not T_2.

Example 8.31. Let X be any infinite set and let $\tau = \{A \subset X : \#(A^c) < \infty\} \cup \{\emptyset\}$ – the so called **cofinite** topology. This topology is T_1 because if $x \neq y$ in X, then $V = \{x\}^c \in \tau$ with $x \notin V$ while $y \in V$. This topology however is not T_2. Indeed if $U, V \in \tau$ are open sets such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ then $U \subset V^c$. But this implies $\#(U) < \infty$ which is impossible unless $U = \emptyset$ which is impossible since $x \in U$.

The uniqueness of limits of sequences which occurs for Hausdorff topologies (see Remark 8.5) need not occur for T_1 – spaces. For example, let $X = \mathbb{N}$ and τ be the cofinite topology on X as in Example 8.31. Then $x_n = n$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ **for all** $x \in \mathbb{N}$. For the most part we will avoid these pathologies in the future by only considering Hausdorff topologies.

Definition 8.32 (Normal Spaces: T_4 – Separation Axiom)**.** A topological space (X, τ) is said to be **normal** or T_4 if:

1. X is Hausdorff and
2. if for any two closed disjoint subsets $A, B \subset X$ there exists disjoint open sets $V, W \subset X$ such that $A \subset V$ and $B \subset W$.

Example 8.33. By Lemma 8.1 and Corollary 8.26 it follows that metric space and locally compact and σ – compact Hausdorff space (in particular compact Hausdorff spaces) are normal. Indeed, in each case if A, B are disjoint closed subsets of X, there exists $f \in C(X, [0,1])$ such that $f = 1$ on A and $f = 0$ on B. Now let $U = \left\{f > \frac{1}{2}\right\}$ and $V = \{f < \frac{1}{2}\}$.

Remark 8.34. A topological space, (X, τ), is normal iff for any $C \subset W \subset X$ with C being closed and W being open there exists an open set $U \subset_o X$ such that

$$C \subset U \subset \bar{U} \subset W.$$

To prove this first suppose X is normal. Since W^c is closed and $C \cap W^c = \emptyset$, there exists disjoint open sets U and V such that $C \subset U$ and $W^c \subset V$. Therefore $C \subset U \subset V^c \subset W$ and since V^c is closed, $C \subset U \subset \bar{U} \subset V^c \subset W$.

For the converse direction suppose A and B are disjoint closed subsets of X. Then $A \subset B^c$ and B^c is open, and so by assumption there exists $U \subset_o X$ such that $A \subset U \subset \bar{U} \subset B^c$ and by the same token there exists $W \subset_o X$ such that $\bar{U} \subset W \subset \bar{W} \subset B^c$. Taking complements of the last expression implies

$$B \subset \bar{W}^c \subset W^c \subset \bar{U}^c.$$

Let $V = \bar{W}^c$. Then $A \subset U \subset_o X$, $B \subset V \subset_o X$ and $U \cap V \subset U \cap W^c = \emptyset$.

[18]If one point subsets are closed and $x \neq y$ in X then $V := \{x\}^c$ is an open set containing y but not x. Conversely if τ is T_1 and $x \in X$ there exists $V_y \in \tau$ such that $y \in V_y$ and $x \notin V_y$ for all $y \neq x$. Therefore, $\{x\}^c = \cup_{y \neq x} V_y \in \tau$.

Theorem 8.35 (Urysohn's Lemma for Normal Spaces)**.** *Let X be a normal space. Assume A, B are disjoint closed subsets of X. Then there exists $f \in C(X,[0,1])$ such that $f = 0$ on A and $f = 1$ on B.*

Proof. To make the notation match Lemma 8.15, let $U = A^c$ and $K = B$. Then $K \subset U$ and it suffices to produce a function $f \in C(X,[0,1])$ such that $f = 1$ on K and $\text{supp}(f) \subset U$. The proof is now identical to that for Lemma 8.15 except we now use Remark 8.34 in place of Proposition 8.13. ■

Theorem 8.36 (Tietze Extension Theorem)**.** *Let (X,τ) be a normal space, D be a closed subset of X, $-\infty < a < b < \infty$ and $f \in C(D,[a,b])$. Then there exists $F \in C(X,[a,b])$ such that $F|_D = f$.*

Proof. The proof is identical to that of Theorem 8.2 except we now use Theorem 8.35 in place of Lemma 8.1. ■

Corollary 8.37. *Suppose that X is a normal topological space, $D \subset X$ is closed, $F \in C(D,\mathbb{R})$. Then there exists $F \in C(X)$ such that $F|_D = f$.*

Proof. Let $g = \arctan(f) \in C(D,(-\frac{\pi}{2},\frac{\pi}{2}))$. Then by the Tietze extension theorem, there exists $G \in C(X,[-\frac{\pi}{2},\frac{\pi}{2}])$ such that $G|_D = g$. Let $B \equiv G^{-1}(\{-\frac{\pi}{2},\frac{\pi}{2}\}) \sqsubset X$, then $B \cap D = \emptyset$. By Urysohn's lemma (Theorem 8.35) there exists $h \in C(X,[0,1])$ such that $h \equiv 1$ on D and $h = 0$ on B and in particular $hG \in C(D,(-\frac{\pi}{2},\frac{\pi}{2}))$ and $(hG)|_D = g$. The function $F \equiv \tan(hG) \in C(X)$ is an extension of f. ■

Notation 8.38. Let $Q := [0,1]^{\mathbb{N}}$ denote the (infinite dimensional) **unit cube** in $\mathbb{R}^{\mathbb{N}}$. For $a, b \in Q$ let

$$d(a,b) := \sum_{n=1}^{\infty} \frac{1}{2^n} |a_n - b_n| .$$

The metric introduced in Exercise 3.27 would be defined, in this context, as $\tilde{d}(a,b) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|a_n - b_n|}{1+|a_n-b_n|}$. Since $1 \le 1+|a_n - b_n| \le 2$, it follows that $\tilde{d} \le d \le 2d$. So the metrics d and $\tilde{d}$ are equivalent and in particular the topologies induced by d and $\tilde{d}$ are the same. By Exercises 4.15, the d – topology on Q is the same as the product topology and by Exercise 3.27, (Q,d) is a compact metric space.

Theorem 8.39 (Urysohn Metrization Theorem)**.** *Every second countable normal space, (X,τ), is metrizable, i.e. there is a metric ρ on X such that $\tau = \tau_\rho$. Moreover, ρ may be chosen so that X is isometric to a subset $Q_0 \subset Q$. In this metric X is totally bounded and hence the completion of X (which is isometric to $\bar{Q}_0 \subset Q$) is compact.*

Proof. Let $\mathcal{B}$ be a countable base for τ and set

$$\Gamma \equiv \{(U,V) \in \mathcal{B} \times \mathcal{B} \mid \bar{U} \subset V\}.$$

To each $O \in \tau$ and $x \in O$ there exist $(U,V) \in \Gamma$ such that $x \in U \subset V \subset O$. Indeed, since $\mathcal{B}$ is a basis for τ, there exists $V \in \mathcal{B}$ such that $x \in V \subset O$. Because $\{x\} \cap V^c = \emptyset$, there exists disjoint open sets $\widetilde{U}$ and W such that $x \in \widetilde{U}$, $V^c \subset W$ and $\widetilde{U} \cap W = \emptyset$. Choose $U \in \mathcal{B}$ such that $x \in U \subset \widetilde{U}$. Since $U \subset \widetilde{U} \subset W^c$, $\overline{U} \subset W^c \subset V$ and hence $(U,V) \in \Gamma$. See Figure 21 below. In particular this shows that $\{U \in \mathcal{B} : (U,V) \in \Gamma \text{ for some } V \in \mathcal{B}\}$ is still a base for τ.

If Γ is a finite set, the previous comment shows that τ only has a finite number of elements as well. Since (X,τ) is Hausdorff, it follows that X is a finite set.

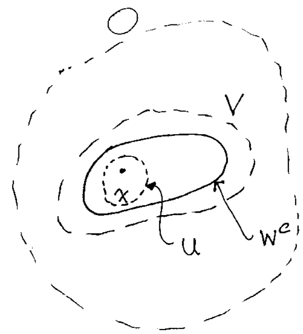

FIGURE 21. Constructing $(U, V) \in \Gamma$.

Letting $\{x_n\}_{n=1}^N$ be an enumeration of X, define $T : X \to Q$ by $T(x_n) = e_n$ for $n = 1, 2, \ldots, N$ where $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, with the 1 ocurring in the n^{th} spot. Then $\rho(x, y) := d(T(x), T(y))$ for $x, y \in X$ is the desired metric. So we will for now on assume that Γ is an infinite set and let $\{(U_n, V_n)\}_{n=1}^\infty$ be an enumeration of Γ.

By Urysohn's Lemma (Theorem 8.35) there exists $f_{U,V} \in C(X, [0, 1])$ such that $f_{U,V} = 0$ on $\bar{U}$ and $f_{U,V} = 1$ on V^c. Let $\mathcal{F} \equiv \{f_{U,V} \mid (U, V) \in \Gamma\}$ and set $f_n := f_{U_n,V_n}$ – an enumeration of $\mathcal{F}$. We will now show that

$$\rho(x, y) := \sum_{n=1}^\infty \frac{1}{2^n} |f_n(x) - f_n(y)|$$

is the desired metric on X. The proof will involve a number of steps.

1. (ρ is a metric on X.) It is routine to show ρ satisfies the triangle inequality and ρ is symmetric. If $x, y \in X$ are distinct points then there exists $(U_{n_0}, V_{n_0}) \in \Gamma$ such that $x \in U_{n_0}$ and $V_{n_0} \subset O := \{y\}^c$. Since $f_{n_0}(x) = 0$ and $f_{n_0}(y) = 1$, it follows that $\rho(x, y) \geq 2^{-n_0} > 0$.
2. (Let $\tau_0 = \tau(f_n : n \in \mathbb{N})$, then $\tau = \tau_0 = \tau_\rho$.) As usual we have $\tau_0 \subset \tau$. Since, for each $x \in X$, $y \to \rho(x, y)$ is τ_0 – continuous (being the uniformly convergent sum of continuous functions), it follows that $B_x(\epsilon) := \{y \in X : \rho(x, y) < \epsilon\} \in \tau_0$ for all $x \in X$ and $\epsilon > 0$. Thus $\tau_\rho \subset \tau_0 \subset \tau$.

   Suppose that $O \in \tau$ and $x \in O$. Let $(U_{n_0}, V_{n_0}) \in \Gamma$ be such that $x \in U_{n_0}$ and $V_{n_0} \subset O$. Then $f_{n_0}(x) = 0$ and $f_{n_0} = 1$ on O^c. Therefore if $y \in X$ and $f_{n_0}(y) < 1$, then $y \in O$ so $x \in \{f_{n_0} < 1\} \subset O$. This shows that O may be written as a union of elements from τ_0 and therefore $O \in \tau_0$. So $\tau \subset \tau_0$ and hence $\tau = \tau_0$. Moreover, if $y \in B_x(2^{-n_0})$ then $2^{-n_0} > \rho(x, y) \geq 2^{-n_0} f_{n_0}(y)$ and therefore $x \in B_x(2^{-n_0}) \subset \{f_{n_0} < 1\} \subset O$. This shows O is ρ – open and hence $\tau_\rho \subset \tau_0 \subset \tau \subset \tau_\rho$.
3. (X is isometric to some $Q_0 \subset Q$.) Let $T : X \to Q$ be defined by $T(x) = (f_1(x), f_2(x), \ldots, f_n(x), \ldots)$. Then T is an isometry by the very definitions of d and ρ and therefore X is isometric to $Q_0 := T(X)$. Since Q_0 is a subset of the compact metric space (Q, d), Q_0 is totally bounded and therefore X is totally bounded.

■

8.4. **Exercises.**

Exercise 8.8. Let (X, τ) be a topological space, $A \subset X$, $i_A : A \to X$ be the inclusion map and $\tau_A := i_A^{-1}(\tau)$ be the relative topology on A. Verify $\tau_A = \{A \cap V : V \in \tau\}$ and show $C \subset A$ is closed in (A, τ_A) iff there exists a closed set $F \subset X$ such that $C = A \cap F$. (If you get stuck, see the remarks after Definition 3.17 where this has already been proved.)

Exercise 8.9. Let (X, τ) and (Y, τ') be a topological spaces, $f : X \to Y$ be a function, $\mathcal{U}$ be an open cover of X and $\{F_j\}_{j=1}^n$ be a finite cover of X by closed sets.

1. If $A \subset X$ is any set and $f : X \to Y$ is (τ, τ') – continuous then $f|_A : A \to Y$ is (τ_A, τ') – continuous.
2. Show $f : X \to Y$ is (τ, τ') – continuous iff $f|_U : U \to Y$ is (τ_U, τ') – continuous for all $U \in \mathcal{U}$.
3. Show $f : X \to Y$ is (τ, τ') – continuous iff $f|_{F_j} : F_j \to Y$ is (τ_{F_j}, τ') – continuous for all $j = 1, 2, \ldots, n$.
4. (A baby form of the Tietze extension Theorem.) Suppose $V \in \tau$ and $f : V \to \mathbb{C}$ is a continuous function such supp$(f) \subset V$, then $F : X \to \mathbb{C}$ defined by
$$F(x) = \begin{cases} f(x) & \text{if} \quad x \in V \\ 0 & \text{otherwise} \end{cases}$$
is continuous.

Exercise 8.10. Prove Theorem 8.16. **Hints:**

1. By Proposition 8.13, there exists a precompact open set V such that $K \subset V \subset \bar{V} \subset U$. Now suppose that $f : K \to [0, \alpha]$ is continuous with $\alpha \in (0, 1]$ and let $A := f^{-1}([0, \frac{1}{3}\alpha])$ and $B := f^{-1}([\frac{2}{3}\alpha, 1])$. Appeal to Lemma 8.15 to find a function $g \in C(X, [0, \alpha/3])$ such that $g = \alpha/3$ on B and supp$(g) \subset V \setminus A$.
2. Now follow the argument in the proof of Theorem 8.2 to construct $F \in C(X, [a, b])$ such that $F|_K = f$.
3. For $c \in [a, b]$, choose $\phi \prec U$ such that $\phi = 1$ on K and replace F by $F_c := \phi F + (1 - \phi)c$.

Exercise 8.11 (Sterographic Projection)**.** Let $X = \mathbb{R}^n$, $X^* := X \cup \{\infty\}$ be the one point compactification of X, $S^n := \{y \in \mathbb{R}^{n+1} : |y| = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$ and $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Define $f : S^n \to X^*$ by $f(N) = \infty$, and for $y \in S^n \setminus \{N\}$ let $f(y) = b \in \mathbb{R}^n$ be the unique point such that $(b, 0)$ is on the line containing N and y, see Figure 22 below. Find a formula for f and show $f : S^n \to X^*$ is a homeomorphism. (So the one point compactification of $\mathbb{R}^n$ is homeomorphic to the n sphere.)

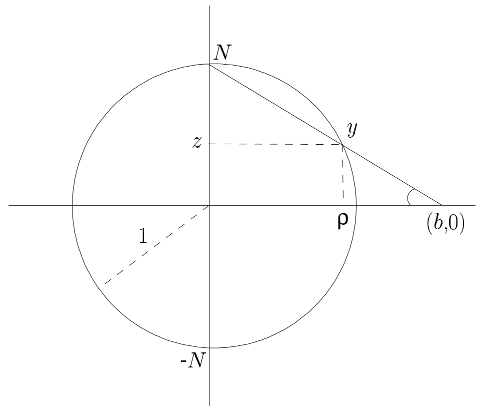

FIGURE 22. Sterographic projection and the one point compactification of $\mathbb{R}^n$.