Exercise 9.12. Again let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$. Further assume that $C_M := \int_{\mathbb{R}^d} e^{M|x|} d\mu(x) < \infty$ for all $M \in (0,\infty)$. Let $\mathcal{P}(\mathbb{R}^d)$ be the space of polynomials, $\rho(x) = \sum_{|\alpha| \le N} \rho_\alpha x^\alpha$ with $\rho_\alpha \in \mathbb{C}$, on $\mathbb{R}^d$. (Notice that $|\rho(x)|^p \le C(\rho, p, M) e^{M|x|}$, so that $\mathcal{P}(\mathbb{R}^d) \subset L^p(\mu)$ for all $1 \le p < \infty$.) Show $\mathcal{P}(\mathbb{R}^d)$ is dense in $L^p(\mu)$ for all $1 \le p < \infty$. Here is a possible outline.

Outline: For $\lambda \in \mathbb{R}^d$ and $n \in \mathbb{N}$ let $f_\lambda^n(x) = (\lambda \cdot x)^n / n!$

1. Use calculus to verify $\sup_{t \ge 0} t^\alpha e^{-Mt} = (\alpha/M)^\alpha e^{-\alpha}$ for all $\alpha \ge 0$ where $(0/M)^0 := 1$. Use this estimate along with the identity
$$|\lambda \cdot x|^{pn} \le |\lambda|^{pn} |x|^{pn} = \left(|x|^{pn} e^{-M|x|}\right) |\lambda|^{pn} e^{M|x|}$$
to find an estimate on $\|f_\lambda^n\|_p$.
2. Use your estimate on $\|f_\lambda^n\|_p$ to show $\sum_{n=0}^\infty \|f_\lambda^n\|_p < \infty$ and conclude
$$\lim_{N \to \infty} \left\| e^{i\lambda \cdot (\cdot)} - \sum_{n=0}^N f_\lambda^n \right\|_p = 0.$$
3. Now finish by appealing to Exercise 9.10.

Exercise 9.13. Again let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$ but now assume there exists an $\epsilon > 0$ such that $C := \int_{\mathbb{R}^d} e^{\epsilon|x|} d\mu(x) < \infty$. Also let $q > 1$ and $h \in L^q(\mu)$ be a function such that $\int_{\mathbb{R}^d} h(x) x^\alpha d\mu(x) = 0$ for all $\alpha \in \mathbb{N}_0^d$. (As mentioned in Exercise 9.13, $\mathcal{P}(\mathbb{R}^d) \subset L^p(\mu)$ for all $1 \le p < \infty$, so $x \to h(x) x^\alpha$ is in $L^1(\mu)$.) Show $h(x) = 0$ for μ– a.e. x using the following outline.

Outline: For $\lambda \in \mathbb{R}^d$ and $n \in \mathbb{N}$ let $f_n^\lambda(x) = (\lambda \cdot x)^n / n!$ and let $p = q/(q-1)$ be the conjugate exponent to q.

1. Use calculus to verify $\sup_{t \ge 0} t^\alpha e^{-\epsilon t} = (\alpha/\epsilon)^\alpha e^{-\alpha}$ for all $\alpha \ge 0$ where $(0/\epsilon)^0 := 1$. Use this estimate along with the identity
$$|\lambda \cdot x|^{pn} \le |\lambda|^{pn} |x|^{pn} = \left(|x|^{pn} e^{-\epsilon|x|}\right) |\lambda|^{pn} e^{\epsilon|x|}$$
to find an estimate on $\left\|f_n^\lambda\right\|_p$.
2. Use your estimate on $\left\|f_n^\lambda\right\|_p$ to show there exists $\delta > 0$ such that $\sum_{n=0}^\infty \left\|f_n^\lambda\right\|_p < \infty$ when $|\lambda| \le \delta$ and conclude for $|\lambda| \le \delta$ that $e^{i\lambda \cdot x} = L^p(\mu)$–$\sum_{n=0}^\infty f_n^\lambda(x)$. Conclude from this that
$$\int_{\mathbb{R}^d} h(x) e^{i\lambda \cdot x} d\mu(x) = 0 \text{ when } |\lambda| \le \delta.$$
3. Let $\lambda \in \mathbb{R}^d$ ($|\lambda|$ not necessarily small) and set $g(t) := \int_{\mathbb{R}^d} e^{it\lambda \cdot x} h(x) d\mu(x)$ for $t \in \mathbb{R}$. Show $g \in C^\infty(\mathbb{R})$ and
$$g^{(n)}(t) = \int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{it\lambda \cdot x} h(x) d\mu(x) \text{ for all } n \in \mathbb{N}.$$
4. Let $T = \sup\{\tau \ge 0 : g|_{[0,\tau]} \equiv 0\}$. By Step 2., $T \ge \delta$. If $T < \infty$, then
$$0 = g^{(n)}(T) = \int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{iT\lambda \cdot x} h(x) d\mu(x) \text{ for all } n \in \mathbb{N}.$$
Use Step 3. with h replaced by $e^{iT\lambda \cdot x} h(x)$ to conclude
$$g(T+t) = \int_{\mathbb{R}^d} e^{i(T+t)\lambda \cdot x} h(x) d\mu(x) = 0 \text{ for all } t \le \delta / |\lambda|.$$

This violates the definition of T and therefore $T = \infty$ and in particular we may take $T = 1$ to learn

$$\int_{\mathbb{R}^d} h(x) e^{i\lambda \cdot x} d\mu(x) = 0 \text{ for all } \lambda \in \mathbb{R}^d.$$

5. Use Exercise 9.10 to conclude that

$$\int_{\mathbb{R}^d} h(x) g(x) d\mu(x) = 0$$

for all $g \in L^p(\mu)$. Now choose g judiciously to finish the proof.