

Bruce K. Driver

# Math 280 (Probability Theory) Lecture Notes

October 6, 2006 *File:prob.tex*



---

# Contents

---

## Part I Background Material

---

<b>1</b>	<b>Limsups, Liminfs and Extended Limits</b> .....	<b>3</b>
<b>2</b>	<b>Basic Probabilistic Notions</b> .....	<b>7</b>

---

## Part II Formal Development

---

<b>3</b>	<b>Preliminaries</b> .....	<b>13</b>
3.1	Set Operations .....	13
3.2	Exercises .....	15
3.3	Algebraic sub-structures of sets .....	15
<b>4</b>	<b>Finitely Additive Measures</b> .....	<b>19</b>
4.1	Finitely Additive Measures .....	19
4.2	Examples of Measures .....	20
4.3	Simple Integration .....	22
4.4	Simple Independence and the Weak Law of Large Numbers .....	24
4.5	Constructing Finitely Additive Measures .....	26
<b>5</b>	<b>Countably Additive Measures</b> .....	<b>29</b>
5.1	Distribution Function for Probability Measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ .....	29
5.2	Construction of Premeasures .....	29
5.3	Regularity and Uniqueness Results .....	31
5.4	Construction of Measures .....	32
5.5	Completions of Measure Spaces .....	34



Background Material



## Limsups, Liminfs and Extended Limits

**Notation 1.1** The *extended real numbers* is the set  $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ , i.e. it is  $\mathbb{R}$  with two new points called  $\infty$  and  $-\infty$ . We use the following conventions,  $\pm\infty \cdot 0 = 0$ ,  $\pm\infty \cdot a = \pm\infty$  if  $a \in \mathbb{R}$  with  $a > 0$ ,  $\pm\infty \cdot a = \mp\infty$  if  $a \in \mathbb{R}$  with  $a < 0$ ,  $\pm\infty + a = \pm\infty$  for any  $a \in \mathbb{R}$ ,  $\infty + \infty = \infty$  and  $-\infty - \infty = -\infty$  while  $\infty - \infty$  is not defined. A sequence  $a_n \in \bar{\mathbb{R}}$  is said to converge to  $\infty$  ( $-\infty$ ) if for all  $M \in \mathbb{R}$  there exists  $m \in \mathbb{N}$  such that  $a_n \geq M$  ( $a_n \leq M$ ) for all  $n \geq m$ .

**Lemma 1.2.** Suppose  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are convergent sequences in  $\bar{\mathbb{R}}$ , then:

1. If  $a_n \leq b_n$  for<sup>1</sup> a.a.  $n$  then  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ .
2. If  $c \in \bar{\mathbb{R}}$ ,  $\lim_{n \rightarrow \infty} (ca_n) = c \lim_{n \rightarrow \infty} a_n$ .
3. If  $\{a_n + b_n\}_{n=1}^{\infty}$  is convergent and

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n \quad (1.1)$$

provided the right side is not of the form  $\infty - \infty$ .

4.  $\{a_n b_n\}_{n=1}^{\infty}$  is convergent and

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n \quad (1.2)$$

provided the right hand side is not of the for  $\pm\infty \cdot 0$  of  $0 \cdot (\pm\infty)$ .

Before going to the proof consider the simple example where  $a_n = n$  and  $b_n = -\alpha n$  with  $\alpha > 0$ . Then

$$\lim (a_n + b_n) = \begin{cases} \infty & \text{if } \alpha < 1 \\ 0 & \text{if } \alpha = 1 \\ -\infty & \text{if } \alpha > 1 \end{cases}$$

while

$$\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = \infty - \infty.$$

This shows that the requirement that the right side of Eq. (1.1) is not of form  $\infty - \infty$  is necessary in Lemma 1.2. Similarly by considering the examples  $a_n = n$

<sup>1</sup> Here we use ‘‘a.a.  $n$ ’’ as an abbreviation for almost all  $n$ . So  $a_n \leq b_n$  a.a.  $n$  iff there exists  $N < \infty$  such that  $a_n \leq b_n$  for all  $n \geq N$ .

and  $b_n = n^{-\alpha}$  with  $\alpha > 0$  shows the necessity for assuming right hand side of Eq. (1.2) is not of the form  $\infty \cdot 0$ .

**Proof.** The proofs of items 1. and 2. are left to the reader.

**Proof of Eq. (1.1).** Let  $a := \lim_{n \rightarrow \infty} a_n$  and  $b = \lim_{n \rightarrow \infty} b_n$ . Case 1., suppose  $b = \infty$  in which case we must assume  $a > -\infty$ . In this case, for every  $M > 0$ , there exists  $N$  such that  $b_n \geq M$  and  $a_n \geq a - 1$  for all  $n \geq N$  and this implies

$$a_n + b_n \geq M + a - 1 \text{ for all } n \geq N.$$

Since  $M$  is arbitrary it follows that  $a_n + b_n \rightarrow \infty$  as  $n \rightarrow \infty$ . The cases where  $b = -\infty$  or  $a = \pm\infty$  are handled similarly. Case 2. If  $a, b \in \mathbb{R}$ , then for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$|a - a_n| \leq \varepsilon \text{ and } |b - b_n| \leq \varepsilon \text{ for all } n \geq N.$$

Therefore,

$$|a + b - (a_n + b_n)| = |a - a_n + b - b_n| \leq |a - a_n| + |b - b_n| \leq 2\varepsilon$$

for all  $n \geq N$ . Since  $n$  is arbitrary, it follows that  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ .

**Proof of Eq. (1.2).** It will be left to the reader to prove the case where  $\lim a_n$  and  $\lim b_n$  exist in  $\mathbb{R}$ . I will only consider the case where  $a = \lim_{n \rightarrow \infty} a_n \neq 0$  and  $\lim_{n \rightarrow \infty} b_n = \infty$  here. Let us also suppose that  $a > 0$  (the case  $a < 0$  is handled similarly) and let  $\alpha := \min(\frac{a}{2}, 1)$ . Given any  $M < \infty$ , there exists  $N \in \mathbb{N}$  such that  $a_n \geq \alpha$  and  $b_n \geq M$  for all  $n \geq N$  and for this choice of  $N$ ,  $a_n b_n \geq M\alpha$  for all  $n \geq N$ . Since  $\alpha > 0$  is fixed and  $M$  is arbitrary it follows that  $\lim_{n \rightarrow \infty} (a_n b_n) = \infty$  as desired. ■

For any subset  $A \subset \bar{\mathbb{R}}$ , let  $\sup A$  and  $\inf A$  denote the least upper bound and greatest lower bound of  $A$  respectively. The convention being that  $\sup A = \infty$  if  $\infty \in A$  or  $A$  is not bounded from above and  $\inf A = -\infty$  if  $-\infty \in A$  or  $A$  is not bounded from below. We will also use the **conventions** that  $\sup \emptyset = -\infty$  and  $\inf \emptyset = +\infty$ .

**Notation 1.3** Suppose that  $\{x_n\}_{n=1}^{\infty} \subset \bar{\mathbb{R}}$  is a sequence of numbers. Then

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf\{x_k : k \geq n\} \text{ and} \quad (1.3)$$

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\}. \quad (1.4)$$

We will also write  $\underline{\lim}$  for  $\liminf_{n \rightarrow \infty}$  and  $\overline{\lim}$  for  $\limsup_{n \rightarrow \infty}$ .

*Remark 1.4.* Notice that if  $a_k := \inf\{x_k : k \geq n\}$  and  $b_k := \sup\{x_k : k \geq n\}$ , then  $\{a_k\}$  is an increasing sequence while  $\{b_k\}$  is a decreasing sequence. Therefore the limits in Eq. (1.3) and Eq. (1.4) always exist in  $\bar{\mathbb{R}}$  and

$$\liminf_{n \rightarrow \infty} x_n = \sup_n \inf\{x_k : k \geq n\} \text{ and}$$

$$\limsup_{n \rightarrow \infty} x_n = \inf_n \sup\{x_k : k \geq n\}.$$

The following proposition contains some basic properties of liminfs and limsups.

**Proposition 1.5.** *Let  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  be two sequences of real numbers. Then*

1.  $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$  and  $\lim_{n \rightarrow \infty} a_n$  exists in  $\bar{\mathbb{R}}$  iff

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n \in \bar{\mathbb{R}}.$$

2. There is a subsequence  $\{a_{n_k}\}_{k=1}^{\infty}$  of  $\{a_n\}_{n=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = \limsup_{n \rightarrow \infty} a_n$ . Similarly, there is a subsequence  $\{a_{n_k}\}_{k=1}^{\infty}$  of  $\{a_n\}_{n=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = \liminf_{n \rightarrow \infty} a_n$ .

- 3.
- $$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n \quad (1.5)$$

whenever the right side of this equation is not of the form  $\infty - \infty$ .

4. If  $a_n \geq 0$  and  $b_n \geq 0$  for all  $n \in \mathbb{N}$ , then

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \limsup_{n \rightarrow \infty} a_n \cdot \limsup_{n \rightarrow \infty} b_n, \quad (1.6)$$

provided the right hand side of (1.6) is not of the form  $0 \cdot \infty$  or  $\infty \cdot 0$ .

**Proof.** Item 1. will be proved here leaving the remaining items as an exercise to the reader. Since

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\} \forall n,$$

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n.$$

Now suppose that  $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = a \in \mathbb{R}$ . Then for all  $\varepsilon > 0$ , there is an integer  $N$  such that

$$a - \varepsilon \leq \inf\{a_k : k \geq N\} \leq \sup\{a_k : k \geq N\} \leq a + \varepsilon,$$

i.e.

$$a - \varepsilon \leq a_k \leq a + \varepsilon \text{ for all } k \geq N.$$

Hence by the definition of the limit,  $\lim_{k \rightarrow \infty} a_k = a$ . If  $\liminf_{n \rightarrow \infty} a_n = \infty$ , then we know for all  $M \in (0, \infty)$  there is an integer  $N$  such that

$$M \leq \inf\{a_k : k \geq N\}$$

and hence  $\lim_{n \rightarrow \infty} a_n = \infty$ . The case where  $\limsup_{n \rightarrow \infty} a_n = -\infty$  is handled similarly.

Conversely, suppose that  $\lim_{n \rightarrow \infty} a_n = A \in \bar{\mathbb{R}}$  exists. If  $A \in \mathbb{R}$ , then for every  $\varepsilon > 0$  there exists  $N(\varepsilon) \in \mathbb{N}$  such that  $|A - a_n| \leq \varepsilon$  for all  $n \geq N(\varepsilon)$ , i.e.

$$A - \varepsilon \leq a_n \leq A + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

From this we learn that

$$A - \varepsilon \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq A + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, it follows that

$$A \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq A,$$

i.e. that  $A = \liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$ . If  $A = \infty$ , then for all  $M > 0$  there exists  $N = N(M)$  such that  $a_n \geq M$  for all  $n \geq N$ . This show that  $\liminf_{n \rightarrow \infty} a_n \geq M$  and since  $M$  is arbitrary it follows that

$$\infty \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n.$$

The proof for the case  $A = -\infty$  is analogous to the  $A = \infty$  case. ■

**Proposition 1.6 (Tonelli's theorem for sums).** *If  $\{a_{kn}\}_{k,n=1}^{\infty}$  is any sequence of non-negative numbers, then*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{kn}.$$

Here we allow for one and hence both sides to be infinite.

**Proof.** Let

$$M := \sup \left\{ \sum_{k=1}^K \sum_{n=1}^N a_{kn} : K, N \in \mathbb{N} \right\} = \sup \left\{ \sum_{n=1}^N \sum_{k=1}^K a_{kn} : K, N \in \mathbb{N} \right\}$$

and

$$L := \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn}.$$



Since

$$L = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn} = \lim_{K \rightarrow \infty} \sum_{k=1}^K \sum_{n=1}^{\infty} a_{kn} = \lim_{K \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{k=1}^K \sum_{n=1}^N a_{kn}$$

and  $\sum_{k=1}^K \sum_{n=1}^N a_{kn} \leq M$  for all  $K$  and  $N$ , it follows that  $L \leq M$ . Conversely,

$$\sum_{k=1}^K \sum_{n=1}^N a_{kn} \leq \sum_{k=1}^K \sum_{n=1}^{\infty} a_{kn} \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn} = L$$

and therefore taking the supremum of the left side of this inequality over  $K$  and  $N$  shows that  $M \leq L$ . Thus we have shown

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn} = M.$$

By symmetry (or by a similar argument), we also have that  $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{kn} = M$  and hence the proof is complete. ■



## Basic Probabilistic Notions

**Definition 2.1.** A sample space  $\Omega$  is a set which represents all possible outcomes of an “experiment.”



*Example 2.2.* 1. The sample space for flipping a coin one time could be taken to be,  $\Omega = \{0, 1\}$ .  
 2. The sample space for flipping a coin  $N$ -times could be taken to be,  $\Omega = \{0, 1\}^N$  and for flipping an infinite number of times,

$$\Omega = \{\omega = (\omega_1, \omega_2, \dots) : \omega_i \in \{0, 1\}\} = \{0, 1\}^{\mathbb{N}}.$$

3. If we have a roulette wheel with 40 entries, then we might take

$$\Omega = \{00, 0, 1, 2, \dots, 36\}$$

for one spin,

$$\Omega = \{00, 0, 1, 2, \dots, 36\}^N$$

for  $N$  spins, and

$$\Omega = \{00, 0, 1, 2, \dots, 36\}^{\mathbb{N}}$$

for an infinite number of spins.

4. If we throw darts at a board of radius  $R$ , we may take

$$\Omega = D_R := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R\}$$

for one throw,

$$\Omega = D_R^{\mathbb{N}}$$

for  $N$  throws, and

$$\Omega = D_R^{\mathbb{N}}$$

for an infinite number of throws.

5. Suppose we release a perfume particle at location  $x \in \mathbb{R}^3$  and follow its motion for all time,  $0 \leq t < \infty$ . In this case, we might take,

$$\Omega = \{\omega \in C([0, \infty), \mathbb{R}^3) : \omega(0) = x\}.$$

**Definition 2.3.** An event is a subset of  $\Omega$ .

*Example 2.4.* Suppose that  $\Omega = \{0, 1\}^{\mathbb{N}}$  is the sample space for flipping a coin an infinite number of times. Here  $\omega_n = 1$  represents the fact that a head was thrown on the  $n^{\text{th}}$ -toss, while  $\omega_n = 0$  represents a tail on the  $n^{\text{th}}$ -toss.

1.  $A = \{\omega \in \Omega : \omega_3 = 1\}$  represents the event that the third toss was a head.
2.  $A = \cup_{i=1}^{\infty} \{\omega \in \Omega : \omega_i = \omega_{i+1} = 1\}$  represents the event that (at least) two heads are tossed twice in a row at some time.
3.  $A = \cap_{N=1}^{\infty} \cup_{n \geq N} \{\omega \in \Omega : \omega_n = 1\}$  is the event where there are infinitely many heads tossed in the sequence.
4.  $A = \cup_{N=1}^{\infty} \cap_{n \geq N} \{\omega \in \Omega : \omega_n = 1\}$  is the event where heads occurs from some time onwards, i.e.  $\omega \in A$  iff there exists,  $N = N(\omega)$  such that  $\omega_n = 1$  for all  $n \geq N$ .

Ideally we would like to assign a probability,  $P(A)$ , to all events  $A \subset \Omega$ . Given a physical experiment, we think of assigning this probability as follows. Run the experiment many times to get sample points,  $\omega(n) \in \Omega$  for each  $n \in \mathbb{N}$ , then try to “define”  $P(A)$  by

$$P(A) = \lim_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq k \leq N : \omega(k) \in A\}. \quad (2.1)$$

That is we think of  $P(A)$  as being the long term relative frequency that the event  $A$  occurred for the sequence of experiments,  $\{\omega(k)\}_{k=1}^{\infty}$ .

Similarly supposed that  $A$  and  $B$  are two events and we wish to know how likely the event  $A$  is given that we now that  $B$  has occurred. Thus we would like to compute:

$$P(A|B) = \lim_{n \rightarrow \infty} \frac{\# \{k : 1 \leq k \leq n \text{ and } \omega_k \in A \cap B\}}{\# \{k : 1 \leq k \leq n \text{ and } \omega_k \in B\}},$$

which represents the frequency that  $A$  occurs given that we know that  $B$  has occurred. This may be rewritten as

$$\begin{aligned} P(A|B) &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \# \{k : 1 \leq k \leq n \text{ and } \omega_k \in A \cap B\}}{\frac{1}{n} \# \{k : 1 \leq k \leq n \text{ and } \omega_k \in B\}} \\ &= \frac{P(A \cap B)}{P(B)}. \end{aligned}$$

**Definition 2.5.** If  $B$  is a non-null event, i.e.  $P(B) > 0$ , define the **conditional probability of  $A$  given  $B$**  by,

$$P(A|B) := \frac{P(A \cap B)}{P(B)}.$$

There are of course a number of problems with this definition of  $P$  in Eq. (2.1) including the fact that it is not mathematical nor necessarily well defined. For example the limit may not exist. But ignoring these technicalities for the moment, let us point out three key properties that  $P$  should have.

1.  $P(A) \in [0, 1]$  for all  $A \subset \Omega$ .
2.  $P(\emptyset) = 0$  and  $P(\Omega) = 1$ .
3. **Additivity.** If  $A$  and  $B$  are disjoint event, i.e.  $A \cap B = AB = \emptyset$ , then

$$\begin{aligned} P(A \cup B) &= \lim_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq k \leq N : \omega(k) \in A \cup B\} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} [\# \{1 \leq k \leq N : \omega(k) \in A\} + \# \{1 \leq k \leq N : \omega(k) \in B\}] \\ &= P(A) + P(B). \end{aligned}$$

*Example 2.6.* Let us consider the tossing of a coin  $N$  times with a fair coin. In this case we would expect that every  $\omega \in \Omega$  is equally likely, i.e.  $P(\{\omega\}) = \frac{1}{2^N}$ . Assuming this we are then forced to define

$$P(A) = \frac{1}{2^N} \#(A).$$

Observe that this probability has the following property. Suppose that  $\sigma \in \{0, 1\}^k$  is a given sequence, then

$$P(\{\omega : (\omega_1, \dots, \omega_k) = \sigma\}) = \frac{1}{2^N} \cdot 2^{N-k} = \frac{1}{2^k}.$$

That is if we ignore the flips after time  $k$ , the resulting probabilities are the same as if we only flipped the coin  $k$  times.

*Example 2.7.* The previous example suggests that if we flip a fair coin an infinite number of times, so that now  $\Omega = \{0, 1\}^{\mathbb{N}}$ , then we should define

$$P(\{\omega \in \Omega : (\omega_1, \dots, \omega_k) = \sigma\}) = \frac{1}{2^k} \quad (2.2)$$

for any  $k \geq 1$  and  $\sigma \in \{0, 1\}^k$ . Assuming there exists a probability,  $P : 2^\Omega \rightarrow [0, 1]$  such that Eq. (2.2) holds, we would like to compute, for example, the probability of the event  $B$  where an infinite number of heads are tossed. To try to compute this, let

$$\begin{aligned} A_n &= \{\omega \in \Omega : \omega_n = 1\} = \{\text{heads at time } n\} \\ B_N &:= \cup_{n \geq N} A_n = \{\text{at least one heads at time } N \text{ or later}\} \end{aligned}$$

and

$$B = \cap_{N=1}^{\infty} B_N = \{A_n \text{ i.o.}\} = \cap_{N=1}^{\infty} \cup_{n \geq N} A_n.$$

Since

$$B_N^c = \cap_{n \geq N} A_n^c \subset \cap_{M \geq n \geq N} A_n^c = \{\omega \in \Omega : \omega_N = \dots = \omega_M = 1\},$$

we see that

$$P(B_N^c) \leq \frac{1}{2^{M-N}} \rightarrow 0 \text{ as } M \rightarrow \infty.$$

Therefore,  $P(B_N) = 1$  for all  $N$ . If we assume that  $P$  is continuous under taking decreasing limits we may conclude, using  $B_N \downarrow B$ , that

$$P(B) = \lim_{N \rightarrow \infty} P(B_N) = 1.$$

Without this continuity assumption we would not be able to compute  $P(B)$ .

The unfortunate fact is that we can not always assign a desired probability function,  $P(A)$ , for all  $A \subset \Omega$ . For example we have the following negative theorem.

**Theorem 2.8 (No-Go Theorem).** Let  $S = \{z \in \mathbb{C} : |z| = 1\}$  be the unit circle. Then there is no probability function,  $P : 2^S \rightarrow [0, 1]$  such that  $P(S) = 1$ ,  $P$  is invariant under rotations, and  $P$  is continuous under taking decreasing limits.

**Proof.** We are going to use the fact proved below in Lemma , that the continuity condition on  $P$  is equivalent to the  $\sigma$ -additivity of  $P$ . For  $z \in S$  and  $N \subset S$  let

$$zN := \{zn \in S : n \in N\}, \quad (2.3)$$

that is to say  $e^{i\theta}N$  is the set  $N$  rotated counter clockwise by angle  $\theta$ . By assumption, we are supposing that

$$P(zN) = P(N) \quad (2.4)$$

for all  $z \in S$  and  $N \subset S$ .

Let

$$R := \{z = e^{i2\pi t} : t \in \mathbb{Q}\} = \{z = e^{i2\pi t} : t \in [0, 1) \cap \mathbb{Q}\}$$

– a countable subgroup of  $S$ . As above  $R$  acts on  $S$  by rotations and divides  $S$  up into equivalence classes, where  $z, w \in S$  are equivalent if  $z = rw$  for some  $r \in R$ . Choose (using the axiom of choice) one representative point  $n$  from each of these equivalence classes and let  $N \subset S$  be the set of these representative points. Then every point  $z \in S$  may be uniquely written as  $z = nr$  with  $n \in N$  and  $r \in R$ . That is to say

$$S = \coprod_{r \in R} (rN) \quad (2.5)$$

where  $\coprod_{\alpha} A_{\alpha}$  is used to denote the union of pair-wise disjoint sets  $\{A_{\alpha}\}$ . By Eqs. (2.4) and (2.5),

$$1 = P(S) = \sum_{r \in R} P(rN) = \sum_{r \in R} P(N). \quad (2.6)$$

We have thus arrived at a contradiction, since the right side of Eq. (2.6) is either equal to 0 or to  $\infty$  depending on whether  $P(N) = 0$  or  $P(N) > 0$ . ■

To avoid this problem, we are going to have to relinquish the idea that  $P$  should necessarily be defined on all of  $2^{\Omega}$ . So we are going to only define  $P$  on particular subsets,  $\mathcal{B} \subset 2^{\Omega}$ . We will develop this below.



Formal Development





## Preliminaries

### 3.1 Set Operations

Let  $\mathbb{N}$  denote the positive integers,  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$  be the non-negative integers and  $\mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N})$  – the positive and negative integers including 0,  $\mathbb{Q}$  the rational numbers,  $\mathbb{R}$  the real numbers, and  $\mathbb{C}$  the complex numbers. We will also use  $\mathbb{F}$  to stand for either of the fields  $\mathbb{R}$  or  $\mathbb{C}$ .

**Notation 3.1** Given two sets  $X$  and  $Y$ , let  $Y^X$  denote the collection of all functions  $f : X \rightarrow Y$ . If  $X = \mathbb{N}$ , we will say that  $f \in Y^{\mathbb{N}}$  is a sequence with values in  $Y$  and often write  $f_n$  for  $f(n)$  and express  $f$  as  $\{f_n\}_{n=1}^{\infty}$ . If  $X = \{1, 2, \dots, N\}$ , we will write  $Y^N$  in place of  $Y^{\{1, 2, \dots, N\}}$  and denote  $f \in Y^N$  by  $f = (f_1, f_2, \dots, f_N)$  where  $f_n = f(n)$ .

**Notation 3.2** More generally if  $\{X_\alpha : \alpha \in A\}$  is a collection of non-empty sets, let  $X_A = \prod_{\alpha \in A} X_\alpha$  and  $\pi_\alpha : X_A \rightarrow X_\alpha$  be the canonical projection map defined by  $\pi_\alpha(x) = x_\alpha$ . If  $X_\alpha = X$  for some fixed space  $X$ , then we will write  $\prod_{\alpha \in A} X_\alpha$  as  $X^A$  rather than  $X_A$ .

Recall that an element  $x \in X_A$  is a “**choice function**,” i.e. an assignment  $x_\alpha := x(\alpha) \in X_\alpha$  for each  $\alpha \in A$ . The **axiom of choice** states that  $X_A \neq \emptyset$  provided that  $X_\alpha \neq \emptyset$  for each  $\alpha \in A$ .

**Notation 3.3** Given a set  $X$ , let  $2^X$  denote the **power set** of  $X$  – the collection of all subsets of  $X$  including the empty set.

The reason for writing the power set of  $X$  as  $2^X$  is that if we think of 2 meaning  $\{0, 1\}$ , then an element of  $a \in 2^X = \{0, 1\}^X$  is completely determined by the set

$$A := \{x \in X : a(x) = 1\} \subset X.$$

In this way elements in  $\{0, 1\}^X$  are in one to one correspondence with subsets of  $X$ .

For  $A \in 2^X$  let

$$A^c := X \setminus A = \{x \in X : x \notin A\}$$

and more generally if  $A, B \subset X$  let

$$B \setminus A := \{x \in B : x \notin A\} = A \cap B^c.$$

We also define the symmetric difference of  $A$  and  $B$  by

$$A \Delta B := (B \setminus A) \cup (A \setminus B).$$

As usual if  $\{A_\alpha\}_{\alpha \in I}$  is an indexed collection of subsets of  $X$  we define the union and the intersection of this collection by

$$\begin{aligned} \cup_{\alpha \in I} A_\alpha &:= \{x \in X : \exists \alpha \in I \ni x \in A_\alpha\} \text{ and} \\ \cap_{\alpha \in I} A_\alpha &:= \{x \in X : x \in A_\alpha \forall \alpha \in I\}. \end{aligned}$$

**Notation 3.4** We will also write  $\prod_{\alpha \in I} A_\alpha$  for  $\cup_{\alpha \in I} A_\alpha$  in the case that  $\{A_\alpha\}_{\alpha \in I}$  are pairwise disjoint, i.e.  $A_\alpha \cap A_\beta = \emptyset$  if  $\alpha \neq \beta$ .

Notice that  $\cup$  is closely related to  $\exists$  and  $\cap$  is closely related to  $\forall$ . For example let  $\{A_n\}_{n=1}^{\infty}$  be a sequence of subsets from  $X$  and define

$$\begin{aligned} \inf_{k \geq n} A_n &:= \cap_{k \geq n} A_k, \\ \sup_{k \geq n} A_n &:= \cup_{k \geq n} A_k, \end{aligned}$$

$$\limsup_{n \rightarrow \infty} A_n := \{A_n \text{ i.o.}\} := \{x \in X : \#\{n : x \in A_n\} = \infty\}$$

and

$$\liminf_{n \rightarrow \infty} A_n := \{A_n \text{ a.a.}\} := \{x \in X : x \in A_n \text{ for all } n \text{ sufficiently large}\}.$$

(One should read  $\{A_n \text{ i.o.}\}$  as  $A_n$  infinitely often and  $\{A_n \text{ a.a.}\}$  as  $A_n$  almost always.) Then  $x \in \{A_n \text{ i.o.}\}$  iff

$$\forall N \in \mathbb{N} \exists n \geq N \ni x \in A_n$$

and this may be expressed as

$$\{A_n \text{ i.o.}\} = \cap_{N=1}^{\infty} \cup_{n \geq N} A_n.$$

Similarly,  $x \in \{A_n \text{ a.a.}\}$  iff

$$\exists N \in \mathbb{N} \ni \forall n \geq N, x \in A_n$$

which may be written as

$$\{A_n \text{ a.a.}\} = \cup_{N=1}^{\infty} \cap_{n \geq N} A_n.$$

**Definition 3.5.** Given a set  $A \subset X$ , let

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

be the characteristic function of  $A$ .

**Lemma 3.6.** We have:

1.  $\{A_n \text{ i.o.}\}^c = \{A_n^c \text{ a.a.}\}$ ,
2.  $\limsup_{n \rightarrow \infty} A_n = \{x \in X : \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty\}$ ,
3.  $\liminf_{n \rightarrow \infty} A_n = \{x \in X : \sum_{n=1}^{\infty} 1_{A_n^c}(x) < \infty\}$ ,
4.  $\sup_{k \geq n} 1_{A_k}(x) = 1_{\cup_{k \geq n} A_k} = 1_{\sup_{k \geq n} A_k}$ ,
5.  $\inf 1_{A_k}(x) = 1_{\cap_{k \geq n} A_k} = 1_{\inf_{k \geq n} A_k}$ ,
6.  $1_{\limsup_{n \rightarrow \infty} A_n} = \limsup_{n \rightarrow \infty} 1_{A_n}$ , and
7.  $1_{\liminf_{n \rightarrow \infty} A_n} = \liminf_{n \rightarrow \infty} 1_{A_n}$ .

**Definition 3.7.** A set  $X$  is said to be **countable** if is empty or there is an injective function  $f : X \rightarrow \mathbb{N}$ , otherwise  $X$  is said to be **uncountable**.

**Lemma 3.8 (Basic Properties of Countable Sets).**

1. If  $A \subset X$  is a subset of a countable set  $X$  then  $A$  is countable.
2. Any infinite subset  $A \subset \mathbb{N}$  is in one to one correspondence with  $\mathbb{N}$ .
3. A non-empty set  $X$  is countable iff there exists a surjective map,  $g : \mathbb{N} \rightarrow X$ .
4. If  $X$  and  $Y$  are countable then  $X \times Y$  is countable.
5. Suppose for each  $m \in \mathbb{N}$  that  $A_m$  is a countable subset of a set  $X$ , then  $A = \cup_{m=1}^{\infty} A_m$  is countable. In short, the countable union of countable sets is still countable.
6. If  $X$  is an infinite set and  $Y$  is a set with at least two elements, then  $Y^X$  is uncountable. In particular  $2^X$  is uncountable for any infinite set  $X$ .

**Proof.** 1. If  $f : X \rightarrow \mathbb{N}$  is an injective map then so is the restriction,  $f|_A$ , of  $f$  to the subset  $A$ . 2. Let  $f(1) = \min A$  and define  $f$  inductively by

$$f(n+1) = \min(A \setminus \{f(1), \dots, f(n)\}).$$

Since  $A$  is infinite the process continues indefinitely. The function  $f : \mathbb{N} \rightarrow A$  defined this way is a bijection.

3. If  $g : \mathbb{N} \rightarrow X$  is a surjective map, let

$$f(x) = \min g^{-1}(\{x\}) = \min \{n \in \mathbb{N} : f(n) = x\}.$$

Then  $f : X \rightarrow \mathbb{N}$  is injective which combined with item

2. (taking  $A = f(X)$ ) shows  $X$  is countable. Conversely if  $f : X \rightarrow \mathbb{N}$  is injective let  $x_0 \in X$  be a fixed point and define  $g : \mathbb{N} \rightarrow X$  by  $g(n) = f^{-1}(n)$  for  $n \in f(X)$  and  $g(n) = x_0$  otherwise.

4. Let us first construct a bijection,  $h$ , from  $\mathbb{N}$  to  $\mathbb{N} \times \mathbb{N}$ . To do this put the elements of  $\mathbb{N} \times \mathbb{N}$  into an array of the form

$$\begin{pmatrix} (1,1) & (1,2) & (1,3) & \dots \\ (2,1) & (2,2) & (2,3) & \dots \\ (3,1) & (3,2) & (3,3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and then “count” these elements by counting the sets  $\{(i,j) : i+j=k\}$  one at a time. For example let  $h(1) = (1,1)$ ,  $h(2) = (2,1)$ ,  $h(3) = (1,2)$ ,  $h(4) = (3,1)$ ,  $h(5) = (2,2)$ ,  $h(6) = (1,3)$  and so on. If  $f : \mathbb{N} \rightarrow X$  and  $g : \mathbb{N} \rightarrow Y$  are surjective functions, then the function  $(f \times g) \circ h : \mathbb{N} \rightarrow X \times Y$  is surjective where  $(f \times g)(m,n) := (f(m), g(n))$  for all  $(m,n) \in \mathbb{N} \times \mathbb{N}$ .

5. If  $A = \emptyset$  then  $A$  is countable by definition so we may assume  $A \neq \emptyset$ . With out loss of generality we may assume  $A_1 \neq \emptyset$  and by replacing  $A_m$  by  $A_1$  if necessary we may also assume  $A_m \neq \emptyset$  for all  $m$ . For each  $m \in \mathbb{N}$  let  $a_m : \mathbb{N} \rightarrow A_m$  be a surjective function and then define  $f : \mathbb{N} \times \mathbb{N} \rightarrow \cup_{m=1}^{\infty} A_m$  by  $f(m,n) := a_m(n)$ . The function  $f$  is surjective and hence so is the composition,  $f \circ h : \mathbb{N} \rightarrow \cup_{m=1}^{\infty} A_m$ , where  $h : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  is the bijection defined above.

6. Let us begin by showing  $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$  is uncountable. For sake of contradiction suppose  $f : \mathbb{N} \rightarrow \{0,1\}^{\mathbb{N}}$  is a surjection and write  $f(n)$  as  $(f_1(n), f_2(n), f_3(n), \dots)$ . Now define  $a \in \{0,1\}^{\mathbb{N}}$  by  $a_n := 1 - f_n(n)$ . By construction  $f_n(n) \neq a_n$  for all  $n$  and so  $a \notin f(\mathbb{N})$ . This contradicts the assumption that  $f$  is surjective and shows  $2^{\mathbb{N}}$  is uncountable. For the general case, since  $Y_0^X \subset Y^X$  for any subset  $Y_0 \subset Y$ , if  $Y_0^X$  is uncountable then so is  $Y^X$ . In this way we may assume  $Y_0$  is a two point set which may as well be  $Y_0 = \{0,1\}$ . Moreover, since  $X$  is an infinite set we may find an injective map  $x : \mathbb{N} \rightarrow X$  and use this to set up an injection,  $i : 2^{\mathbb{N}} \rightarrow 2^X$  by setting  $i(A) := \{x_n : n \in \mathbb{N}\} \subset X$  for all  $A \subset \mathbb{N}$ . If  $2^X$  were countable we could find a surjective map  $f : 2^X \rightarrow \mathbb{N}$  in which case  $f \circ i : 2^{\mathbb{N}} \rightarrow \mathbb{N}$  would be surjective as well. However this is impossible since we have already seen that  $2^{\mathbb{N}}$  is uncountable. ■

We end this section with some notation which will be used frequently in the sequel.

**Notation 3.9** If  $f : X \rightarrow Y$  is a function and  $\mathcal{E} \subset 2^Y$  let

$$f^{-1}\mathcal{E} := f^{-1}(\mathcal{E}) := \{f^{-1}(E) | E \in \mathcal{E}\}.$$

If  $\mathcal{G} \subset 2^X$ , let

$$f_*\mathcal{G} := \{A \in 2^Y | f^{-1}(A) \in \mathcal{G}\}.$$

**Definition 3.10.** Let  $\mathcal{E} \subset 2^X$  be a collection of sets,  $A \subset X$ ,  $i_A : A \rightarrow X$  be the **inclusion map** ( $i_A(x) = x$  for all  $x \in A$ ) and

$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{A \cap E : E \in \mathcal{E}\}.$$

### 3.2 Exercises

Let  $f : X \rightarrow Y$  be a function and  $\{A_i\}_{i \in I}$  be an indexed family of subsets of  $Y$ , verify the following assertions.

**Exercise 3.1.**  $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$ .

**Exercise 3.2.** Suppose that  $B \subset Y$ , show that  $B \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (B \setminus A_i)$ .

**Exercise 3.3.**  $f^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^{-1}(A_i)$ .

**Exercise 3.4.**  $f^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f^{-1}(A_i)$ .

**Exercise 3.5.** Find a counterexample which shows that  $f(C \cap D) = f(C) \cap f(D)$  need not hold.

*Example 3.11.* Let  $X = \{a, b, c\}$  and  $Y = \{1, 2\}$  and define  $f(a) = f(b) = 1$  and  $f(c) = 2$ . Then  $\emptyset = f(\{a\} \cap \{b\}) \neq f(\{a\}) \cap f(\{b\}) = \{1\}$  and  $\{1, 2\} = f(\{a\}^c) \neq f(\{a\})^c = \{2\}$ .

### 3.3 Algebraic sub-structures of sets

**Definition 3.12.** A collection of subsets  $\mathcal{A}$  of a set  $X$  is a  $\pi$ -system or **multiplicative system** if  $\mathcal{A}$  is closed under taking finite intersections.

**Definition 3.13.** A collection of subsets  $\mathcal{A}$  of a set  $X$  is an **algebra (Field)** if

1.  $\emptyset, X \in \mathcal{A}$
  2.  $A \in \mathcal{A}$  implies that  $A^c \in \mathcal{A}$
  3.  $\mathcal{A}$  is closed under finite unions, i.e. if  $A_1, \dots, A_n \in \mathcal{A}$  then  $A_1 \cup \dots \cup A_n \in \mathcal{A}$ .
- In view of conditions 1. and 2., 3. is equivalent to
- 3'.  $\mathcal{A}$  is closed under finite intersections.

**Definition 3.14.** A collection of subsets  $\mathcal{B}$  of  $X$  is a  $\sigma$ -algebra (or sometimes called a  $\sigma$ -field) if  $\mathcal{B}$  is an algebra which also closed under countable unions, i.e. if  $\{A_i\}_{i=1}^\infty \subset \mathcal{B}$ , then  $\bigcup_{i=1}^\infty A_i \in \mathcal{B}$ . (Notice that since  $\mathcal{B}$  is also closed under taking complements,  $\mathcal{B}$  is also closed under taking countable intersections.)

*Example 3.15.* Here are some examples of algebras.

1.  $\mathcal{B} = 2^X$ , then  $\mathcal{B}$  is a  $\sigma$ -algebra.
2.  $\mathcal{B} = \{\emptyset, X\}$  is a  $\sigma$ -algebra called the trivial  $\sigma$ -field.

3. Let  $X = \{1, 2, 3\}$ , then  $\mathcal{A} = \{\emptyset, X, \{1\}, \{2, 3\}\}$  is an algebra while,  $\mathcal{S} := \{\emptyset, X, \{2, 3\}\}$  is not an algebra but is a  $\pi$ -system.

**Proposition 3.16.** Let  $\mathcal{E}$  be any collection of subsets of  $X$ . Then there exists a unique smallest algebra  $\mathcal{A}(\mathcal{E})$  and  $\sigma$ -algebra  $\sigma(\mathcal{E})$  which contains  $\mathcal{E}$ .

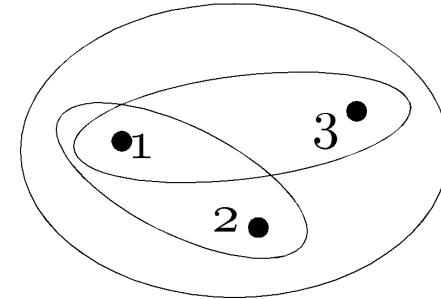
**Proof.** Simply take

$$\mathcal{A}(\mathcal{E}) := \bigcap \{ \mathcal{A} : \mathcal{A} \text{ is an algebra such that } \mathcal{E} \subset \mathcal{A} \}$$

and

$$\sigma(\mathcal{E}) := \bigcap \{ \mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra such that } \mathcal{E} \subset \mathcal{M} \}.$$

*Example 3.17.* Suppose  $X = \{1, 2, 3\}$  and  $\mathcal{E} = \{\emptyset, X, \{1, 2\}, \{1, 3\}\}$ , see Figure 3.1. Then



**Fig. 3.1.** A collection of subsets.

$$\mathcal{A}(\mathcal{E}) = \sigma(\mathcal{E}) = 2^X.$$

On the other hand if  $\mathcal{E} = \{\{1, 2\}\}$ , then  $\mathcal{A}(\mathcal{E}) = \{\emptyset, X, \{1, 2\}, \{3\}\}$ .

**Exercise 3.6.** Suppose that  $\mathcal{E}_i \subset 2^X$  for  $i = 1, 2$ . Show that  $\mathcal{A}(\mathcal{E}_1) = \mathcal{A}(\mathcal{E}_2)$  iff  $\mathcal{E}_1 \subset \mathcal{A}(\mathcal{E}_2)$  and  $\mathcal{E}_2 \subset \mathcal{A}(\mathcal{E}_1)$ . Similarly show,  $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$  iff  $\mathcal{E}_1 \subset \sigma(\mathcal{E}_2)$  and  $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1)$ . Give a simple example where  $\mathcal{A}(\mathcal{E}_1) = \mathcal{A}(\mathcal{E}_2)$  while  $\mathcal{E}_1 \neq \mathcal{E}_2$ .

**Definition 3.18.** Let  $X$  be a set. We say that a family of sets  $\mathcal{F} \subset 2^X$  is a **partition** of  $X$  if distinct members of  $\mathcal{F}$  are disjoint and if  $X$  is the union of the sets in  $\mathcal{F}$ .

*Example 3.19.* Let  $X$  be a set and  $\mathcal{E} = \{A_1, \dots, A_n\}$  where  $A_1, \dots, A_n$  is a partition of  $X$ . In this case

$$\mathcal{A}(\mathcal{E}) = \sigma(\mathcal{E}) = \{\cup_{i \in \Lambda} A_i : \Lambda \subset \{1, 2, \dots, n\}\}$$

where  $\cup_{i \in \Lambda} A_i := \emptyset$  when  $\Lambda = \emptyset$ . Notice that

$$\#(\mathcal{A}(\mathcal{E})) = \#(2^{\{1, 2, \dots, n\}}) = 2^n.$$

*Example 3.20.* Suppose that  $X$  is a finite set and that  $\mathcal{A} \subset 2^X$  is an algebra. For each  $x \in X$  let

$$A_x = \cap \{A \in \mathcal{A} : x \in A\} \in \mathcal{A},$$

wherein we have used  $\mathcal{A}$  is finite to insure  $A_x \in \mathcal{A}$ . Hence  $A_x$  is the smallest set in  $\mathcal{A}$  which contains  $x$ . Let  $C = A_x \cap A_y \in \mathcal{A}$ . I claim that if  $C \neq \emptyset$ , then  $A_x = A_y$ . To see this, let us first consider the case where  $\{x, y\} \subset C$ . In this case we must have  $A_x \subset C$  and  $A_y \subset C$  and therefore  $A_x = A_y$ . Now suppose either  $x$  or  $y$  is not in  $C$ . For definiteness, say  $x \notin C$ , i.e.  $x \notin y$ . Then  $x \in A_x \setminus A_y \in \mathcal{A}$  from which it follows that  $A_x = A_x \setminus A_y$ , i.e.  $A_x \cap A_y = \emptyset$ .

Let us now define  $\{B_i\}_{i=1}^k$  to be an enumeration of  $\{A_x\}_{x \in X}$ . It is now a straightforward exercise to show

$$\mathcal{A} = \{\cup_{i \in \Lambda} B_i : \Lambda \subset \{1, 2, \dots, k\}\}.$$

**Proposition 3.21.** *Suppose that  $\mathcal{B} \subset 2^X$  is a  $\sigma$ -algebra and  $\mathcal{B}$  is at most a countable set. Then there exists a unique **finite** partition  $\mathcal{F}$  of  $X$  such that  $\mathcal{F} \subset \mathcal{B}$  and every element  $B \in \mathcal{B}$  is of the form*

$$B = \cup \{A \in \mathcal{F} : A \subset B\}. \quad (3.1)$$

*In particular  $\mathcal{B}$  is actually a finite set and  $\#(\mathcal{B}) = 2^n$  for some  $n \in \mathbb{N}$ .*

**Proof.** We proceed as in Example 3.20. For each  $x \in X$  let

$$A_x = \cap \{A \in \mathcal{B} : x \in A\} \in \mathcal{B},$$

wherein we have used  $\mathcal{B}$  is a countable  $\sigma$ -algebra to insure  $A_x \in \mathcal{B}$ . Just as above either  $A_x \cap A_y = \emptyset$  or  $A_x = A_y$  and therefore  $\mathcal{F} = \{A_x : x \in X\} \subset \mathcal{B}$  is a (necessarily countable) partition of  $X$  for which Eq. (3.1) holds for all  $B \in \mathcal{B}$ .

Enumerate the elements of  $\mathcal{F}$  as  $\mathcal{F} = \{P_n\}_{n=1}^N$  where  $N \in \mathbb{N}$  or  $N = \infty$ . If  $N = \infty$ , then the correspondence

$$a \in \{0, 1\}^{\mathbb{N}} \rightarrow A_a = \cup \{P_n : a_n = 1\} \in \mathcal{B}$$

is bijective and therefore, by Lemma 3.8,  $\mathcal{B}$  is uncountable. Thus any countable  $\sigma$ -algebra is necessarily finite. This finishes the proof modulo the uniqueness assertion which is left as an exercise to the reader. ■

*Example 3.22 (Countable/Co-countable  $\sigma$ -Field).* Let  $X = \mathbb{R}$  and  $\mathcal{E} := \{\{x\} : x \in \mathbb{R}\}$ . Then  $\sigma(\mathcal{E})$  consists of those subsets,  $A \subset \mathbb{R}$ , such that  $A$  is countable or  $A^c$  is countable. Similarly,  $\mathcal{A}(\mathcal{E})$  consists of those subsets,  $A \subset \mathbb{R}$ , such that  $A$  is finite or  $A^c$  is finite. More generally we have the following exercise.

**Exercise 3.7.** Let  $X$  be a set,  $I$  be an **infinite** index set, and  $\mathcal{E} = \{A_i\}_{i \in I}$  be a partition of  $X$ . Prove the algebra,  $\mathcal{A}(\mathcal{E})$ , and that  $\sigma$ -algebra,  $\sigma(\mathcal{E})$ , generated by  $\mathcal{E}$  are given by

$$\mathcal{A}(\mathcal{E}) = \{\cup_{i \in \Lambda} A_i : \Lambda \subset I \text{ with } \#(\Lambda) < \infty \text{ or } \#(\Lambda^c) < \infty\}$$

and

$$\sigma(\mathcal{E}) = \{\cup_{i \in \Lambda} A_i : \Lambda \subset I \text{ with } \Lambda \text{ countable or } \Lambda^c \text{ countable}\}$$

respectively. Here we are using the convention that  $\cup_{i \in \Lambda} A_i := \emptyset$  when  $\Lambda = \emptyset$ .

**Proposition 3.23.** *Let  $X$  be a set and  $\mathcal{E} \subset 2^X$ . Let  $\mathcal{E}^c := \{A^c : A \in \mathcal{E}\}$  and  $\mathcal{E}_c := \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$ . Then*

$$\mathcal{A}(\mathcal{E}) := \{\text{finite unions of finite intersections of elements from } \mathcal{E}_c\}. \quad (3.2)$$

**Proof.** Let  $\mathcal{A}$  denote the right member of Eq. (3.2). From the definition of an algebra, it is clear that  $\mathcal{E} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{E})$ . Hence to finish that proof it suffices to show  $\mathcal{A}$  is an algebra. The proof of these assertions are routine except for possibly showing that  $\mathcal{A}$  is closed under complementation. To check  $\mathcal{A}$  is closed under complementation, let  $Z \in \mathcal{A}$  be expressed as

$$Z = \bigcup_{i=1}^N \bigcap_{j=1}^K A_{ij}$$

where  $A_{ij} \in \mathcal{E}_c$ . Therefore, writing  $B_{ij} = A_{ij}^c \in \mathcal{E}_c$ , we find that

$$Z^c = \bigcap_{i=1}^N \bigcup_{j=1}^K B_{ij} = \bigcup_{j_1, \dots, j_N=1}^K (B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}) \in \mathcal{A}$$

wherein we have used the fact that  $B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}$  is a finite intersection of sets from  $\mathcal{E}_c$ . ■

*Remark 3.24.* One might think that in general  $\sigma(\mathcal{E})$  may be described as the countable unions of countable intersections of sets in  $\mathcal{E}^c$ . However this is in general **false**, since if

$$Z = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{ij}$$

with  $A_{ij} \in \mathcal{E}_c$ , then

$$Z^c = \bigcup_{j_1=1, j_2=1, \dots, j_N=1, \dots}^{\infty} \left( \bigcap_{\ell=1}^{\infty} A_{\ell, j_\ell}^c \right)$$

which is now an **uncountable** union. Thus the above description is not correct. In general it is complicated to explicitly describe  $\sigma(\mathcal{E})$ , see Proposition 1.23 on page 39 of Folland for details. Also see Proposition 3.21.

**Exercise 3.8.** Let  $\tau$  be a topology on a set  $X$  and  $\mathcal{A} = \mathcal{A}(\tau)$  be the algebra generated by  $\tau$ . Show  $\mathcal{A}$  is the collection of subsets of  $X$  which may be written as finite union of sets of the form  $F \cap V$  where  $F$  is closed and  $V$  is open.

**Solution to Exercise (3.8).** In this case  $\tau_c$  is the collection of sets which are either open or closed. Now if  $V_i \subset_o X$  and  $F_j \subset X$  for each  $j$ , then  $(\bigcap_{i=1}^n V_i) \cap (\bigcap_{j=1}^m F_j)$  is simply a set of the form  $V \cap F$  where  $V \subset_o X$  and  $F \subset X$ . Therefore the result is an immediate consequence of Proposition 3.23.

**Definition 3.25.** The Borel  $\sigma$ -field,  $\mathcal{B} = \mathcal{B}_{\mathbb{R}} = \mathcal{B}(\mathbb{R})$ , on  $\mathbb{R}$  is the smallest  $\sigma$ -field containing all of the open subsets of  $\mathbb{R}$ .

**Exercise 3.9.** Verify the  $\sigma$ -algebra,  $\mathcal{B}_{\mathbb{R}}$ , is generated by any of the following collection of sets:

1.  $\{(a, \infty) : a \in \mathbb{R}\}$ , 2.  $\{(a, \infty) : a \in \mathbb{Q}\}$  or 3.  $\{[a, \infty) : a \in \mathbb{Q}\}$ .

**Hint:** make use of Exercise 3.6.

**Exercise 3.10.** Suppose  $f : X \rightarrow Y$  is a function,  $\mathcal{F} \subset 2^Y$  and  $\mathcal{B} \subset 2^X$ . Show  $f^{-1}\mathcal{F}$  and  $f_*\mathcal{B}$  (see Notation 3.9) are algebras ( $\sigma$ -algebras) provided  $\mathcal{F}$  and  $\mathcal{B}$  are algebras ( $\sigma$ -algebras).

**Lemma 3.26.** Suppose that  $f : X \rightarrow Y$  is a function and  $\mathcal{E} \subset 2^Y$  and  $A \subset Y$  then

$$\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E})) \text{ and} \quad (3.3)$$

$$(\sigma(\mathcal{E}))_A = \sigma(\mathcal{E}_A), \quad (3.4)$$

where  $\mathcal{B}_A := \{B \cap A : B \in \mathcal{B}\}$ . (Similar assertion hold with  $\sigma(\cdot)$  being replaced by  $\mathcal{A}(\cdot)$ .)

**Proof.** By Exercise 3.10,  $f^{-1}(\sigma(\mathcal{E}))$  is a  $\sigma$ -algebra and since  $\mathcal{E} \subset \mathcal{F}$ ,  $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$ . It now follows that

$$\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E})).$$

For the reverse inclusion, notice that

$$f_*\sigma(f^{-1}(\mathcal{E})) := \{B \subset Y : f^{-1}(B) \in \sigma(f^{-1}(\mathcal{E}))\}$$

is a  $\sigma$ -algebra which contains  $\mathcal{E}$  and thus  $\sigma(\mathcal{E}) \subset f_*\sigma(f^{-1}(\mathcal{E}))$ . Hence for every  $B \in \sigma(\mathcal{E})$  we know that  $f^{-1}(B) \in \sigma(f^{-1}(\mathcal{E}))$ , i.e.

$$f^{-1}(\sigma(\mathcal{E})) \subset \sigma(f^{-1}(\mathcal{E})).$$

Applying Eq. (3.3) with  $X = A$  and  $f = i_A$  being the inclusion map implies

$$(\sigma(\mathcal{E}))_A = i_A^{-1}(\sigma(\mathcal{E})) = \sigma(i_A^{-1}(\mathcal{E})) = \sigma(\mathcal{E}_A).$$

■

*Example 3.27.* Let  $\mathcal{E} = \{(a, b] : -\infty < a < b < \infty\}$  and  $\mathcal{B} = \sigma(\mathcal{E})$  be the Borel  $\sigma$ -field on  $\mathbb{R}$ . Then

$$\mathcal{E}_{(0,1]} = \{(a, b] : 0 \leq a < b \leq 1\}$$

and we have

$$\mathcal{B}_{(0,1]} = \sigma(\mathcal{E}_{(0,1]}).$$

In particular, if  $A \in \mathcal{B}$  such that  $A \subset (0, 1]$ , then  $A \in \sigma(\mathcal{E}_{(0,1]})$ .

**Definition 3.28.** A function,  $f : \Omega \rightarrow Y$  is said to be **simple** if  $f(\Omega) \subset Y$  is a finite set. If  $\mathcal{A} \subset 2^\Omega$  is an algebra, we say that a simple function  $f : \Omega \rightarrow Y$  is **measurable** if  $\{f = y\} := f^{-1}(\{y\}) \in \mathcal{A}$  for all  $y \in Y$ . A measurable simple function,  $f : \Omega \rightarrow \mathbb{C}$ , is called a **simple random variable** relative to  $\mathcal{A}$ .

**Notation 3.29** Given an algebra,  $\mathcal{A} \subset 2^\Omega$ , let  $\mathbb{S}(\mathcal{A})$  denote the collection of simple random variables from  $\Omega$  to  $\mathbb{C}$ . For example if  $A \in \mathcal{A}$ , then  $1_A \in \mathbb{S}(\mathcal{A})$  is a measurable simple function.

**Lemma 3.30.** For every algebra  $\mathcal{A} \subset 2^\Omega$ , the set simple random variables,  $\mathbb{S}(\mathcal{A})$ , forms an algebra.

**Proof.** Let us observe that  $1_\Omega = 1$  and  $1_\emptyset = 0$  are in  $\mathbb{S}(\mathcal{A})$ . If  $f, g \in \mathbb{S}(\mathcal{A})$  and  $c \in \mathbb{C} \setminus \{0\}$ , then

$$\{f + cg = \lambda\} = \bigcup_{a, b \in \mathbb{C}: a+cb=\lambda} (\{f = a\} \cap \{g = b\}) \in \mathcal{A} \quad (3.5)$$

and

$$\{f \cdot g = \lambda\} = \bigcup_{a, b \in \mathbb{C}: a \cdot b = \lambda} (\{f = a\} \cap \{g = b\}) \in \mathcal{A} \quad (3.6)$$

from which it follows that  $f + cg$  and  $f \cdot g$  are back in  $\mathbb{S}(\mathcal{A})$ . ■

**Definition 3.31.** A *simple function algebra*,  $\mathbb{S}$ , is a subalgebra of the bounded complex functions on  $X$  such that  $1 \in \mathbb{S}$  and each function,  $f \in \mathbb{S}$ , is a simple function. If  $\mathbb{S}$  is a simple function algebra, let

$$\mathcal{A}(\mathbb{S}) := \{A \subset X : 1_A \in \mathbb{S}\}.$$

(It is easily checked that  $\mathcal{A}(\mathbb{S})$  is a sub-algebra of  $2^X$ .)

**Lemma 3.32.** Suppose that  $\mathbb{S}$  is a simple function algebra,  $f \in \mathbb{S}$  and  $\alpha \in f(X)$ . Then  $\{f = \alpha\} \in \mathcal{A}(\mathbb{S})$ .

**Proof.** Let  $\{\lambda_i\}_{i=0}^n$  be an enumeration of  $f(X)$  with  $\lambda_0 = \alpha$ . Then

$$g := \left[ \prod_{i=1}^n (\alpha - \lambda_i) \right]^{-1} \prod_{i=1}^n (f - \lambda_i 1) \in \mathbb{S}.$$

Moreover, we see that  $g = 0$  on  $\cup_{i=1}^n \{f = \lambda_i\}$  while  $g = 1$  on  $\{f = \alpha\}$ . So we have shown  $g = 1_{\{f = \alpha\}} \in \mathbb{S}$  and therefore that  $\{f = \alpha\} \in \mathcal{A}$ . ■

**Exercise 3.11.** Continuing the notation introduced above:

1. Show  $\mathcal{A}(\mathbb{S})$  is an algebra of sets.
2. Show  $\mathbb{S}(\mathcal{A})$  is a simple function algebra.
3. Show that the map

$$\mathcal{A} \in \{\text{Algebras} \subset 2^X\} \rightarrow \mathbb{S}(\mathcal{A}) \in \{\text{simple function algebras on } X\}$$

is bijective and the map,  $\mathbb{S} \rightarrow \mathcal{A}(\mathbb{S})$ , is the inverse map.

**Solution to Exercise (3.11).**

1. Since  $0 = 1_\emptyset, 1 = 1_X \in \mathbb{S}$ , it follows that  $\emptyset$  and  $X$  are in  $\mathcal{A}(\mathbb{S})$ . If  $A \in \mathcal{A}(\mathbb{S})$ , then  $1_{A^c} = 1 - 1_A \in \mathbb{S}$  and so  $A^c \in \mathcal{A}(\mathbb{S})$ . Finally, if  $A, B \in \mathcal{A}(\mathbb{S})$  then  $1_{A \cap B} = 1_A \cdot 1_B \in \mathbb{S}$  and thus  $A \cap B \in \mathcal{A}(\mathbb{S})$ .
2. If  $f, g \in \mathbb{S}(\mathcal{A})$  and  $c \in \mathbb{F}$ , then

$$\{f + cg = \lambda\} = \bigcup_{a, b \in \mathbb{F}: a + cb = \lambda} (\{f = a\} \cap \{g = b\}) \in \mathcal{A}$$

and

$$\{f \cdot g = \lambda\} = \bigcup_{a, b \in \mathbb{F}: a \cdot b = \lambda} (\{f = a\} \cap \{g = b\}) \in \mathcal{A}$$

from which it follows that  $f + cg$  and  $f \cdot g$  are back in  $\mathbb{S}(\mathcal{A})$ .

3. If  $f : \Omega \rightarrow \mathbb{C}$  is a simple function such that  $1_{\{f = \lambda\}} \in \mathbb{S}$  for all  $\lambda \in \mathbb{C}$ , then  $f = \sum_{\lambda \in \mathbb{C}} \lambda 1_{\{f = \lambda\}} \in \mathbb{S}$ . Conversely, by Lemma 3.32, if  $f \in \mathbb{S}$  then  $1_{\{f = \lambda\}} \in \mathbb{S}$  for all  $\lambda \in \mathbb{C}$ . Therefore, a simple function,  $f : X \rightarrow \mathbb{C}$  is in  $\mathbb{S}$  iff  $1_{\{f = \lambda\}} \in \mathbb{S}$  for all  $\lambda \in \mathbb{C}$ . With this preparation, we are now ready to complete the verification.

First off,

$$A \in \mathcal{A}(\mathbb{S}(\mathcal{A})) \iff 1_A \in \mathbb{S}(\mathcal{A}) \iff A \in \mathcal{A}$$

which shows that  $\mathcal{A}(\mathbb{S}(\mathcal{A})) = \mathcal{A}$ . Similarly,

$$\begin{aligned} f \in \mathbb{S}(\mathcal{A}(\mathbb{S})) &\iff \{f = \lambda\} \in \mathcal{A}(\mathbb{S}) \quad \forall \lambda \in \mathbb{C} \\ &\iff 1_{\{f = \lambda\}} \in \mathbb{S} \quad \forall \lambda \in \mathbb{C} \\ &\iff f \in \mathbb{S} \end{aligned}$$

which shows  $\mathbb{S}(\mathcal{A}(\mathbb{S})) = \mathbb{S}$ .

## Finitely Additive Measures

**Definition 4.1.** Suppose that  $\mathcal{E} \subset 2^X$  is a collection of subsets of  $X$  and  $\mu : \mathcal{E} \rightarrow [0, \infty]$  is a function. Then

1.  $\mu$  is **monotonic** if  $\mu(A) \leq \mu(B)$  for all  $A, B \in \mathcal{E}$  with  $A \subset B$ .
2.  $\mu$  is **sub-additive (finitely sub-additive)** on  $\mathcal{E}$  if

$$\mu(E) \leq \sum_{i=1}^n \mu(E_i)$$

whenever  $E = \bigcup_{i=1}^n E_i \in \mathcal{E}$  with  $n \in \mathbb{N} \cup \{\infty\}$  ( $n \in \mathbb{N}$ ).

3.  $\mu$  is **super-additive (finitely super-additive)** on  $\mathcal{E}$  if

$$\mu(E) \geq \sum_{i=1}^n \mu(E_i) \quad (4.1)$$

whenever  $E = \prod_{i=1}^n E_i \in \mathcal{E}$  with  $n \in \mathbb{N} \cup \{\infty\}$  ( $n \in \mathbb{N}$ ).

4.  $\mu$  is **additive or finitely additive** on  $\mathcal{E}$  if

$$\mu(E) = \sum_{i=1}^n \mu(E_i) \quad (4.2)$$

whenever  $E = \prod_{i=1}^n E_i \in \mathcal{E}$  with  $E_i \in \mathcal{E}$  for  $i = 1, 2, \dots, n < \infty$ .

5. If  $\mathcal{E} = \mathcal{A}$  is an algebra,  $\mu(\emptyset) = 0$ , and  $\mu$  is finitely additive on  $\mathcal{A}$ , then  $\mu$  is said to be a **finitely additive measure**.
6.  $\mu$  is  $\sigma$  - **additive (or countable additive)** on  $\mathcal{E}$  if item 4. holds even when  $n = \infty$ .
7. If  $\mathcal{E} = \mathcal{A}$  is an algebra,  $\mu(\emptyset) = 0$ , and  $\mu$  is  $\sigma$  - additive on  $\mathcal{A}$  then  $\mu$  is called a **premeasure on  $\mathcal{A}$** .
8. A **measure** is a premeasure,  $\mu : \mathcal{B} \rightarrow [0, \infty]$ , where  $\mathcal{B}$  is a  $\sigma$  - algebra. We say that  $\mu$  is a **probability measure** if  $\mu(X) = 1$ .

### 4.1 Finitely Additive Measures

**Proposition 4.2 (Basic properties of finitely additive measures).** Suppose  $\mu$  is a finitely additive measure on an algebra,  $\mathcal{A} \subset 2^X$ ,  $E, F \in \mathcal{A}$  with  $E \subset F$  and  $\{E_j\}_{j=1}^n \subset \mathcal{A}$ , then :

1. ( $\mu$  is **monotone**)  $\mu(E) \leq \mu(F)$  if  $E \subset F$ .
2. For  $A, B \in \mathcal{A}$ , the following **strong additivity formula** holds;

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B). \quad (4.3)$$

3. ( $\mu$  is **finitely subadditive**)  $\mu(\bigcup_{j=1}^n E_j) \leq \sum_{j=1}^n \mu(E_j)$ .
4.  $\mu$  is sub-additive on  $\mathcal{A}$  iff

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i) \text{ for } A = \prod_{i=1}^{\infty} A_i \quad (4.4)$$

where  $A \in \mathcal{A}$  and  $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$  are pairwise disjoint sets.

5. ( $\mu$  is **countably superadditive**) If  $A = \prod_{i=1}^{\infty} A_i$  with  $A_i, A \in \mathcal{A}$ , then

$$\mu\left(\prod_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i).$$

6. A finitely additive measure,  $\mu$ , is a premeasure iff  $\mu$  is sub-additive.

#### Proof.

1. Since  $F$  is the disjoint union of  $E$  and  $(F \setminus E)$  and  $F \setminus E = F \cap E^c \in \mathcal{A}$  it follows that

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

2. Since

$$A \cup B = [A \setminus (A \cap B)] \prod [B \setminus (A \cap B)] \prod A \cap B,$$

$$\begin{aligned} \mu(A \cup B) &= \mu(A \cup B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A \setminus (A \cap B)) + \mu(B \setminus (A \cap B)) + \mu(A \cap B). \end{aligned}$$

Adding  $\mu(A \cap B)$  to both sides of this equation proves Eq. (4.3).

3. Let  $\tilde{E}_j = E_j \setminus (E_1 \cup \dots \cup E_{j-1})$  so that the  $\tilde{E}_j$ 's are pair-wise disjoint and  $E = \bigcup_{j=1}^n \tilde{E}_j$ . Since  $\tilde{E}_j \subset E_j$  it follows from the monotonicity of  $\mu$  that

$$\mu(E) = \sum \mu(\tilde{E}_j) \leq \sum \mu(E_j).$$

4. If  $A = \bigcup_{i=1}^{\infty} B_i$  with  $A \in \mathcal{A}$  and  $B_i \in \mathcal{A}$ , then  $A = \prod_{i=1}^{\infty} A_i$  where  $A_i := B_i \setminus (B_1 \cup \dots \cup B_{i-1}) \in \mathcal{A}$  and  $B_0 = \emptyset$ . Therefore using the monotonicity of  $\mu$  and Eq. (4.4)

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i) \leq \sum_{i=1}^{\infty} \mu(B_i).$$

5. Suppose that  $A = \prod_{i=1}^{\infty} A_i$  with  $A_i, A \in \mathcal{A}$ , then  $\prod_{i=1}^n A_i \subset A$  for all  $n$  and so by the monotonicity and finite additivity of  $\mu$ ,  $\sum_{i=1}^n \mu(A_i) \leq \mu(A)$ . Letting  $n \rightarrow \infty$  in this equation shows  $\mu$  is superadditive.
6. This is a combination of items 5. and 6. ■

**Proposition 4.3.** *Suppose that  $P$  is a finitely additive probability measure on an algebra,  $\mathcal{A} \subset 2^{\Omega}$ . Then the following are equivalent:*

1.  $P$  is  $\sigma$ -additive on  $\mathcal{A}$ .
2. For all  $A_n \in \mathcal{A}$  such that  $A_n \uparrow A \in \mathcal{A}$ ,  $P(A_n) \uparrow P(A)$ .
3. For all  $A_n \in \mathcal{A}$  such that  $A_n \downarrow A \in \mathcal{A}$ ,  $P(A_n) \downarrow P(A)$ .
4. For all  $A_n \in \mathcal{A}$  such that  $A_n \uparrow \Omega$ ,  $P(A_n) \uparrow 1$ .
5. For all  $A_n \in \mathcal{A}$  such that  $A_n \downarrow \Omega$ ,  $P(A_n) \downarrow 1$ .

**Proof.** We will start by showing  $1 \iff 2 \iff 3$ .

$1 \implies 2$ . Suppose  $A_n \in \mathcal{A}$  such that  $A_n \uparrow A \in \mathcal{A}$ . Let  $A'_n := A_n \setminus A_{n-1}$  with  $A_0 := \emptyset$ . Then  $\{A'_n\}_{n=1}^{\infty}$  are disjoint,  $A_n = \bigcup_{k=1}^n A'_k$  and  $A = \bigcup_{k=1}^{\infty} A'_k$ . Therefore,

$$P(A) = \sum_{k=1}^{\infty} P(A'_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n P(A'_k) = \lim_{n \rightarrow \infty} P(\bigcup_{k=1}^n A'_k) = \lim_{n \rightarrow \infty} P(A_n).$$

$2 \implies 1$ . If  $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$  are disjoint and  $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ , then  $\bigcup_{n=1}^N A_n \uparrow A$ . Therefore,

$$P(A) = \lim_{N \rightarrow \infty} P(\bigcup_{n=1}^N A_n) = \lim_{N \rightarrow \infty} \sum_{n=1}^N P(A_n) = \sum_{n=1}^{\infty} P(A_n).$$

$2 \implies 3$ . If  $A_n \in \mathcal{A}$  such that  $A_n \downarrow A \in \mathcal{A}$ , then  $A_n^c \uparrow A^c$  and therefore,

$$\lim_{n \rightarrow \infty} (1 - P(A_n)) = \lim_{n \rightarrow \infty} P(A_n^c) = P(A^c) = 1 - P(A).$$

$3 \implies 2$ . If  $A_n \in \mathcal{A}$  such that  $A_n \uparrow A \in \mathcal{A}$ , then  $A_n^c \downarrow A^c$  and therefore we again have,

$$\lim_{n \rightarrow \infty} (1 - P(A_n)) = \lim_{n \rightarrow \infty} P(A_n^c) = P(A^c) = 1 - P(A).$$

It is clear that  $2 \implies 4$  and that  $3 \implies 5$ . To finish the proof we will show  $5 \implies 2$  and  $5 \implies 3$ .

$5 \implies 2$ . If  $A_n \in \mathcal{A}$  such that  $A_n \uparrow A \in \mathcal{A}$ , then  $A \setminus A_n \downarrow \emptyset$  and therefore

$$\lim_{n \rightarrow \infty} [P(A) - P(A_n)] = \lim_{n \rightarrow \infty} P(A \setminus A_n) = 0.$$

$5 \implies 3$ . If  $A_n \in \mathcal{A}$  such that  $A_n \downarrow A \in \mathcal{A}$ , then  $A_n \setminus A \downarrow \emptyset$ . Therefore,

$$\lim_{n \rightarrow \infty} [P(A_n) - P(A)] = \lim_{n \rightarrow \infty} P(A_n \setminus A) = 0.$$

*Remark 4.4.* Observe that the equivalence of items 1. and 2. in the above proposition hold without the restriction that  $P(\Omega) = 1$  and in fact  $P(\Omega) = \infty$  may be allowed for this equivalence.

**Definition 4.5.** Let  $(\Omega, \mathcal{B})$  be a *measurable space*, i.e.  $\mathcal{B} \subset 2^{\Omega}$  is a  $\sigma$ -algebra. A **probability measure on  $(\Omega, \mathcal{B})$**  is a finitely additive probability measure,  $P : \mathcal{B} \rightarrow [0, 1]$  such that any and hence all of the continuity properties in Proposition 4.3 hold. We will call  $(\Omega, \mathcal{B}, P)$  a *probability space*.

**Lemma 4.6.** *Suppose that  $(\Omega, \mathcal{B}, P)$  is a probability space, then  $P$  is countably sub-additive.*

**Proof.** Suppose that  $A_n \in \mathcal{B}$  and let  $A'_1 := A_1$  and for  $n \geq 2$ , let  $A'_n := A_n \setminus (A_1 \cup \dots \cup A_{n-1}) \in \mathcal{B}$ . Then

$$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} A'_n) = \sum_{n=1}^{\infty} P(A'_n) \leq \sum_{n=1}^{\infty} P(A_n).$$

## 4.2 Examples of Measures

Most  $\sigma$ -algebras and  $\sigma$ -additive measures are somewhat difficult to describe and define. However, there are a few special cases where we can describe explicitly what is going on.

*Example 4.7.* Suppose that  $\Omega$  is a finite set,  $\mathcal{B} := 2^{\Omega}$ , and  $p : \Omega \rightarrow [0, 1]$  is a function such that

$$\sum_{\omega \in \Omega} p(\omega) = 1.$$

Then

$$P(A) := \sum_{\omega \in A} p(\omega) \text{ for all } A \subset \Omega$$

defines a measure on  $2^{\Omega}$ .



*Example 4.8.* Suppose that  $X$  is any set and  $x \in X$  is a point. For  $A \subset X$ , let

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then  $\mu = \delta_x$  is a measure on  $X$  called the Dirac delta measure at  $x$ .

*Example 4.9.* Suppose that  $\mu$  is a measure on  $X$  and  $\lambda > 0$ , then  $\lambda \cdot \mu$  is also a measure on  $X$ . Moreover, if  $\{\mu_j\}_{j \in J}$  are all measures on  $X$ , then  $\mu = \sum_{j=1}^{\infty} \mu_j$ , i.e.

$$\mu(A) = \sum_{j=1}^{\infty} \mu_j(A) \text{ for all } A \subset X$$

is a measure on  $X$ . (See Section 3.1 for the meaning of this sum.) To prove this we must show that  $\mu$  is countably additive. Suppose that  $\{A_i\}_{i=1}^{\infty}$  is a collection of pair-wise disjoint subsets of  $X$ , then

$$\begin{aligned} \mu(\cup_{i=1}^{\infty} A_i) &= \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_j(A_i) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \mu_j(A_i) = \sum_{j=1}^{\infty} \mu_j(\cup_{i=1}^{\infty} A_i) \\ &= \mu(\cup_{i=1}^{\infty} A_i) \end{aligned}$$

wherein the third equality we used Theorem 1.6 and in the fourth we used that fact that  $\mu_j$  is a measure.

*Example 4.10.* Suppose that  $X$  is a set  $\lambda : X \rightarrow [0, \infty]$  is a function. Then

$$\mu := \sum_{x \in X} \lambda(x) \delta_x$$

is a measure, explicitly

$$\mu(A) = \sum_{x \in A} \lambda(x)$$

for all  $A \subset X$ .

*Example 4.11.* Suppose that  $\mathcal{F} \subset 2^X$  is a countable or finite partition of  $X$  and  $\mathcal{B} \subset 2^X$  is the  $\sigma$ -algebra which consists of the collection of sets  $A \subset X$  such that

$$A = \cup \{\alpha \in \mathcal{F} : \alpha \subset A\}. \quad (4.5)$$

Any measure  $\mu : \mathcal{B} \rightarrow [0, \infty]$  is determined uniquely by its values on  $\mathcal{F}$ . Conversely, if we are given any function  $\lambda : \mathcal{F} \rightarrow [0, \infty]$  we may define, for  $A \in \mathcal{B}$ ,

$$\mu(A) = \sum_{\alpha \in \mathcal{F} \ni \alpha \subset A} \lambda(\alpha) = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A}$$

where  $1_{\alpha \subset A}$  is one if  $\alpha \subset A$  and zero otherwise. We may check that  $\mu$  is a measure on  $\mathcal{B}$ . Indeed, if  $A = \coprod_{i=1}^{\infty} A_i$  and  $\alpha \in \mathcal{F}$ , then  $\alpha \subset A$  iff  $\alpha \subset A_i$  for one and hence exactly one  $A_i$ . Therefore  $1_{\alpha \subset A} = \sum_{i=1}^{\infty} 1_{\alpha \subset A_i}$  and hence

$$\begin{aligned} \mu(A) &= \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A} = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) \sum_{i=1}^{\infty} 1_{\alpha \subset A_i} \\ &= \sum_{i=1}^{\infty} \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A_i} = \sum_{i=1}^{\infty} \mu(A_i) \end{aligned}$$

as desired. Thus we have shown that there is a one to one correspondence between measures  $\mu$  on  $\mathcal{B}$  and functions  $\lambda : \mathcal{F} \rightarrow [0, \infty]$ .

The following example explains what is going on in a more typical case of interest to us in the sequel.

*Example 4.12.* Suppose that  $\Omega = \mathbb{R}$ ,  $\mathcal{A}$  consists of those sets,  $A \subset \mathbb{R}$  which may be written as finite disjoint unions from

$$\mathcal{S} := \{(a, b] \cap \mathbb{R} : -\infty \leq a \leq b \leq \infty\}.$$

We will show below the following:

1.  $\mathcal{A}$  is an algebra. (Recall that  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A})$ .)
2. To every increasing function,  $F : \mathbb{R} \rightarrow [0, 1]$  such that

$$F(-\infty) := \lim_{x \rightarrow -\infty} F(x) = 0 \text{ and}$$

$$F(+\infty) := \lim_{x \rightarrow \infty} F(x) = 1$$

there exists a finitely additive probability measure,  $P = P_F$  on  $\mathcal{A}$  such that

$$P((a, b] \cap \mathbb{R}) = F(b) - F(a) \text{ for all } -\infty \leq a \leq b \leq \infty.$$

3.  $P$  is  $\sigma$ -additive on  $\mathcal{A}$  iff  $F$  is right continuous.
4.  $P$  extends to a probability measure on  $\mathcal{B}_{\mathbb{R}}$  iff  $F$  is right continuous.

Let us observe directly that if  $F(a+) := \lim_{x \downarrow a} F(x) \neq F(a)$ , then  $(a, a + 1/n] \downarrow \emptyset$  while

$$P((a, a + 1/n]) = F(a + 1/n) - F(a) \downarrow F(a+) - F(a) > 0.$$

Hence  $P$  can not be  $\sigma$ -additive on  $\mathcal{A}$  in this case.

### 4.3 Simple Integration

**Definition 4.13 (Simple Integral).** Suppose now that  $P$  is a finitely additive probability measure on an algebra  $\mathcal{A} \subset 2^X$ . For  $f \in \mathbb{S}(\mathcal{A})$  the **integral or expectation**,  $\mathbb{E}(f) = \mathbb{E}_P(f)$ , is defined by

$$\mathbb{E}_P(f) = \sum_{y \in \mathbb{C}} y P(f = y). \quad (4.6)$$

*Example 4.14.* Suppose that  $A \in \mathcal{A}$ , then

$$\mathbb{E}1_A = 0 \cdot P(A^c) + 1 \cdot P(A) = P(A). \quad (4.7)$$

*Remark 4.15.* Let us recall that our intuitive notion of  $P(A)$  was given as in Eq. (2.1) by

$$P(A) = \lim_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq k \leq N : \omega(k) \in A\}$$

where  $\omega(k) \in \Omega$  was the result of the  $k^{\text{th}}$  “independent” experiment. If we use this interpretation back in Eq. (4.6), we arrive at

$$\begin{aligned} \mathbb{E}(f) &= \sum_{y \in \mathbb{C}} y P(f = y) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{y \in \mathbb{C}} y \cdot \# \{1 \leq k \leq N : f(\omega(k)) = y\} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{y \in \mathbb{C}} y \cdot \sum_{k=1}^N 1_{f(\omega(k))=y} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \sum_{y \in \mathbb{C}} f(\omega(k)) \cdot 1_{f(\omega(k))=y} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(\omega(k)). \end{aligned}$$

Thus informally,  $\mathbb{E}f$  should represent the average of the values of  $f$  over many “independent” experiments.

**Proposition 4.16.** The expectation operator,  $\mathbb{E} = \mathbb{E}_P$ , satisfies:

1. If  $f \in \mathbb{S}(\mathcal{A})$  and  $\lambda \in \mathbb{C}$ , then

$$\mathbb{E}(\lambda f) = \lambda \mathbb{E}(f). \quad (4.8)$$

2. If  $f, g \in \mathbb{S}(\mathcal{A})$ , then

$$\mathbb{E}(f + g) = \mathbb{E}(f) + \mathbb{E}(g). \quad (4.9)$$

3.  $\mathbb{E}$  is **positive**, i.e.  $\mathbb{E}(f) \geq 0$  if  $f$  is a non-negative measurable simple function.

4. For all  $f \in \mathbb{S}(\mathcal{A})$ ,

$$|\mathbb{E}f| \leq \mathbb{E}|f|. \quad (4.10)$$

**Proof.**

1. If  $\lambda \neq 0$ , then

$$\begin{aligned} \mathbb{E}(\lambda f) &= \sum_{y \in \mathbb{C} \cup \{\infty\}} y P(\lambda f = y) = \sum_{y \in \mathbb{C} \cup \{\infty\}} y P(f = y/\lambda) \\ &= \sum_{z \in \mathbb{C} \cup \{\infty\}} \lambda z P(f = z) = \lambda \mathbb{E}(f). \end{aligned}$$

The case  $\lambda = 0$  is trivial.

2. Writing  $\{f = a, g = b\}$  for  $f^{-1}(\{a\}) \cap g^{-1}(\{b\})$ , then

$$\begin{aligned} \mathbb{E}(f + g) &= \sum_{z \in \mathbb{C}} z P(f + g = z) \\ &= \sum_{z \in \mathbb{C}} z P(\cup_{a+b=z} \{f = a, g = b\}) \\ &= \sum_{z \in \mathbb{C}} z \sum_{a+b=z} P(\{f = a, g = b\}) \\ &= \sum_{z \in \mathbb{C}} \sum_{a+b=z} (a + b) P(\{f = a, g = b\}) \\ &= \sum_{a,b} (a + b) P(\{f = a, g = b\}). \end{aligned}$$

But

$$\begin{aligned} \sum_{a,b} a P(\{f = a, g = b\}) &= \sum_a a \sum_b P(\{f = a, g = b\}) \\ &= \sum_a a P(\cup_b \{f = a, g = b\}) \\ &= \sum_a a P(\{f = a\}) = \mathbb{E}f \end{aligned}$$

and similarly,

$$\sum_{a,b} b P(\{f = a, g = b\}) = \mathbb{E}g.$$

Equation (4.9) is now a consequence of the last three displayed equations.

3. If  $f \geq 0$  then

$$\mathbb{E}(f) = \sum_{a \geq 0} a P(f = a) \geq 0.$$

4. First observe that

$$|f| = \sum_{\lambda \in \mathbb{C}} |\lambda| 1_{f=\lambda}$$

and therefore,

$$\mathbb{E}|f| = \mathbb{E} \sum_{\lambda \in \mathbb{C}} |\lambda| 1_{f=\lambda} = \sum_{\lambda \in \mathbb{C}} |\lambda| \mathbb{E} 1_{f=\lambda} = \sum_{\lambda \in \mathbb{C}} |\lambda| P(f = \lambda) \leq \max |\lambda|.$$

On the other hand,

$$|\mathbb{E}f| = \left| \sum_{\lambda \in \mathbb{C}} \lambda P(f = \lambda) \right| \leq \sum_{\lambda \in \mathbb{C}} |\lambda| P(f = \lambda) = \mathbb{E}|f|.$$

■

*Remark 4.17.* Every simple measurable function,  $f : \Omega \rightarrow \mathbb{C}$ , may be written as  $f = \sum_{j=1}^N \lambda_j 1_{A_j}$  for some  $\lambda_j \in \mathbb{C}$  and some  $A_j \in \mathcal{A}$ . Moreover if  $f$  is represented this way, then

$$\mathbb{E}f = \mathbb{E} \left[ \sum_{j=1}^N \lambda_j 1_{A_j} \right] = \sum_{j=1}^N \lambda_j \mathbb{E} 1_{A_j} = \sum_{j=1}^N \lambda_j P(A_j).$$

*Remark 4.18 (Chebyshev's Inequality).* Suppose that  $f \in \mathbb{S}(\mathcal{A})$ ,  $\varepsilon > 0$ , and  $p > 0$ , then

$$P(\{|f| \geq \varepsilon\}) = \mathbb{E}[1_{|f| \geq \varepsilon}] \leq \mathbb{E} \left[ \frac{|f|^p}{\varepsilon^p} 1_{|f| \geq \varepsilon} \right] \leq \varepsilon^{-p} \mathbb{E}|f|^p. \quad (4.11)$$

Observe that

$$|f|^p = \sum_{\lambda \in \mathbb{C}} |\lambda|^p 1_{\{f=\lambda\}}$$

is a simple random variable and  $\{|f| \geq \varepsilon\} = \sum_{|\lambda| \geq \varepsilon} \{f = \lambda\} \in \mathcal{A}$  as well. Therefore,  $\frac{|f|^p}{\varepsilon^p} 1_{|f| \geq \varepsilon}$  is still a simple random variable.

**Lemma 4.19 (Inclusion Exclusion Formula).** *If  $A_n \in \mathcal{A}$  for  $n = 1, 2, \dots, M$  such that  $\mu(\cup_{n=1}^M A_n) < \infty$ , then*

$$\mu(\cup_{n=1}^M A_n) = \sum_{k=1}^M (-1)^{k+1} \sum_{1 \leq n_1 < n_2 < \dots < n_k \leq M} \mu(A_{n_1} \cap \dots \cap A_{n_k}). \quad (4.12)$$

**Proof.** This may be proved inductively from Eq. (4.3). We will give a different and perhaps more illuminating proof here. Let  $A := \cup_{n=1}^M A_n$ .

Since  $A^c = (\cup_{n=1}^M A_n)^c = \cap_{n=1}^M A_n^c$ , we have

$$\begin{aligned} 1 - 1_A &= 1_{A^c} = \prod_{n=1}^M 1_{A_n^c} = \prod_{n=1}^M (1 - 1_{A_n}) \\ &= \sum_{k=0}^M (-1)^k \sum_{0 \leq n_1 < n_2 < \dots < n_k \leq M} 1_{A_{n_1}} \cdots 1_{A_{n_k}} \\ &= \sum_{k=0}^M (-1)^k \sum_{0 \leq n_1 < n_2 < \dots < n_k \leq M} 1_{A_{n_1} \cap \dots \cap A_{n_k}} \end{aligned}$$

from which it follows that

$$1_{\cup_{n=1}^M A_n} = 1_A = \sum_{k=1}^M (-1)^{k+1} \sum_{1 \leq n_1 < n_2 < \dots < n_k \leq M} 1_{A_{n_1} \cap \dots \cap A_{n_k}}. \quad (4.13)$$

Taking expectations of this equation then gives Eq. (4.12). ■

*Remark 4.20.* Here is an alternate proof of Eq. (4.13). Let  $\omega \in \Omega$  and by relabeling the sets  $\{A_n\}$  if necessary, we may assume that  $\omega \in A_1 \cap \dots \cap A_m$  and  $\omega \notin A_{m+1} \cup \dots \cup A_M$  for some  $0 \leq m \leq M$ . (When  $m = 0$ , both sides of Eq. (4.13) are zero and so we will only consider the case where  $1 \leq m \leq M$ .) With this notation we have

$$\begin{aligned} &\sum_{k=1}^M (-1)^{k+1} \sum_{1 \leq n_1 < n_2 < \dots < n_k \leq M} 1_{A_{n_1} \cap \dots \cap A_{n_k}}(\omega) \\ &= \sum_{k=1}^m (-1)^{k+1} \sum_{1 \leq n_1 < n_2 < \dots < n_k \leq m} 1_{A_{n_1} \cap \dots \cap A_{n_k}}(\omega) \\ &= \sum_{k=1}^m (-1)^{k+1} \binom{m}{k} \\ &= 1 - \sum_{k=0}^m (-1)^k (1)^{m-k} \binom{m}{k} \\ &= 1 - (1-1)^m = 1. \end{aligned}$$

This verifies Eq. (4.13) since  $1_{\cup_{n=1}^M A_n}(\omega) = 1$ .

*Example 4.21 (Coincidences).* Let  $\Omega$  be the set of permutations (think of card shuffling),  $\omega : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ , and define  $P(A) := \frac{\#(A)}{n!}$  to be the uniform distribution (Haar measure) on  $\Omega$ . We wish to compute the probability of the event,  $B$ , that a random permutation fixes some index  $i$ . To do this, let  $A_i := \{\omega \in \Omega : \omega(i) = i\}$  and observe that  $B = \cup_{i=1}^n A_i$ . So by the Inclusion Exclusion Formula, we have

$$P(B) = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < i_3 < \dots < i_k \leq n} P(A_{i_1} \cap \dots \cap A_{i_k}).$$

Since

$$\begin{aligned} P(A_{i_1} \cap \dots \cap A_{i_k}) &= P(\{\omega \in \Omega : \omega(i_1) = i_1, \dots, \omega(i_k) = i_k\}) \\ &= \frac{(n-k)!}{n!} \end{aligned}$$

and

$$\#\{1 \leq i_1 < i_2 < i_3 < \dots < i_k \leq n\} = \binom{n}{k},$$

we find

$$P(B) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^n (-1)^{k+1} \frac{1}{k!}.$$

For large  $n$  this gives,

$$P(B) = - \sum_{k=1}^n (-1)^k \frac{1}{k!} \cong - (e^{-1} - 1) \cong 0.632.$$

*Example 4.22.* Continue the notation in Example 4.21. We now wish to compute the expected number of fixed points of a random permutation,  $\omega$ , i.e. how many cards in the shuffled stack have not moved on average. To this end, let

$$X_i = 1_{A_i}$$

and observe that

$$N(\omega) = \sum_{i=1}^n X_i(\omega) = \sum_{i=1}^n 1_{\omega(i)=i} = \#\{i : \omega(i) = i\}.$$

denote the number of fixed points of  $\omega$ . Hence we have

$$\mathbb{E}N = \sum_{i=1}^n \mathbb{E}X_i = \sum_{i=1}^n P(A_i) = \sum_{i=1}^n \frac{(n-1)!}{n!} = 1.$$

Let us check the above formula when  $n = 6$ . In this case we have

$\omega$	$N(\omega)$
1 2 3	3
1 3 2	1
2 1 3	1
2 3 1	0
3 1 2	0
3 2 1	1

and so

$$P(\exists \text{ a fixed point}) = \frac{4}{6} = \frac{2}{3}$$

while

$$\sum_{k=1}^3 (-1)^{k+1} \frac{1}{k!} = 1 - \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

and

$$\mathbb{E}N = \frac{1}{6} (3 + 1 + 1 + 0 + 0 + 1) = 1.$$

#### 4.4 Simple Independence and the Weak Law of Large Numbers

For the next two problems, let  $A$  be a finite set,  $n \in \mathbb{N}$ ,  $\Omega = A^n$ , and  $X_i : \Omega \rightarrow A$  be defined by  $X_i(\omega) = \omega_i$  for  $\omega \in \Omega$  and  $i = 1, 2, \dots, n$ . We further suppose  $p : \Omega \rightarrow [0, 1]$  is a function such that

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

and  $P : 2^\Omega \rightarrow [0, 1]$  is the probability measure defined by

$$P(A) := \sum_{\omega \in A} p(\omega) \text{ for all } A \in 2^\Omega. \quad (4.14)$$

**Exercise 4.1 (Simple Independence 1.).** Suppose  $q_i : A \rightarrow [0, 1]$  are functions such that  $\sum_{\lambda \in A} q_i(\lambda) = 1$  for  $i = 1, 2, \dots, n$  and If  $p(\omega) = \prod_{i=1}^n q_i(\omega_i)$ . Show for any functions,  $f_i : A \rightarrow \mathbb{R}$  that

$$\mathbb{E}_P \left[ \prod_{i=1}^n f_i(X_i) \right] = \prod_{i=1}^n \mathbb{E}_P [f_i(X_i)] = \prod_{i=1}^n \mathbb{E}_{Q_i} f_i$$

where  $Q_i(\gamma) = \sum_{\lambda \in \gamma} q_i(\lambda)$  for all  $\gamma \subset A$ .

**Exercise 4.2 (Simple Independence 2.).** Prove the converse of the previous exercise. Namely, if

$$\mathbb{E}_P \left[ \prod_{i=1}^n f_i(X_i) \right] = \prod_{i=1}^n \mathbb{E}_P [f_i(X_i)] \quad (4.15)$$

for any functions,  $f_i : A \rightarrow \mathbb{R}$ , then there exists functions  $q_i : A \rightarrow [0, 1]$  with  $\sum_{\lambda \in A} q_i(\lambda) = 1$ , such that  $p(\omega) = \prod_{i=1}^n q_i(\omega_i)$ .

**Exercise 4.3 (A Weak Law of Large Numbers).** Suppose that  $A \subset \mathbb{R}$  is a finite set,  $n \in \mathbb{N}$ ,  $\Omega = A^n$ ,  $p(\omega) = \prod_{i=1}^n q(\omega_i)$  where  $q : A \rightarrow [0, 1]$  such that  $\sum_{\lambda \in A} q(\lambda) = 1$ , and let  $P : 2^\Omega \rightarrow [0, 1]$  be the probability measure defined as in Eq. (4.14). Further let  $X_i(\omega) = \omega_i$  for  $i = 1, 2, \dots, n$ ,  $\xi := \mathbb{E}X_i$ ,  $\sigma^2 := \mathbb{E}(X_i - \xi)^2$ , and

$$S_n = \frac{1}{n} (X_1 + \dots + X_n).$$

1. Show,  $\xi = \sum_{\lambda \in A} \lambda q(\lambda)$  and

$$\sigma^2 = \sum_{\lambda \in A} (\lambda - \xi)^2 q(\lambda) = \sum_{\lambda \in A} \lambda^2 q(\lambda) - \xi^2. \quad (4.16)$$

2. Show,  $\mathbb{E}S_n = \xi$ .

3. Let  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ . Show

$$\mathbb{E}[(X_i - \xi)(X_j - \xi)] = \delta_{ij}\sigma^2.$$

4. Using  $S_n - \xi$  may be expressed as,  $\frac{1}{n} \sum_{i=1}^n (X_i - \xi)$ , show

$$\mathbb{E}(S_n - \xi)^2 = \frac{1}{n}\sigma^2. \quad (4.17)$$

5. Conclude using Eq. (4.17) and Remark 4.18 that

$$P(|S_n - \xi| \geq \varepsilon) \leq \frac{1}{n\varepsilon^2}\sigma^2. \quad (4.18)$$

So for large  $n$ ,  $S_n$  is concentrated near  $\xi = \mathbb{E}X_i$  with probability approaching 1 for  $n$  large. This is a version of the weak law of large numbers.

**Exercise 4.4 (Bernoulli Random Variables).** Let  $A = \{0, 1\}$ ,  $X : A \rightarrow \mathbb{R}$  be defined by  $X(0) = 0$  and  $X(1) = 1$ ,  $x \in [0, 1]$ , and define  $Q = x\delta_1 + (1-x)\delta_0$ , i.e.  $Q(\{0\}) = 1-x$  and  $Q(\{1\}) = x$ . Verify,

$$\begin{aligned} \xi(x) &:= \mathbb{E}_Q X = x \text{ and} \\ \sigma^2(x) &:= \mathbb{E}_Q (X - x)^2 = (1-x)x \leq 1/4. \end{aligned}$$

**Theorem 4.23 (Weierstrass Approximation Theorem via Bernstein's Polynomials).** Suppose that  $f \in C([0, 1], \mathbb{C})$  and

$$p_n(x) := \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Then

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, 1]} |f(x) - p_n(x)| = 0.$$

**Proof.** Let  $x \in [0, 1]$ ,  $A = \{0, 1\}$ ,  $q(0) = 1-x$ ,  $q(1) = x$ ,  $\Omega = A^n$ , and

$$P_x(\{\omega\}) = q(\omega_1) \dots q(\omega_n) = x^{\sum_{i=1}^n \omega_i} \cdot (1-x)^{1-\sum_{i=1}^n \omega_i}.$$

As above, let  $S_n = \frac{1}{n} (X_1 + \dots + X_n)$ , where  $X_i(\omega) = \omega_i$  and observe that

$$P_x\left(S_n = \frac{k}{n}\right) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Therefore, writing  $\mathbb{E}_x$  for  $\mathbb{E}_{P_x}$ , we have

$$\mathbb{E}_x[f(S_n)] = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = p_n(x).$$

Hence we find

$$\begin{aligned} |p_n(x) - f(x)| &= |\mathbb{E}_x f(S_n) - f(x)| = |\mathbb{E}_x [f(S_n) - f(x)]| \\ &\leq \mathbb{E}_x |f(S_n) - f(x)| \\ &= \mathbb{E}_x [|f(S_n) - f(x)| : |S_n - x| \geq \varepsilon] \\ &\quad + \mathbb{E}_x [|f(S_n) - f(x)| : |S_n - x| < \varepsilon] \\ &\leq 2M \cdot P_x(|S_n - x| \geq \varepsilon) + \delta(\varepsilon) \end{aligned}$$

where

$$M := \max_{y \in [0, 1]} |f(y)| \text{ and}$$

$$\delta(\varepsilon) := \sup \{|f(y) - f(x)| : x, y \in [0, 1] \text{ and } |y - x| \leq \varepsilon\}$$

is the modulus of continuity of  $f$ . Now by the above exercises,

$$P_x(|S_n - x| \geq \varepsilon) \leq \frac{1}{4n\varepsilon^2} \quad (\text{see Figure 4.1})$$

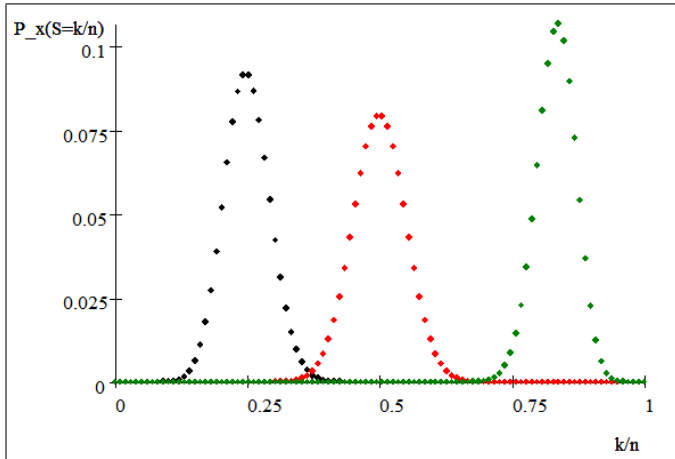
and hence we may conclude that

$$\max_{x \in [0, 1]} |p_n(x) - f(x)| \leq \frac{M}{2n\varepsilon^2} + \delta(\varepsilon)$$

and therefore, that

$$\limsup_{n \rightarrow \infty} \max_{x \in [0, 1]} |p_n(x) - f(x)| \leq \delta(\varepsilon).$$

This completes the proof, since by uniform continuity of  $f$ ,  $\delta(\varepsilon) \downarrow 0$  as  $\varepsilon \downarrow 0$ . ■



**Fig. 4.1.** Plots of  $P_x(S_n = k/n)$  versus  $k/n$  for  $n = 100$  with  $x = 1/4$  (black),  $x = 1/2$  (red), and  $x = 5/6$  (green).

### 4.5 Constructing Finitely Additive Measures

**Definition 4.24.** A set  $\mathcal{S} \subset 2^X$  is said to be an *semialgebra or elementary class* provided that

- $\emptyset \in \mathcal{S}$
- $\mathcal{S}$  is closed under finite intersections
- if  $E \in \mathcal{S}$ , then  $E^c$  is a finite disjoint union of sets from  $\mathcal{S}$ . (In particular  $X = \emptyset^c$  is a finite disjoint union of elements from  $\mathcal{S}$ .)

*Example 4.25.* Let  $X = \mathbb{R}$ , then

$$\begin{aligned} \mathcal{S} &:= \{(a, b] \cap \mathbb{R} : a, b \in \bar{\mathbb{R}}\} \\ &= \{(a, b] : a \in [-\infty, \infty) \text{ and } a < b < \infty\} \cup \{\emptyset, \mathbb{R}\} \end{aligned}$$

is a semiring.

**Exercise 4.5.** Let  $\mathcal{A} \subset 2^X$  and  $\mathcal{B} \subset 2^Y$  be semirings. Show the collection

$$\mathcal{E} := \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is also a semiring.

**Proposition 4.26.** Suppose  $\mathcal{S} \subset 2^X$  is a semiring, then  $\mathcal{A} = \mathcal{A}(\mathcal{S})$  consists of sets which may be written as finite disjoint unions of sets from  $\mathcal{S}$ .

**Proof.** Let  $\mathcal{A}$  denote the collection of sets which may be written as finite disjoint unions of sets from  $\mathcal{S}$ . Clearly  $\mathcal{S} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{S})$  so it suffices to show  $\mathcal{A}$  is an algebra since  $\mathcal{A}(\mathcal{S})$  is the smallest algebra containing  $\mathcal{S}$ . By the properties of  $\mathcal{S}$ , we know that  $\emptyset, X \in \mathcal{A}$ . Now suppose that  $A_i = \coprod_{F \in \Lambda_i} F \in \mathcal{A}$  where, for  $i = 1, 2, \dots, n$ ,  $\Lambda_i$  is a finite collection of disjoint sets from  $\mathcal{S}$ . Then

$$\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \left( \coprod_{F \in \Lambda_i} F \right) = \bigcup_{(F_1, \dots, F_n) \in \Lambda_1 \times \dots \times \Lambda_n} (F_1 \cap F_2 \cap \dots \cap F_n)$$

and this is a disjoint (you check) union of elements from  $\mathcal{S}$ . Therefore  $\mathcal{A}$  is closed under finite intersections. Similarly, if  $A = \coprod_{F \in \Lambda} F$  with  $\Lambda$  being a finite collection of disjoint sets from  $\mathcal{S}$ , then  $A^c = \bigcap_{F \in \Lambda} F^c$ . Since by assumption  $F^c \in \mathcal{A}$  for  $F \in \Lambda \subset \mathcal{S}$  and  $\mathcal{A}$  is closed under finite intersections, it follows that  $A^c \in \mathcal{A}$ . ■

*Example 4.27.* Let  $X = \mathbb{R}$  and  $\mathcal{S} := \{(a, b] \cap \mathbb{R} : a, b \in \bar{\mathbb{R}}\}$  be as in Example 4.25. Then  $\mathcal{A}(\mathcal{S})$  may be described as being those sets which are finite disjoint unions of sets from  $\mathcal{S}$ .

**Proposition 4.28 (Construction of Finitely Additive Measures).** Suppose  $\mathcal{S} \subset 2^X$  is a semi-algebra (see Definition 4.24) and  $\mathcal{A} = \mathcal{A}(\mathcal{S})$  is the algebra generated by  $\mathcal{S}$ . Then every additive function  $\mu : \mathcal{S} \rightarrow [0, \infty]$  such that  $\mu(\emptyset) = 0$  extends uniquely to an additive measure (which we still denote by  $\mu$ ) on  $\mathcal{A}$ .

**Proof.** Since (by Proposition 4.26) every element  $A \in \mathcal{A}$  is of the form  $A = \coprod_i E_i$  for a finite collection of  $E_i \in \mathcal{S}$ , it is clear that if  $\mu$  extends to a measure then the extension is unique and must be given by

$$\mu(A) = \sum_i \mu(E_i). \tag{4.19}$$

To prove existence, the main point is to show that  $\mu(A)$  in Eq. (4.19) is well defined; i.e. if we also have  $A = \coprod_j F_j$  with  $F_j \in \mathcal{S}$ , then we must show

$$\sum_i \mu(E_i) = \sum_j \mu(F_j). \tag{4.20}$$

But  $E_i = \coprod_j (E_i \cap F_j)$  and the additivity of  $\mu$  on  $\mathcal{S}$  implies  $\mu(E_i) = \sum_j \mu(E_i \cap F_j)$  and hence

$$\sum_i \mu(E_i) = \sum_i \sum_j \mu(E_i \cap F_j) = \sum_{i,j} \mu(E_i \cap F_j).$$

Similarly,

$$\sum_j \mu(F_j) = \sum_{i,j} \mu(E_i \cap F_j)$$

which combined with the previous equation shows that Eq. (4.20) holds. It is now easy to verify that  $\mu$  extended to  $\mathcal{A}$  as in Eq. (4.19) is an additive measure on  $\mathcal{A}$ . ■

**Proposition 4.29.** *Let  $X = \mathbb{R}$ ,  $\mathcal{S}$  be a semi-algebra*

$$\mathcal{S} = \{(a, b] \cap \mathbb{R} : -\infty \leq a \leq b \leq \infty\}, \quad (4.21)$$

and  $\mathcal{A} = \mathcal{A}(\mathcal{S})$  be the algebra formed by taking finite disjoint unions of elements from  $\mathcal{S}$ , see Proposition 4.26. To each finitely additive probability measures  $\mu : \mathcal{A} \rightarrow [0, \infty]$ , there is a unique increasing function  $F : \bar{\mathbb{R}} \rightarrow [0, 1]$  such that  $F(-\infty) = 0$ ,  $F(\infty) = 1$  and

$$\mu((a, b] \cap \mathbb{R}) = F(b) - F(a) \quad \forall a \leq b \text{ in } \bar{\mathbb{R}}. \quad (4.22)$$

Conversely, given an increasing function  $F : \bar{\mathbb{R}} \rightarrow [0, 1]$  such that  $F(-\infty) = 0$ ,  $F(\infty) = 1$  there is a unique finitely additive measure  $\mu = \mu_F$  on  $\mathcal{A}$  such that the relation in Eq. (4.22) holds.

**Proof.** Given a finitely additive probability measure  $\mu$ , let

$$F(x) := \mu((-\infty, x] \cap \mathbb{R}) \text{ for all } x \in \bar{\mathbb{R}}.$$

Then  $F(\infty) = 1$ ,  $F(-\infty) = 0$  and for  $b > a$ ,

$$F(b) - F(a) = \mu((-\infty, b] \cap \mathbb{R}) - \mu((-\infty, a] \cap \mathbb{R}) = \mu((a, b] \cap \mathbb{R}).$$

Conversely, suppose  $F : \bar{\mathbb{R}} \rightarrow [0, 1]$  as in the statement of the theorem is given. Define  $\mu$  on  $\mathcal{S}$  using the formula in Eq. (4.22). The argument will be completed by showing  $\mu$  is additive on  $\mathcal{S}$  and hence, by Proposition 4.28, has a unique extension to a finitely additive measure on  $\mathcal{A}$ . Suppose that

$$(a, b] = \bigsqcup_{i=1}^n (a_i, b_i].$$

By reordering  $(a_i, b_i]$  if necessary, we may assume that

$$a = a_1 < b_1 = a_2 < b_2 = a_3 < \cdots < b_{n-1} = a_n < b_n = b.$$

Therefore, by the telescoping series argument,

$$\mu((a, b] \cap \mathbb{R}) = F(b) - F(a) = \sum_{i=1}^n [F(b_i) - F(a_i)] = \sum_{i=1}^n \mu((a_i, b_i] \cap \mathbb{R}).$$

■





## Countably Additive Measures

### 5.1 Distribution Function for Probability Measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

**Definition 5.1.** Given a probability measure,  $P$  on  $\mathcal{B}_{\mathbb{R}}$ , the **cumulative distribution function (CDF)** of  $P$  is defined as the function,  $F = F_P : \mathbb{R} \rightarrow [0, 1]$  given as

$$F(x) := P((-\infty, x]).$$

*Example 5.2.* Suppose that

$$P = p\delta_{-1} + q\delta_1 + r\delta_{\pi}$$

with  $p, q, r > 0$  and  $p + q + r = 1$ . In this case,

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ p & \text{for } -1 \leq x < 1 \\ p + q & \text{for } 1 \leq x < \pi \\ 1 & \text{for } \pi \leq x < \infty \end{cases}.$$

**Lemma 5.3.** If  $F = F_P : \mathbb{R} \rightarrow [0, 1]$  is a distribution function for a probability measure,  $P$ , on  $\mathcal{B}_{\mathbb{R}}$ , then:

1.  $F(-\infty) := \lim_{x \rightarrow -\infty} F(x) = 0$ ,
2.  $F(\infty) := \lim_{x \rightarrow \infty} F(x) = 1$ ,
3.  $F$  is non-decreasing, and
4.  $F$  is right continuous.

**Theorem 5.4.** To each function  $F : \mathbb{R} \rightarrow [0, 1]$  satisfying properties 1. – 4. in Lemma 5.3, there exists a unique probability measure,  $P_F$ , on  $\mathcal{B}_{\mathbb{R}}$  such that

$$P_F((a, b]) = F(b) - F(a) \text{ for all } -\infty < a \leq b < \infty.$$

**Proof.** The uniqueness assertion in the theorem is covered in Exercise 5.1 below. The existence portion of the Theorem follows from Proposition 5.7 and Theorem 5.18 below. ■

*Example 5.5 (Uniform Distribution).* The function,

$$F(x) := \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x < \infty \end{cases},$$

is the distribution function for a measure,  $m$  on  $\mathcal{B}_{\mathbb{R}}$  which is concentrated on  $(0, 1]$ . The measure,  $m$  is called the **uniform distribution** or **Lebesgue measure** on  $(0, 1]$ .

Recall from Definition 3.14 that  $\mathcal{B} \subset 2^X$  is a  $\sigma$ -algebra on  $X$  if  $\mathcal{B}$  is an algebra which is closed under countable unions and intersections.

### 5.2 Construction of Premeasures

**Proposition 5.6.** Suppose that  $\mathcal{S} \subset 2^X$  is a semi-algebra,  $\mathcal{A} = \mathcal{A}(\mathcal{S})$  and  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a finitely additive measure. Then  $\mu$  is a premeasure on  $\mathcal{A}$  iff  $\mu$  is sub-additive on  $\mathcal{S}$ .

**Proof.** Clearly if  $\mu$  is a premeasure on  $\mathcal{A}$  then  $\mu$  is  $\sigma$ -additive and hence sub-additive on  $\mathcal{S}$ . Because of Proposition 4.2, to prove the converse it suffices to show that the sub-additivity of  $\mu$  on  $\mathcal{S}$  implies the sub-additivity of  $\mu$  on  $\mathcal{A}$ .

So suppose  $A = \bigsqcup_{n=1}^{\infty} A_n$  with  $A \in \mathcal{A}$  and each  $A_n \in \mathcal{A}$  which we express as  $A = \bigsqcup_{j=1}^k E_j$  with  $E_j \in \mathcal{S}$  and  $A_n = \bigsqcup_{i=1}^{N_n} E_{n,i}$  with  $E_{n,i} \in \mathcal{S}$ . Then

$$E_j = A \cap E_j = \bigsqcup_{n=1}^{\infty} A_n \cap E_j = \bigsqcup_{n=1}^{\infty} \bigsqcup_{i=1}^{N_n} E_{n,i} \cap E_j$$

which is a countable union and hence by assumption,

$$\mu(E_j) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(E_{n,i} \cap E_j).$$

Summing this equation on  $j$  and using the finite additivity of  $\mu$  shows

$$\begin{aligned}\mu(A) &= \sum_{j=1}^k \mu(E_j) \leq \sum_{j=1}^k \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(E_{n,i} \cap E_j) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \sum_{j=1}^k \mu(E_{n,i} \cap E_j) = \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(E_{n,i}) = \sum_{n=1}^{\infty} \mu(A_n),\end{aligned}$$

which proves (using Proposition 4.2) the sub-additivity of  $\mu$  on  $\mathcal{A}$ .  $\blacksquare$

Now suppose that  $F : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing function,  $F(\pm\infty) := \lim_{x \rightarrow \pm\infty} F(x)$  and  $\mu = \mu_F$  be the finitely additive measure on  $(\mathbb{R}, \mathcal{A})$  described in Proposition 4.29. If  $\mu$  happens to be a premeasure on  $\mathcal{A}$ , then, letting  $A_n = (a, b_n]$  with  $b_n \downarrow b$  as  $n \rightarrow \infty$ , implies

$$F(b_n) - F(a) = \mu((a, b_n]) \downarrow \mu((a, b]) = F(b) - F(a).$$

Since  $\{b_n\}_{n=1}^{\infty}$  was an arbitrary sequence such that  $b_n \downarrow b$ , we have shown  $\lim_{y \downarrow b} F(y) = F(b)$ , i.e.  $F$  is right continuous. The next proposition shows the converse is true as well. Hence premeasures on  $\mathcal{A}$  which are finite on bounded sets are in one to one correspondences with right continuous increasing functions which vanish at 0.

**Proposition 5.7.** *To each right continuous increasing function  $F : \mathbb{R} \rightarrow \mathbb{R}$  there exists a unique premeasure  $\mu = \mu_F$  on  $\mathcal{A}$  such that*

$$\mu_F((a, b]) = F(b) - F(a) \quad \forall \quad -\infty < a < b < \infty.$$

**Proof.** As above, let  $F(\pm\infty) := \lim_{x \rightarrow \pm\infty} F(x)$  and  $\mu = \mu_F$  be as in Proposition 4.29. Because of Proposition 5.6, to finish the proof it suffices to show  $\mu$  is sub-additive on  $\mathcal{S}$ .

First suppose that  $-\infty < a < b < \infty$ ,  $J = (a, b]$ ,  $J_n = (a_n, b_n]$  such that  $J = \bigsqcup_{n=1}^{\infty} J_n$ . We wish to show

$$\mu(J) \leq \sum_{n=1}^{\infty} \mu(J_n). \quad (5.1)$$

To do this choose numbers  $\tilde{a} > a$ ,  $\tilde{b}_n > b_n$  in which case  $I := (\tilde{a}, \tilde{b}] \subset J$ ,

$$\tilde{J}_n := (a_n, \tilde{b}_n] \supset \tilde{J}_n^o := (a_n, \tilde{b}_n) \supset J_n.$$

Since  $\bar{I} = [\tilde{a}, \tilde{b}]$  is compact and  $\bar{I} \subset J \subset \bigcup_{n=1}^{\infty} \tilde{J}_n^o$  there exists  $N < \infty$  such that

$$I \subset \bar{I} \subset \bigcup_{n=1}^N \tilde{J}_n^o \subset \bigcup_{n=1}^N \tilde{J}_n.$$

Hence by **finite** sub-additivity of  $\mu$ ,

$$F(b) - F(\tilde{a}) = \mu(I) \leq \sum_{n=1}^N \mu(\tilde{J}_n) \leq \sum_{n=1}^{\infty} \mu(\tilde{J}_n).$$

Using the right continuity of  $F$  and letting  $\tilde{a} \downarrow a$  in the above inequality,

$$\begin{aligned}\mu(J) &= \mu((a, b]) = F(b) - F(a) \leq \sum_{n=1}^{\infty} \mu(\tilde{J}_n) \\ &= \sum_{n=1}^{\infty} \mu(J_n) + \sum_{n=1}^{\infty} \mu(\tilde{J}_n \setminus J_n).\end{aligned} \quad (5.2)$$

Given  $\varepsilon > 0$ , we may use the right continuity of  $F$  to choose  $\tilde{b}_n$  so that

$$\mu(\tilde{J}_n \setminus J_n) = F(\tilde{b}_n) - F(b_n) \leq \varepsilon 2^{-n} \quad \forall n \in \mathbb{N}.$$

Using this in Eq. (5.2) shows

$$\mu(J) = \mu((a, b]) \leq \sum_{n=1}^{\infty} \mu(J_n) + \varepsilon$$

which verifies Eq. (5.1) since  $\varepsilon > 0$  was arbitrary.

The hard work is now done but we still have to check the cases where  $a = -\infty$  or  $b = \infty$ . For example, suppose that  $b = \infty$  so that

$$J = (a, \infty) = \bigsqcup_{n=1}^{\infty} J_n$$

with  $J_n = (a_n, b_n] \cap \mathbb{R}$ . Then

$$I_M := (a, M] = J \cap I_M = \bigsqcup_{n=1}^{\infty} J_n \cap I_M$$

and so by what we have already proved,

$$F(M) - F(a) = \mu(I_M) \leq \sum_{n=1}^{\infty} \mu(J_n \cap I_M) \leq \sum_{n=1}^{\infty} \mu(J_n).$$

Now let  $M \rightarrow \infty$  in this last inequality to find that

$$\mu((a, \infty)) = F(\infty) - F(a) \leq \sum_{n=1}^{\infty} \mu(J_n).$$

The other cases where  $a = -\infty$  and  $b \in \mathbb{R}$  and  $a = -\infty$  and  $b = \infty$  are handled similarly. ■

Before continuing our development of the existence of measures, we will pause to show that measures are often uniquely determined by their values on a generating sub-algebra. This detour will also have the added benefit of motivating Carathéodory's existence proof to be given below.

### 5.3 Regularity and Uniqueness Results

**Definition 5.8.** Given a collection of subsets,  $\mathcal{E}$ , of  $X$ , let  $\mathcal{E}_\sigma$  denote the collection of subsets of  $X$  which are finite or countable unions of sets from  $\mathcal{E}$ . Similarly let  $\mathcal{E}_\delta$  denote the collection of subsets of  $X$  which are finite or countable intersections of sets from  $\mathcal{E}$ . We also write  $\mathcal{E}_{\sigma\delta} = (\mathcal{E}_\sigma)_\delta$  and  $\mathcal{E}_{\delta\sigma} = (\mathcal{E}_\delta)_\sigma$ , etc.

**Lemma 5.9.** Suppose that  $\mathcal{A} \subset 2^X$  is an algebra. Then:

1.  $\mathcal{A}_\sigma$  is closed under taking countable unions and finite intersections.
2.  $\mathcal{A}_\delta$  is closed under taking countable intersections and finite unions.
3.  $\{A^c : A \in \mathcal{A}_\sigma\} = \mathcal{A}_\delta$  and  $\{A^c : A \in \mathcal{A}_\delta\} = \mathcal{A}_\sigma$ .

**Proof.** By construction  $\mathcal{A}_\sigma$  is closed under countable unions. Moreover if  $A = \cup_{i=1}^\infty A_i$  and  $B = \cup_{j=1}^\infty B_j$  with  $A_i, B_j \in \mathcal{A}$ , then

$$A \cap B = \cup_{i,j=1}^\infty A_i \cap B_j \in \mathcal{A}_\sigma,$$

which shows that  $\mathcal{A}_\sigma$  is also closed under finite intersections. Item 3. is straight forward and item 2. follows from items 1. and 3. ■

**Theorem 5.10 (Finite Regularity Result).** Suppose  $\mathcal{A} \subset 2^X$  is an algebra,  $\mathcal{B} = \sigma(\mathcal{A})$  and  $\mu : \mathcal{B} \rightarrow [0, \infty)$  is a finite measure, i.e.  $\mu(X) < \infty$ . Then for every  $\varepsilon > 0$  and  $B \in \mathcal{B}$  there exists  $A \in \mathcal{A}_\delta$  and  $C \in \mathcal{A}_\sigma$  such that  $A \subset B \subset C$  and  $\mu(C \setminus A) < \varepsilon$ .

**Proof.** Let  $\mathcal{B}_0$  denote the collection of  $B \in \mathcal{B}$  such that for every  $\varepsilon > 0$  there here exists  $A \in \mathcal{A}_\delta$  and  $C \in \mathcal{A}_\sigma$  such that  $A \subset B \subset C$  and  $\mu(C \setminus A) < \varepsilon$ . It is now clear that  $\mathcal{A} \subset \mathcal{B}_0$  and that  $\mathcal{B}_0$  is closed under complementation. Now suppose that  $B_i \in \mathcal{B}_0$  for  $i = 1, 2, \dots$  and  $\varepsilon > 0$  is given. By assumption there exists  $A_i \in \mathcal{A}_\delta$  and  $C_i \in \mathcal{A}_\sigma$  such that  $A_i \subset B_i \subset C_i$  and  $\mu(C_i \setminus A_i) < 2^{-i}\varepsilon$ .

Let  $A := \cup_{i=1}^\infty A_i$ ,  $A^N := \cup_{i=1}^N A_i \in \mathcal{A}_\delta$ ,  $B := \cup_{i=1}^\infty B_i$ , and  $C := \cup_{i=1}^\infty C_i \in \mathcal{A}_\sigma$ . Then  $A^N \subset A \subset B \subset C$  and

$$C \setminus A = [\cup_{i=1}^\infty C_i] \setminus A = \cup_{i=1}^\infty [C_i \setminus A] \subset \cup_{i=1}^\infty [C_i \setminus A_i].$$

Therefore,

$$\mu(C \setminus A) = \mu(\cup_{i=1}^\infty [C_i \setminus A]) \leq \sum_{i=1}^\infty \mu(C_i \setminus A) \leq \sum_{i=1}^\infty \mu(C_i \setminus A_i) < \varepsilon.$$

Since  $C \setminus A^N \downarrow C \setminus A$ , it also follows that  $\mu(C \setminus A^N) < \varepsilon$  for sufficiently large  $N$  and this shows  $B = \cup_{i=1}^\infty B_i \in \mathcal{B}_0$ . Hence  $\mathcal{B}_0$  is a sub- $\sigma$ -algebra of  $\mathcal{B} = \sigma(\mathcal{A})$  which contains  $\mathcal{A}$  which shows  $\mathcal{B}_0 = \mathcal{B}$ . ■

**Corollary 5.11 ( $\sigma$  – Finite Regularity Result).** Theorem 5.10 continues to hold under the weaker assumption that  $\mu : \mathcal{B} \rightarrow [0, \infty]$  is a measure which is  $\sigma$  – finite on  $\mathcal{A}$ , i.e. there exists  $X_n \in \mathcal{A}$  such that  $\mu(X_n) < \infty$  for all  $n$ .

**Proof.** Since  $A \in \mathcal{B} \rightarrow \mu_i(A) := \mu(X_i \cap A)$  is a finite measure on  $A \in \mathcal{B}$  for each  $i$ , by Theorem 5.10, for every  $B \in \mathcal{B}$  there exists  $C_i \in \mathcal{A}_\sigma$  such that  $B \subset C_i$  and  $\mu(X_i \cap [C_i \setminus B]) < 2^{-i}\varepsilon$ . Now let  $C := \cup_{i=1}^\infty [X_i \cap C_i] \in \mathcal{A}_\sigma$  and observe that  $B \subset C$  and

$$\begin{aligned} \mu(C \setminus B) &= \mu(\cup_{i=1}^\infty ([X_i \cap C_i] \setminus B)) \\ &\leq \sum_{i=1}^\infty \mu([X_i \cap C_i] \setminus B) = \sum_{i=1}^\infty \mu(X_i \cap [C_i \setminus B]) < \varepsilon. \end{aligned}$$

Applying this result to  $B^c$  shows there exists  $D \in \mathcal{A}_\sigma$  such that  $B^c \subset D$  and

$$\mu(B \setminus D^c) = \mu(D \setminus B^c) < \varepsilon.$$

So if we let  $A := D^c \in \mathcal{A}_\delta$ , then  $A \subset B \subset C$  and

$$\mu(C \setminus A) = \mu([B \setminus A] \cup [(C \setminus B) \setminus A]) \leq \mu(B \setminus A) + \mu(C \setminus B) < 2\varepsilon$$

and the result is proved. ■

**Exercise 5.1.** Suppose  $\mathcal{A} \subset 2^X$  is an algebra and  $\mu$  and  $\nu$  are two measures on  $\mathcal{B} = \sigma(\mathcal{A})$ .

- a. Suppose that  $\mu$  and  $\nu$  are finite measures such that  $\mu = \nu$  on  $\mathcal{A}$ . Show  $\mu = \nu$ .
- b. Generalize the previous assertion to the case where you only assume that  $\mu$  and  $\nu$  are  $\sigma$  – finite on  $\mathcal{A}$ .

**Corollary 5.12.** Suppose  $\mathcal{A} \subset 2^X$  is an algebra and  $\mu : \mathcal{B} = \sigma(\mathcal{A}) \rightarrow [0, \infty]$  is a measure which is  $\sigma$  – finite on  $\mathcal{A}$ . Then for all  $B \in \mathcal{B}$ , there exists  $A \in \mathcal{A}_{\delta\sigma}$  and  $C \in \mathcal{A}_{\sigma\delta}$  such that  $A \subset B \subset C$  and  $\mu(C \setminus A) = 0$ .

**Proof.** By Theorem 5.10, given  $B \in \mathcal{B}$ , we may choose  $A_n \in \mathcal{A}_\delta$  and  $C_n \in \mathcal{A}_\sigma$  such that  $A_n \subset B \subset C_n$  and  $\mu(C_n \setminus B) \leq 1/n$  and  $\mu(B \setminus A_n) \leq 1/n$ . By replacing  $A_N$  by  $\cup_{n=1}^N A_n$  and  $C_N$  by  $\cap_{n=1}^N C_n$ , we may assume that  $A_n \uparrow$  and  $C_n \downarrow$  as  $n$  increases. Let  $A = \cup A_n \in \mathcal{A}_{\delta\sigma}$  and  $C = \cap C_n \in \mathcal{A}_{\sigma\delta}$ , then  $A \subset B \subset C$  and

$$\begin{aligned} \mu(C \setminus A) &= \mu(C \setminus B) + \mu(B \setminus A) \leq \mu(C_n \setminus B) + \mu(B \setminus A_n) \\ &\leq 2/n \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

■

**Exercise 5.2.** Let  $\mathcal{B} = \mathcal{B}_{\mathbb{R}^n} = \sigma(\{\text{open subsets of } \mathbb{R}^n\})$  be the Borel  $\sigma$ -algebra on  $\mathbb{R}^n$  and  $\mu$  be a probability measure on  $\mathcal{B}$ . Further, let  $\mathcal{B}_0$  denote those sets  $B \in \mathcal{B}$  such that for every  $\varepsilon > 0$  there exists  $F \subset B \subset V$  such that  $F$  is closed,  $V$  is open, and  $\mu(V \setminus F) < \varepsilon$ . Show:

1.  $\mathcal{B}_0$  contains all closed subsets of  $\mathcal{B}$ . **Hint:** given a closed subset,  $F \subset \mathbb{R}^n$  and  $k \in \mathbb{N}$ , let  $V_k := \cup_{x \in F} B(x, 1/k)$ , where  $B(x, \delta) := \{y \in \mathbb{R}^n : |y - x| < \delta\}$ . Show,  $V_k \downarrow F$  as  $k \rightarrow \infty$ .
2. Show  $\mathcal{B}_0$  is a  $\sigma$ -algebra and use this along with the first part of this exercise to conclude  $\mathcal{B} = \mathcal{B}_0$ . **Hint:** follow closely the method used in the first step of the proof of Theorem 5.10.
3. Show for every  $\varepsilon > 0$  and  $B \in \mathcal{B}$ , there exist a compact subset,  $K \subset \mathbb{R}^n$ , such that  $K \subset B$  and  $\mu(B \setminus K) < \varepsilon$ . **Hint:** take  $K := F \cap \{x \in \mathbb{R}^n : |x| \leq n\}$  for some sufficiently large  $n$ .

## 5.4 Construction of Measures

*Remark 5.13.* Let us recall from Proposition 4.3 and Remark 4.4 that a finitely additive measure  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a premeasure on  $\mathcal{A}$  iff  $\mu(A_n) \uparrow \mu(A)$  for all  $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$  such that  $A_n \uparrow A \in \mathcal{A}$ . Furthermore if  $\mu(X) < \infty$ , then  $\mu$  is a premeasure on  $\mathcal{A}$  iff  $\mu(A_n) \downarrow 0$  for all  $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$  such that  $A_n \downarrow \emptyset$ .

**Proposition 5.14.** *Let  $\mu$  be a premeasure on an algebra  $\mathcal{A}$ , then  $\mu$  has a unique extension (still called  $\mu$ ) to a function on  $\mathcal{A}_\sigma$  satisfying the following properties.*

1. (**Continuity**) If  $A_n \in \mathcal{A}$  and  $A_n \uparrow A \in \mathcal{A}_\sigma$ , then  $\mu(A_n) \uparrow \mu(A)$  as  $n \rightarrow \infty$ .
2. (**Monotonicity**) If  $A, B \in \mathcal{A}_\sigma$  with  $A \subset B$  then  $\mu(A) \leq \mu(B)$ .
3. (**Strong Additivity**) If  $A, B \in \mathcal{A}_\sigma$ , then

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B). \quad (5.3)$$

4. (**Sub-Additivity on  $\mathcal{A}_\sigma$** ) The function  $\mu$  is sub-additive on  $\mathcal{A}_\sigma$ , i.e. if  $\{A_n\}_{n=1}^\infty \subset \mathcal{A}_\sigma$ , then

$$\mu(\cup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu(A_n). \quad (5.4)$$

5. ( **$\sigma$ -Additivity on  $\mathcal{A}_\sigma$** ) The function  $\mu$  is countably additive on  $\mathcal{A}_\sigma$ .

**Proof.** Let  $A, B$  be sets in  $\mathcal{A}_\sigma$  such that  $A \subset B$  and suppose  $\{A_n\}_{n=1}^\infty$  and  $\{B_n\}_{n=1}^\infty$  are sequences in  $\mathcal{A}$  such that  $A_n \uparrow A$  and  $B_n \uparrow B$  as  $n \rightarrow \infty$ . Since  $B_m \cap A_n \uparrow A_n$  as  $m \rightarrow \infty$ , the continuity of  $\mu$  on  $\mathcal{A}$  implies,

$$\mu(A_n) = \lim_{m \rightarrow \infty} \mu(B_m \cap A_n) \leq \lim_{m \rightarrow \infty} \mu(B_m).$$

We may let  $n \rightarrow \infty$  in this inequality to find,

$$\lim_{n \rightarrow \infty} \mu(A_n) \leq \lim_{m \rightarrow \infty} \mu(B_m). \quad (5.5)$$

Using this equation when  $B = A$ , implies,  $\lim_{n \rightarrow \infty} \mu(A_n) = \lim_{m \rightarrow \infty} \mu(B_m)$  whenever  $A_n \uparrow A$  and  $B_n \uparrow A$ . Therefore it is unambiguous to define  $\mu(A)$  by;

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$$

for any sequence  $\{A_n\}_{n=1}^\infty \subset \mathcal{A}$  such that  $A_n \uparrow A$ . With this definition, the continuity of  $\mu$  is clear and the monotonicity of  $\mu$  follows from Eq. (5.5).

Suppose that  $A, B \in \mathcal{A}_\sigma$  and  $\{A_n\}_{n=1}^\infty$  and  $\{B_n\}_{n=1}^\infty$  are sequences in  $\mathcal{A}$  such that  $A_n \uparrow A$  and  $B_n \uparrow B$  as  $n \rightarrow \infty$ . Then passing to the limit as  $n \rightarrow \infty$  in the identity,

$$\mu(A_n \cup B_n) + \mu(A_n \cap B_n) = \mu(A_n) + \mu(B_n)$$

proves Eq. (5.3). In particular, it follows that  $\mu$  is finitely additive on  $\mathcal{A}_\sigma$ .

Let  $\{A_n\}_{n=1}^\infty$  be any sequence in  $\mathcal{A}_\sigma$  and choose  $\{A_{n,i}\}_{i=1}^\infty \subset \mathcal{A}$  such that  $A_{n,i} \uparrow A_n$  as  $i \rightarrow \infty$ . Then we have,

$$\mu(\cup_{n=1}^N A_{n,N}) \leq \sum_{n=1}^N \mu(A_{n,N}) \leq \sum_{n=1}^N \mu(A_n) \leq \sum_{n=1}^\infty \mu(A_n). \quad (5.6)$$

Since  $\mathcal{A} \ni \cup_{n=1}^N A_{n,N} \uparrow \cup_{n=1}^\infty A_n \in \mathcal{A}_\sigma$ , we may let  $N \rightarrow \infty$  in Eq. (5.6) to conclude Eq. (5.4) holds.

If we further assume that  $\{A_n\}_{n=1}^\infty \subset \mathcal{A}_\sigma$  is a disjoint sequence, by the finite additivity and monotonicity of  $\mu$  on  $\mathcal{A}_\sigma$ , we have

$$\sum_{n=1}^\infty \mu(A_n) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(A_n) = \lim_{N \rightarrow \infty} \mu(\cup_{n=1}^N A_n) \leq \mu(\cup_{n=1}^\infty A_n).$$

The previous two inequalities show  $\mu$  is  $\sigma$ -additive on  $\mathcal{A}_\sigma$ .  $\blacksquare$

Suppose  $\mu$  is a finite premeasure on an algebra,  $\mathcal{A} \subset 2^X$ , and  $A \in \mathcal{A}_\delta \cap \mathcal{A}_\sigma$ . Since  $A, A^c \in \mathcal{A}_\sigma$  and  $X = A \cup A^c$ , it follows that  $\mu(X) = \mu(A) + \mu(A^c)$ . From this observation we may extend  $\mu$  to a function on  $\mathcal{A}_\delta \cup \mathcal{A}_\sigma$  by defining

$$\mu(A) := \mu(X) - \mu(A^c) \text{ for all } A \in \mathcal{A}_\delta. \quad (5.7)$$

**Lemma 5.15.** *Suppose  $\mu$  is a finite premeasure on an algebra,  $\mathcal{A} \subset 2^X$ , and  $\mu$  has been extended to  $\mathcal{A}_\delta \cup \mathcal{A}_\sigma$  as described in Proposition 5.14 and Eq. (5.7) above.*

1. If  $A \in \mathcal{A}_\delta$  and  $A_n \in \mathcal{A}$  such that  $A_n \downarrow A$ , then  $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$ .
2.  $\mu$  is additive when restricted to  $\mathcal{A}_\delta$ .
3. If  $A \in \mathcal{A}_\delta$  and  $C \in \mathcal{A}_\sigma$  such that  $A \subset C$ , then  $\mu(C \setminus A) = \mu(C) - \mu(A)$ .

**Proof.**

1. Since  $A_n^c \uparrow A^c \in \mathcal{A}_\sigma$ , by the definition of  $\mu(A)$  and Proposition 5.14 it follows that

$$\begin{aligned} \mu(A) &= \mu(X) - \mu(A^c) = \mu(X) - \lim_{n \rightarrow \infty} \mu(A_n^c) \\ &= \lim_{n \rightarrow \infty} [\mu(X) - \mu(A_n^c)] = \lim_{n \rightarrow \infty} \mu(A_n). \end{aligned}$$

2. Suppose  $A, B \in \mathcal{A}_\delta$  are disjoint sets and  $A_n, B_n \in \mathcal{A}$  such that  $A_n \downarrow A$  and  $B_n \downarrow B$ , then  $A_n \cup B_n \downarrow A \cup B$  and therefore,

$$\begin{aligned} \mu(A \cup B) &= \lim_{n \rightarrow \infty} \mu(A_n \cup B_n) = \lim_{n \rightarrow \infty} [\mu(A_n) + \mu(B_n) - \mu(A_n \cap B_n)] \\ &= \mu(A) + \mu(B) \end{aligned}$$

wherein the last equality we have used Proposition 4.3.

3. By assumption,  $X = A^c \cup C$ . So applying the strong additivity of  $\mu$  on  $\mathcal{A}_\sigma$  in Eq. (5.3) with  $A \rightarrow A^c \in \mathcal{A}_\sigma$  and  $B \rightarrow C \in \mathcal{A}_\sigma$  shows

$$\begin{aligned} \mu(X) + \mu(C \setminus A) &= \mu(A^c \cup C) + \mu(A^c \cap C) \\ &= \mu(A^c) + \mu(C) = \mu(X) - \mu(A) + \mu(C). \end{aligned}$$

$\blacksquare$

**Definition 5.16 (Measurable Sets).** *Suppose  $\mu$  is a finite premeasure on an algebra  $\mathcal{A} \subset 2^X$ . We say that  $B \subset X$  is **measurable** if for all  $\varepsilon > 0$  there exists  $A \in \mathcal{A}_\delta$  and  $C \in \mathcal{A}_\sigma$  such that  $A \subset B \subset C$  and  $\mu(C \setminus A) < \varepsilon$ . We will denote the collection of measurable subsets of  $X$  by  $\mathcal{B} = \mathcal{B}(\mu)$ . We also define  $\bar{\mu} : \mathcal{B} \rightarrow [0, \mu(X)]$  by*

$$\bar{\mu}(B) = \inf \{ \mu(C) : B \subset C \in \mathcal{A}_\sigma \}. \quad (5.8)$$

*Remark 5.17.* If  $B \in \mathcal{B}$ ,  $\varepsilon > 0$ ,  $A \in \mathcal{A}_\delta$  and  $C \in \mathcal{A}_\sigma$  are such that  $A \subset B \subset C$  and  $\mu(C \setminus A) < \varepsilon$ , then  $\mu(A) \leq \bar{\mu}(B) \leq \mu(C)$  and in particular,

$$0 \leq \bar{\mu}(B) - \mu(A) < \varepsilon, \text{ and } 0 \leq \mu(C) - \bar{\mu}(B) < \varepsilon. \quad (5.9)$$

Indeed, if  $C' \in \mathcal{A}_\sigma$  with  $B \subset C'$ , then  $A \subset C'$  and so by Lemma 5.15,

$$\mu(A) \leq \mu(C' \setminus A) + \mu(A) = \mu(C')$$

from which it follows that  $\mu(A) \leq \bar{\mu}(B)$ . The fact that  $\bar{\mu}(B) \leq \mu(C)$  follows directly from Eq. (5.8).

**Theorem 5.18 (Finite Premeasure Extension Theorem).** *Suppose  $\mu$  is a finite premeasure on an algebra  $\mathcal{A} \subset 2^X$ . Then  $\mathcal{B}$  is a  $\sigma$ -algebra on  $X$  which contains  $\mathcal{A}$  and  $\bar{\mu}$  is a  $\sigma$ -additive measure on  $\mathcal{B}$ . Moreover,  $\bar{\mu}$  is the unique measure on  $\mathcal{B}$  such that  $\bar{\mu}|_{\mathcal{A}} = \mu$ .*

**Proof.** It is clear that  $\mathcal{A} \subset \mathcal{B}$  and that  $\mathcal{B}$  is closed under complementation. Now suppose that  $B_i \in \mathcal{B}$  for  $i = 1, 2$  and  $\varepsilon > 0$  is given. We may then choose  $A_i \subset B_i \subset C_i$  such that  $A_i \in \mathcal{A}_\delta$ ,  $C_i \in \mathcal{A}_\sigma$ , and  $\mu(C_i \setminus A_i) < \varepsilon$  for  $i = 1, 2$ . Then with  $A = A_1 \cup A_2$ ,  $B = B_1 \cup B_2$  and  $C = C_1 \cup C_2$ , we have  $\mathcal{A}_\delta \ni A \subset B \subset C \in \mathcal{A}_\sigma$ . Since

$$C \setminus A = (C_1 \setminus A) \cup (C_2 \setminus A) \subset (C_1 \setminus A_1) \cup (C_2 \setminus A_2),$$

it follows from the sub-additivity of  $\mu$  that with

$$\mu(C \setminus A) \leq \mu(C_1 \setminus A_1) + \mu(C_2 \setminus A_2) < 2\varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, we have shown that  $B \in \mathcal{B}$ . Hence we now know that  $\mathcal{B}$  is an algebra.

Because  $\mathcal{B}$  is an algebra, to verify that  $\mathcal{B}$  is a  $\sigma$ -algebra it suffices to show that  $B = \prod_{n=1}^{\infty} B_n \in \mathcal{B}$  whenever  $\{B_n\}_{n=1}^{\infty}$  is a disjoint sequence in  $\mathcal{B}$ . To prove  $B \in \mathcal{B}$ , let  $\varepsilon > 0$  be given and choose  $A_i \subset B_i \subset C_i$  such that  $A_i \in \mathcal{A}_\delta$ ,  $C_i \in \mathcal{A}_\sigma$ , and  $\mu(C_i \setminus A_i) < \varepsilon 2^{-i}$  for all  $i$ . Since the  $\{A_i\}_{i=1}^{\infty}$  are pairwise disjoint we may use Lemma 5.15 to show,

$$\begin{aligned} \sum_{i=1}^n \mu(C_i) &= \sum_{i=1}^n (\mu(A_i) + \mu(C_i \setminus A_i)) \\ &= \mu(\cup_{i=1}^n A_i) + \sum_{i=1}^n \mu(C_i \setminus A_i) \leq \mu(X) + \sum_{i=1}^n \varepsilon 2^{-i}. \end{aligned}$$

Passing to the limit,  $n \rightarrow \infty$ , in this equation then shows

$$\sum_{i=1}^{\infty} \mu(C_i) \leq \mu(X) + \varepsilon < \infty. \quad (5.10)$$

Let  $B = \cup_{i=1}^{\infty} B_i$ ,  $C := \cup_{i=1}^{\infty} C_i \in \mathcal{A}_\sigma$  and for  $n \in \mathbb{N}$  let  $A^n := \prod_{i=1}^n A_i \in \mathcal{A}_\delta$ . Then  $\mathcal{A}_\delta \ni A^n \subset B \subset C \in \mathcal{A}_\sigma$ ,  $C \setminus A^n \in \mathcal{A}_\sigma$  and

$$C \setminus A^n = \cup_{i=1}^{\infty} (C_i \setminus A^n) \subset [\cup_{i=1}^n (C_i \setminus A_i)] \cup [\cup_{i=n+1}^{\infty} C_i] \in \mathcal{A}_\sigma.$$

Therefore, using the sub-additivity of  $\mu$  on  $\mathcal{A}_\sigma$  and the estimate (5.10),

$$\begin{aligned} \mu(C \setminus A^n) &\leq \sum_{i=1}^n \mu(C_i \setminus A_i) + \sum_{i=n+1}^{\infty} \mu(C_i) \\ &\leq \varepsilon + \sum_{i=n+1}^{\infty} \mu(C_i) \rightarrow \varepsilon \text{ as } n \rightarrow \infty. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, it follows that  $B \in \mathcal{B}$ . Moreover by repeated use of Remark 5.17, we find

$$\begin{aligned} |\bar{\mu}(B) - \mu(A^n)| &< \varepsilon + \sum_{i=n+1}^{\infty} \mu(C_i) \text{ and} \\ \left| \sum_{i=1}^n \bar{\mu}(B_i) - \mu(A^n) \right| &= \left| \sum_{i=1}^n [\bar{\mu}(B_i) - \mu(A_i)] \right| \leq \sum_{i=1}^n |\bar{\mu}(B_i) - \mu(A_i)| \leq \varepsilon \sum_{i=1}^n 2^{-i} < \varepsilon. \end{aligned}$$

Combining these estimates shows

$$\left| \bar{\mu}(B) - \sum_{i=1}^n \bar{\mu}(B_i) \right| < 2\varepsilon + \sum_{i=n+1}^{\infty} \mu(C_i)$$

which upon letting  $n \rightarrow \infty$  gives,

$$\left| \bar{\mu}(B) - \sum_{i=1}^{\infty} \bar{\mu}(B_i) \right| \leq 2\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we have shown  $\bar{\mu}(B) = \sum_{i=1}^{\infty} \bar{\mu}(B_i)$ . This completes the proof that  $\mathcal{B}$  is a  $\sigma$ -algebra and that  $\bar{\mu}$  is a measure on  $\mathcal{B}$ . ■

Many theorems in the sequel will require some control on the size of a measure  $\mu$ . The relevant notion for our purposes (and most purposes) is that of a  $\sigma$ -finite measure defined next.

**Definition 5.19.** Suppose  $X$  is a set,  $\mathcal{E} \subset \mathcal{B} \subset 2^X$  and  $\mu : \mathcal{B} \rightarrow [0, \infty]$  is a function. The function  $\mu$  is  $\sigma$ -finite on  $\mathcal{E}$  if there exists  $E_n \in \mathcal{E}$  such that  $\mu(E_n) < \infty$  and  $X = \cup_{n=1}^{\infty} E_n$ . If  $\mathcal{B}$  is a  $\sigma$ -algebra and  $\mu$  is a measure on  $\mathcal{B}$  which is  $\sigma$ -finite on  $\mathcal{B}$  we will say  $(X, \mathcal{B}, \mu)$  is a  $\sigma$ -finite measure space.

The reader should check that if  $\mu$  is a finitely additive measure on an algebra,  $\mathcal{B}$ , then  $\mu$  is  $\sigma$ -finite on  $\mathcal{B}$  iff there exists  $X_n \in \mathcal{B}$  such that  $X_n \uparrow X$  and  $\mu(X_n) < \infty$ .

**Theorem 5.20.** Suppose that  $\mu$  is a  $\sigma$ -finite premeasure on an algebra  $\mathcal{A}$ . Then

$$\bar{\mu}(B) := \inf \{ \mu(C) : B \subset C \in \mathcal{A}_\sigma \} \quad \forall B \in \sigma(\mathcal{A}) \quad (5.11)$$

defines a measure on  $\sigma(\mathcal{A})$  and this measure is the unique extension of  $\mu$  on  $\mathcal{A}$  to a measure on  $\sigma(\mathcal{A})$ .

**Proof.** Let  $\{X_n\}_{n=1}^{\infty} \subset \mathcal{A}$  be chosen so that  $\mu(X_n) < \infty$  for all  $n$  and  $X_n \uparrow X$  as  $n \rightarrow \infty$  and let

$$\mu_n(A) := \mu_n(A \cap X_n) \text{ for all } A \in \mathcal{A}.$$

Each  $\mu_n$  is a premeasure (as is easily verified) on  $\mathcal{A}$  and hence by Theorem 5.18 each  $\mu_n$  has an extension,  $\bar{\mu}_n$ , to a measure on  $\sigma(\mathcal{A})$ . Since the measure  $\bar{\mu}_n$  are increasing,  $\bar{\mu} := \lim_{n \rightarrow \infty} \bar{\mu}_n$  is a measure which extends  $\mu$ .

The proof will be completed by verifying that Eq. (5.11) holds. Let  $B \in \sigma(\mathcal{A})$ ,  $B_m = X_m \cap B$  and  $\varepsilon > 0$  be given. By Theorem 5.18, there exists  $C_m \in \mathcal{A}_\sigma$  such that  $B_m \subset C_m \subset X_m$  and  $\bar{\mu}(C_m \setminus B_m) = \bar{\mu}_m(C_m \setminus B_m) < \varepsilon 2^{-n}$ . Then  $C := \cup_{m=1}^{\infty} C_m \in \mathcal{A}_\sigma$  and

$$\bar{\mu}(C \setminus B) \leq \bar{\mu} \left( \bigcup_{m=1}^{\infty} (C_m \setminus B) \right) \leq \sum_{m=1}^{\infty} \bar{\mu}(C_m \setminus B) \leq \sum_{m=1}^{\infty} \bar{\mu}(C_m \setminus B_m) < \varepsilon.$$

Thus

$$\bar{\mu}(B) \leq \bar{\mu}(C) = \bar{\mu}(B) + \bar{\mu}(C \setminus B) \leq \bar{\mu}(B) + \varepsilon$$

which, since  $\varepsilon > 0$  is arbitrary, shows  $\bar{\mu}$  satisfies Eq. (5.11). The uniqueness of the extension  $\bar{\mu}$  is proved in Exercise 5.1. ■

## 5.5 Completions of Measure Spaces

**Definition 5.21.** A set  $E \subset X$  is a **null set** if  $E \in \mathcal{B}$  and  $\mu(E) = 0$ . If  $P$  is some “property” which is either true or false for each  $x \in X$ , we will use the terminology  $P$  a.e. (to be read  $P$  almost everywhere) to mean

$$E := \{x \in X : P \text{ is false for } x\}$$

is a null set. For example if  $f$  and  $g$  are two measurable functions on  $(X, \mathcal{B}, \mu)$ ,  $f = g$  a.e. means that  $\mu(f \neq g) = 0$ .

**Definition 5.22.** A measure space  $(X, \mathcal{B}, \mu)$  is **complete** if every subset of a null set is in  $\mathcal{B}$ , i.e. for all  $F \subset X$  such that  $F \subset E \in \mathcal{B}$  with  $\mu(E) = 0$  implies that  $F \in \mathcal{B}$ .

**Proposition 5.23 (Completion of a Measure).** Let  $(X, \mathcal{B}, \mu)$  be a measure space. Set

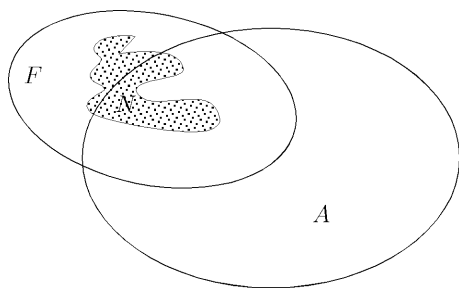
$$\mathcal{N} = \mathcal{N}^\mu := \{N \subset X : \exists F \in \mathcal{B} \text{ such that } N \subset F \text{ and } \mu(F) = 0\},$$

$$\mathcal{B} = \bar{\mathcal{B}}^\mu := \{A \cup N : A \in \mathcal{B} \text{ and } N \in \mathcal{N}\} \text{ and}$$

$$\bar{\mu}(A \cup N) := \mu(A) \text{ for } A \in \mathcal{B} \text{ and } N \in \mathcal{N},$$

see Fig. 5.1. Then  $\bar{\mathcal{B}}$  is a  $\sigma$ -algebra,  $\bar{\mu}$  is a well defined measure on  $\bar{\mathcal{B}}$ ,  $\bar{\mu}$  is the unique measure on  $\bar{\mathcal{B}}$  which extends  $\mu$  on  $\mathcal{B}$ , and  $(X, \bar{\mathcal{B}}, \bar{\mu})$  is complete measure space. The  $\sigma$ -algebra,  $\bar{\mathcal{B}}$ , is called the **completion** of  $\mathcal{B}$  relative to  $\mu$  and  $\bar{\mu}$ , is called the **completion of  $\mu$** .

**Proof.** Clearly  $X, \emptyset \in \bar{\mathcal{B}}$ . Let  $A \in \mathcal{B}$  and  $N \in \mathcal{N}$  and choose  $F \in \mathcal{B}$  such



**Fig. 5.1.** Completing a  $\sigma$ -algebra.

that  $N \subset F$  and  $\mu(F) = 0$ . Since  $N^c = (F \setminus N) \cup F^c$ ,

$$\begin{aligned} (A \cup N)^c &= A^c \cap N^c = A^c \cap (F \setminus N \cup F^c) \\ &= [A^c \cap (F \setminus N)] \cup [A^c \cap F^c] \end{aligned}$$

where  $[A^c \cap (F \setminus N)] \in \mathcal{N}$  and  $[A^c \cap F^c] \in \mathcal{B}$ . Thus  $\bar{\mathcal{B}}$  is closed under complements. If  $A_i \in \mathcal{B}$  and  $N_i \subset F_i \in \mathcal{B}$  such that  $\mu(F_i) = 0$  then  $\cup(A_i \cup N_i) = (\cup A_i) \cup (\cup N_i) \in \bar{\mathcal{B}}$  since  $\cup A_i \in \mathcal{B}$  and  $\cup N_i \subset \cup F_i$  and  $\mu(\cup F_i) \leq \sum \mu(F_i) = 0$ . Therefore,  $\bar{\mathcal{B}}$  is a  $\sigma$ -algebra. Suppose  $A \cup N_1 = B \cup N_2$  with  $A, B \in \mathcal{B}$  and  $N_1, N_2 \in \mathcal{N}$ . Then  $A \subset A \cup N_1 \subset A \cup N_1 \cup F_2 = B \cup F_2$  which shows that

$$\mu(A) \leq \mu(B) + \mu(F_2) = \mu(B).$$

Similarly, we show that  $\mu(B) \leq \mu(A)$  so that  $\mu(A) = \mu(B)$  and hence  $\bar{\mu}(A \cup N) := \mu(A)$  is well defined. It is left as an exercise to show  $\bar{\mu}$  is a measure, i.e. that it is countable additive. ■