UNIVERSITY OF CALIFORNIA, SAN DIEGO

Smooth Densities For Solutions To Differential Equations Driven by Fractional Brownian Motion

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The dissertation of Patrick R. Driscoll is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2011

DEDICATION

To M.A.B. and K.O.

EPIGRAPH

Only that at times like this,
when you're directionless in a dark wood,
trust to the abstract deductive...
Leap
like a knight of faith
into the arms of Peano,
Leibniz,
Hilbert,
L'Hôpital.
You will be lifted up.
—David Foster Wallace

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ABSTRACT OF THE DISSERTATION

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by

Patrick R. Driscoll

Doctor of Philosophy in Mathematics

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Professor Bruce K. Driver, Chair

The fractional Brownian motions are a family of stochastic processes which resemble Brownian motion in many key ways, yet lack the quality of independence of increments. This dissertation focuses on proving smoothness of densities for solutions to differential equations driven by fractional Brownian motion, provided the vector fields satisfy a particular stratification condition. This result is acheived using the methods of Malliavin calculus. Examples of such solutions include the area process for any two-dimensional projection of the fractional Brownian motion.

Chapter 1

Introduction

1.1 Background

We begin the discussion by recalling a definition from the theory of differential equations. Suppose $\mathcal{L}$ be a differential operator on some open $U \subset \mathbb{R}^n$; then $\mathcal{L}$ is called *hypoelliptic* if, for every distribution φ supported on some open $V \subset U$, $\mathcal{L}\varphi \in C^\infty(V)$ implies $\varphi \in C^\infty(V)$. It is easy to check that every elliptic operator is also hypoelliptic. A celebrated theorem of Hörmander (see [Hör67]) gives us one method of constructing a hypoellptic operator, as follows: suppose that $\{X_i\}_{i=0}^m$ is a collection of vector fields on U with the *bracket-generating property*:

$$\mathrm{sp}\{X_i, [X_i, X_j], [X_i, [X_j, X_k]], \ldots : i = 1, \ldots, m\} = \mathbb{R}^n \tag{1.1.1}$$

where $[\cdot, \cdot]$ is the standard Lie bracket operator on vector fields. Then the operator given by

$$\mathcal{L} := X_0 + \sum_{i=1}^m X_i^2 \tag{1.1.2}$$

is hypoelliptic.

One of the pleasant properties of hypoelliptic operators is that they admit an associated smooth kernel. More specifically, there exists a one-parameter semigroup of functions $\{P_t\}_{t\geq 0}$ for which, given the Cauchy problem

$$\begin{cases} u_t(t,x) &= \mathcal{L}u \\ \lim\limits_{t\downarrow 0} u(t,x) &= f(x) \end{cases} \tag{1.1.3}$$

with bounded and continuous initial data f, one has the solution

$$u(t,x) = \int P_t(x,y)f(y)dy.$$

There is a well-known connection between the above deterministic problem and the theory of stochastic differential equations. Let $B = \{B_1, \ldots, B_m\}$ be m-dimensional Brownian motion and $\{X_i\}$ be a collection of bracket-generating vector fields on $\mathbb{R}^n$ as above.Suppose we are given the following stochastic differential equation:

$$\begin{cases} dY_t &= X_0(Y_s)dt + \sum\limits_{i=1}^{m} X_i(Y_s)dB_s^i \\ Y_0 &= x \qquad (x \in \mathbb{R}^d) \end{cases} \tag{1.1.4}$$

where the expression "dB_s^i" indicates stochastic integration in the manner of Stratonovich. One can verify that the infinitesimal generator associated to Y_t^x, whose action on a bounded and twice differentiable function f is given as

$$\lim_{t\to 0} \frac{1}{t} \left(\mathbb{E}[f(Y_t^x)] - f(x) \right)$$

is in fact the differential operator $\mathcal{L}$ as defined in (1.1.2). If one defines the function $u(t,x) := \mathbb{E}[f(Y_t^x)]$, it can be confirmed that u is the solution to (1.1.3). From this, it follows that if our vector fields satisfy the bracket-generating condition, we may conclude that each Y_t^x has a density P_t with respect to Lebesgue measure which is smooth. Further details regarding this connection may be found in [Øks03], among numerous others.

The purpose of this work is to extend these sorts of results to the case when our driving signal is replaced by a more general class of processes known as *fractional Brownian motions*. The fractional Brownian motions are a one-parameter family of Gaussian processes $\{B_t^H\}$, with $H \in [0,1]$. These processes were first suggested by Kolmogorov in [Kol40] and investigated in earnest by Mandelbrot and Van Ness in [MVN68]. Recall that the standard Brownian motion $\{B_t\}$ is characterized by the following properties:

1. for each $s < t$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $|t - s|^{2H}$;

2. increments are independent; that is to say, for each $t_1 < t_2 < \ldots < t_n < \infty$, the set
$$\{B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\}$$
is a collection of independent random variables.

The fractional Brownian motions, in essence, are processes for which the second property does not necessarily hold; that is to say, these processes have some measure of correlation between increments The degree to which the process increments

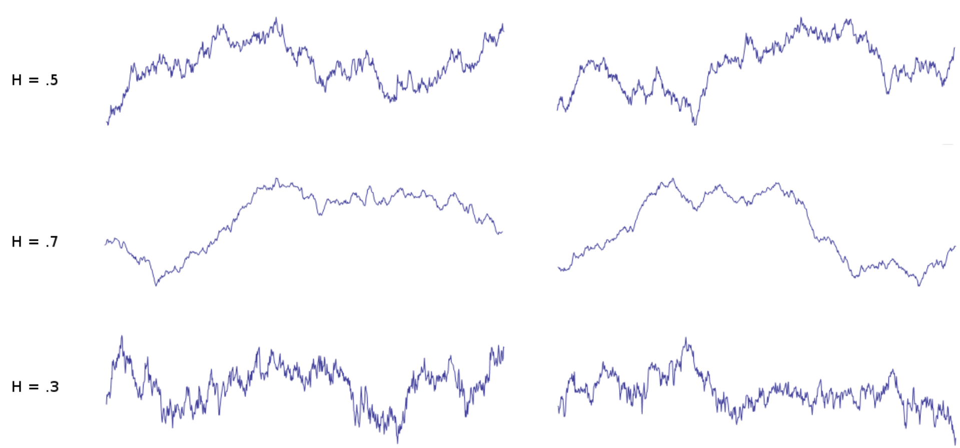

Figure 1.1: Sample Paths for Fractional Brownian Motion

are correlated (and hence, as Gaussians, dependent) is a function of the value of the parameter H, known as the *Hurst parameter*. For $\frac{1}{2} < H \leq 1$, increments are positively correlated, and for $0 \leq H < \frac{1}{2}$, increments are negatively correlated. When $H = \frac{1}{2}$, one recovers the standard Brownian motion. Heuristically, one may consider the Hurst parameter H as a measurement of the "roughness" of the sample paths of B_t^H, with smaller values of H corresponding to greater amounts of roughness; this notion is best understood visually, as one can see from Figure 1, but we will make it more precise in the sequel. From a practical standpoint, differential equations driven by fractional Brownian motion are interesting objects of study as the increment correlation property makes the process a suitable model for many phenomena. Examples include the analysis of global temperature anomaly [RR10], river water runoff [Hur51], electricity markets [Sim03], foreign exchange markets [GBP+96], and logarithmic returns on stocks and volatilities [Shi99].

One fact of great importance for this discussion is that for $H \neq \frac{1}{2}$, the fractional Brownian motion process is not a semimartingale. As a result, the application of usual stochastic integration techniques a la Itô are not available for use, and alternate methods are required in order to define equations of the form (1.1.4). When $H > \frac{1}{2}$, the positive correlation of increments of sample paths allows for the solution to be defined pathwise through Riemann-Stieltjies integrals; under such a regime, existence of a density to a solution of (1.1.4) is proven in [NS09], and smoothness is proven in [BH07]. In the case that $H < \frac{1}{2}$, a different approach known as *rough path theory* may be used to make sense of (1.1.4). In essence, we may consider solutions as the limits of solutions to (1.1.4) driven by signals which approximate our process under a particular topology. For any path $f : [0,T] \to \mathbb{R}^d$ of bounded variation, one may construct the step-N signature $\mathbf{f}$ with domain $\Delta := \{(s,t) : 0 \leq s \leq t \leq T\}$:

$$\mathbf{f}_{s,t} := \Bigg(\int_s^t df(\tau), \int_s^t \int_s^{\tau_2} df(\tau_2) \otimes df(\tau_1), \ldots, \\ \int_{s<\tau_1<\ldots<\tau_{N-1}<t} df(\tau_N) \otimes \ldots \otimes df(\tau_1) \Bigg).$$

For any such f and suitable vector fields X one has the existence of a unique solution to the equation

$$dy = X(y)\, df$$

Now suppose we have some sequence $\{f_n\}$ of smooth functions for which the associated step-N signatures are appropriately uniformly bounded and converge in particular variational spaces to some map $\mathbf{f}$ on Δ. Then one may show that the associated sequence of solutions $\{y_n\}$ to the differential equation above driven by $\{f_n\}$ converge to a limit y, which we will refer to the as the solution to the *rough differential equation* $dy = X(y)d\mathbf{f}$.

While the above treatment of rough path theory is admittedly vague, it is mentioned here for two main reasons. The first is to motivate the construction of solutions to differential equations driven by fractional Brownian motions as limiting processes of solutions to the same equations when driven by processes which adequately approximate the fractional Brownian motion. The second reason is to highlight the fact that the step-2 signature (s, t) of a smooth path f may be expressed in terms of the path increments $f(t) - f(s)$ enhanced with the signed area contained within two-dimensional projections of the path and the chord connecting $f(t)$ with $f(s)$. As we will see, the solutions that will be the focus of this discussion include such enhanced signatures of the fractional Brownian motion.

Progress regarding densities of solutions for $H < \frac{1}{2}$ has been more limited. Existence has been shown for $\frac{1}{4} < H < \frac{1}{2}$ in the case when the vector fields are elliptic by Cass, Friz, and Victoir in [CFV09]; the result was extended to the hypoelliptic case under the same Hurst parameter condition by Cass and Friz in [CF10].

1.2 Statement of Results

Suppose that $\circ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a multilinear form on $\mathbb{R}^n$ such that $G := (\mathbb{R}^n, \circ)$ is a Lie group with associated Lie algebra $\mathfrak{g}$. We say that such a group is *(step-r) stratified* if there exists a decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots \oplus V_r$$

such that $[V_1, V_{i-1}] = V_i$ for $i \in \{2, \ldots, r\}$ and $[V_1, V_r] = 0$. We will be focusing on the case when G is step-2 stratified; that is, when $\mathfrak{g} = V_1 \oplus V_2$ with $V_2 = [V_1, V_1]$. Clearly, the Lie algebra for a step-2 stratified group satisfies the Hörmander bracket-generating condition (1.1.1).

Example 1. *The Heisenberg group.*

Let $\wedge : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote the standard wedge product or "signed area" operator; i.e., $(x_1, x_2) \wedge (y_1, y_2) = x_1 y_2 - x_2 y_1$. We define the Lie group $G = \mathbb{R}^2 \times \mathbb{R}$ with multiplication operation $\circ : G \times G \to G$ given by:

$$(\mathbf{v}, x) \circ (\mathbf{w}, y) = \left(\mathbf{v} + \mathbf{w}, x + y + \frac{1}{2} (\mathbf{v} \wedge \mathbf{w}) \right).$$

(We write $\mathbb{R}^2 \times \mathbb{R}$ rather than $\mathbb{R}^3$ in this case in order to simplify notation throughout.) This group is known as the *Heisenberg group*, and is often denoted $\mathbb{H}^1$. The associated Lie algebra for the group is denoted by $\mathfrak{h} \cong T_0 \mathbb{H}^1 \cong \mathbb{H}^1$. To explicitly define the Lie algebra structure, we will perform the usual operations of calculting the left-invariant vector fields for G. Fix $\xi = (\mathbf{v}, x) \in \mathfrak{h}$. For each $g = (\mathbf{w}, y) \in G$, let $L_g : G \to G$ be multiplication on the left by g; i.e., $L_g(v) = g \circ v$. Then the (left-invariant) vector field $\tilde{\xi}$ defined by

$$\tilde{\xi}_g = (L_g)_* \xi$$

has action given on each $f \in C^\infty(G)$ by

$$\tilde{\xi}_g f = \left. \frac{d}{dt} \right|_{t=0} (f \circ (g \cdot \sigma(t)))$$

where $\sigma(t)$ is any smooth curve in G with $\sigma(0) = 0$ and $\sigma'(0) = \xi$ (for example, $\sigma(t) := t\xi$). More explicitly, we have that

$$\tilde{\xi}_g f = \left. \frac{d}{dt} \right|_{t=0} f \left(\mathbf{w} + t\mathbf{v}, y + tx + \frac{1}{2} (\mathbf{w} \wedge t\mathbf{v}) \right) = f'(g)(\mathbf{v}, x + \frac{1}{2} (\mathbf{w} \wedge \mathbf{v})).$$

and if $\eta = (\mathbf{v}', x') \in \mathfrak{h}$,

$$\begin{aligned} \tilde{\eta}_g \tilde{\xi}_g f &= \tilde{\eta}_g f'(g)(\mathbf{v}, x + \frac{1}{2} (\mathbf{w} \wedge \mathbf{v})) \\ &= f''(g)(\mathbf{v}, x' + \frac{1}{2} (\mathbf{w}' \wedge \mathbf{v}))(\mathbf{v}, x + \frac{1}{2} (\mathbf{w} \wedge \mathbf{v})) \\ &\qquad + f'(g) \left. \frac{d}{dt} \right|_{t=0} (\mathbf{v}, x + \frac{1}{2} ((\mathbf{w} + t\mathbf{v}') \wedge \mathbf{v})) \\ &= f''(g)(\mathbf{v}, x' + \frac{1}{2} (\mathbf{w}' \wedge \mathbf{v}))(\mathbf{v}, x + \frac{1}{2} (\mathbf{w} \wedge \mathbf{v})) + f'(g)(0, \frac{1}{2} (\mathbf{v}' \wedge \mathbf{v})). \end{aligned}$$

Hence, the Lie bracket on $\mathfrak{h}$ is given by

$$[\xi,\eta] = \widetilde{[\xi,\eta]_0} = [\tilde{\xi},\tilde{\eta}]_0 = (0, \mathbf{v}' \wedge \mathbf{v})\,.$$

Given the standard Euclidean basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, there are associated Jacobian vector fields $X_i := \widetilde{(e_i, 0)}$, with

$$\begin{aligned}
(X_1)_{(\mathbf{w},y)} &= \left(e_1, \frac{1}{2}(\mathbf{w}, e_1)\right) = \left(1, 0, -\frac{1}{2}(\mathbf{w})_2\right).\\
(X_2)_{(\mathbf{w},y)} &= \left(e_2, \frac{1}{2}(\mathbf{w}, e_2)\right) = \left(0, 1, \frac{1}{2}(\mathbf{w})_1\right).\\
(X_3)_{(\mathbf{w},y)} &= (0,1)\,.
\end{aligned}$$

From these calculations, we may conclude that

(i) At each point in $g \in \mathbb{H}^1$, the vectors $\{(X_i)_g\}$ span the tangent space;

(ii) $[X_1, X_2] = X_3, [X_1, X_3] = [X_2, X_3] = 0$.

This is the simplest (non-trivial) case of a step-2 stratified Lie group on a Euclidean space.

Now, we suppose that $B = (B^1, B^2)$ is a 2-dimensional fractional Brownian motion on $[0,T]$ with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$. One would like to give meaning to the differential equation

$$dY_T = X_1(Y_T)dB_T^1 + X_2(Y_T)dB_T^2$$

where X_1 and X_2 are the Heisenberg vector fields defined in the example above. Working heuristically, one has that

$$dY_T^1 = dB_T^1, \qquad dY_T^2 = dB_T^2, \qquad dY_T^3 = \frac{1}{2}\left(Y_T^1 dB_T^2 - Y_T^2 dB_T^1\right).$$

The solution to this equation is given as

$$Y_T = \left(B_T, \frac{1}{2}\int_0^T B_t^1\, dB_t^2 - B_t^2\, dB_t^1\right)$$

In the case of standard Brownian motion, the expression $A_T := \int_0^T B_t^1 \; dB_t^2 - B_t^2 \, dB_t^1$ is the well-known Lévy area process. In our current situation, we will need to use a bit of care in order to atttain a rigorous definition for Y. Let

$$(B_m)_T := \left((B_m^1)_T, (B_m^2)_T\right)$$

denote the m-th dyadic approximation of B (as defined below in Section 2.3). Define the area processes

$$(A_m)_T := \frac{1}{2}\left[\int_0^T (B_m^1)_t d(B_m^2)_t - (B_m^2)_t d(B_m^1)_t\right]. \tag{1.2.1}$$

where the integral may be understood pathwise as Riemann-Stietjies integration. It is easy to see that $Y_m := (B_m, A_m)$ is a solution to the differential equation

$$\begin{cases} d(Y_m)_T & = X_1\left((Y_m)_T\right) d(B_m^1)_T + X_2\left((Y_m)_T\right) d(B_m^2)_T, \\ (Y_m)_0 & = 0. \end{cases}$$

As a result of Theorem 2 of [CQ02], there exists a process A given by

$$A_T := \lim_{m\to\infty} (A_m)_T$$

where the convergence is almost sure with regards to the law of B.

Theorem 1.2.1. *Define the random process $\{Y\}_{0\leq t\leq T}$, taking values in $\mathbb{R}^3$, by*

$$\begin{aligned} &Y_0 = 0, \\ &Y_t = (B_t, A_t). \qquad (t \in (0,T]) \end{aligned}$$

Then for all $t \in [0,T]$, the density of Y_t with respect to Lebesgue measure is C^∞.

The above theorem is a special case of the following result.

Theorem 1.2.2. *Let G be a step-2 stratified group on $\mathbb{R}^N$ with stratification $\mathfrak{g} = V_1 \oplus V_2$. Suppose that $\{X_1, \ldots X_n\}$ are Jacobian generators for V_1. Let Y denote the almost sure limit of solutions Y_m to the differential equation*

$$d(Y_m)_t = \sum_{i=1}^n X_i((Y_m)_t)d(B_m^i)_t,$$

where the driving processes B_m^i are dyadic approximators of a fractional Brownian motion. Then Y_T has a smooth density with respect to Lebesgue measure for each $t \in (0,T]$.

We will begin by devoting Chapter 2 to laying the groundwork for proving the above results. In particular, we will discuss the notion of solutions to stochastic differential equations driven by fractional Brownian motion and develop the tools used in the proofs, including a suitable characterization of the abstract Wiener space associated to the process B. Chapter 3 will focus on the proof of Theorem 1.2.1, using Malliavin calculus techniques. Finally, in Chapter 4 we will prove Theorem 1.2.2. This proof, while similar in spirit to that of the previous chapter, will require the proof of an analogue of Norris' Lemma.

Chapter 2

Background

2.1 Fractional Brownian Motion

We begin by defining the primary object of interest in this body of work. Recall that a stochastic process is called Gaussian if any finite linear combination of time samples of the process is a normally distributed random variable. A Gaussian process $\{B_t^H\}_{t\in[0,T]}$ is called a fractional Brownian motion with *Hurst parameter* $H \in [0, 1]$ if

- the samples paths $t \mapsto B_t^H$ are continuous,
- the process is centered; i.e., $\mathbb{E}[B_t^H] = 0$ for all $t \in [0, T]$,
- the process has its covariance given by

$$\mathbb{E}[B_s^H B_t^H] = R_H(s,t) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{2.1.1}$$

General information regarding this process may be found in [BHØZ08] or [MVN68]. We will generally be working with fractional Brownian motions of some fixed Hurst parameter at any given point in the sequel; hence, we will drop the parameter from our notation whenever it is possible to do so without causing confusion.

An n-dimensional fractional Brownian motion is a stochastic process $\{B_t = (B_t^1, \ldots, B_t^n);\ t \in [0,T]\}$ is a continuous-time process comprised of n independent copies of one-dimensional fractional Brownian motion, each having the same Hurst parameter H.

It is easy to see from the definition of $R(s,t)$ that B has homogeneous increments: for $0 < s < t \leq T$,

$$(B_t - B_s) \stackrel{d}{=} B_{t-s}.$$

Also, since

$$\mathbb{E}[B_{\alpha t}^2] = (\alpha t)^{2H} = \alpha^{2H}\mathbb{E}[B_t^2],$$

we have that fractional Brownian motion is self-similar with exponent H; i.e., $B_{\alpha t} \stackrel{d}{=} \alpha^H B_t$.

One additional property of the fractional Brownian motion that will be of great use to us in the sequel is presented below without proof.

Lemma 2.1.1 (Theorem 1.6.1 of [BHØZ08])**.** *The sample paths $t \mapsto B_t$ are almost surely Hölder continuous of order α for all $\alpha < H$.*

We mention at this point one of the most fundamental features of the fractional Brownian motion: the correlation of process increments. We will only present the case most pertinent to us; the proof for the regime of $H > \frac{1}{2}$ follows similarly.

Lemma 2.1.2. *Let $0 < H < \frac{1}{2}$, and suppose B is a fractional Brownian motion of Hurst exponent H. Suppose $0 \leq s < t < u < v \leq T$. Then the process increments $B_v - B_u$ and $B_t - B_s$ are negatively correlated.*

Proof. We begin by noting that the following relations hold:

1. $(v - s) + (u - t) = (v - t) + (u - s)$.

2. $(v - s) > [(v - t) \vee (u - s)] := \max[(v - t), (u - s)]$.

3. $(u - t) < [(v - t) \wedge (u - s)] := \min[(v - t), (u - s)]$.

Let $f(\alpha, \beta) = \alpha^{2H} + \beta^{2H}$. One may check that

$$\mathbb{E}[(B_v - B_u)(B_t - B_s)] = f(v - s, u - t) - f(v - t, u - s).$$

For any positive constant C, the function f attains its maximum on the region $\{(\alpha,\beta) : \alpha \geq 0, \beta \geq 0, \alpha+\beta = C\}$ at $(\frac{C}{2}, \frac{C}{2})$. Thus, $f(v-s, u-t) < f(v-t, u-s)$, and the claim is proven. $\square$

2.2 p-variation and Young's integration

2.2.1 One-dimensional case

For any $T > 0$, we will denote by $\mathcal{P}[0,T]$ the collection of finite partitions of $[0,T]$; i.e., sets of the form $\{0 = t_0 < t_1 < \ldots < t_n = T\}$. Suppose we are given a path $f \in \mathcal{C}([0,T], \mathbb{R}^d)$; then for each $1 \leq p < \infty$ and $\Pi = \{0 = t_1 < t_2 < \ldots < t_N = T\} \in \mathcal{P}[0,T]$, one may define the quantities

$$\begin{aligned}
\Delta_i f &:= f(t_i) - f(t_{i-1}), \\
V_p(f : \Pi) &:= \left(\sum_{i=1}^{N} |\Delta_i f|^p \right)^{\frac{1}{p}}, \\
\|f\|_p &:= \sup_{\Pi \in \mathcal{P}[0,T]} V_p(f : \Pi).
\end{aligned}$$

We shall define the space $\mathcal{C}_p([0,T], \mathbb{R}^d) := \{f \in \mathcal{C}([0,T], \mathbb{R}^d);\ \|f\|_p < \infty\}$. We will repeatedly refer to this space simply as $\mathcal{C}_p$ when the domain and image spaces are both clear from context. The somewhat leading notation of $\|\cdot\|_p$ above is justified, as we see from the following result.

Proposition 2.2.1. *For each $1 \leq p < \infty$, the function $f \to \|f\|_p$ is a seminorm on $\mathcal{C}_p$.*

Proof. It is immediate that for each $\lambda \in \mathbb{R}$,

$$\|\lambda f\|_p = \sup_{\Pi \in \mathcal{P}[0,T]} V_p(\lambda f : \Pi) = |\lambda| \left(\sup_{\Pi \in \mathcal{P}[0,T]} V_p(f : \Pi) \right) = |\lambda| \|f\|_p.$$

Also, Minkowski's Inequality gives that for each $\Pi \in \mathcal{P}[0,T]$,

$$\begin{aligned}
V_p(f+g:\Pi) &= \left(\sum_{i=1}^{N} |\Delta_i(f+g)|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{N} (|\Delta_i f| + |\Delta_i g|)^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{N} |\Delta_i f|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{N} |\Delta_i g|^p\right)^{\frac{1}{p}} \\
&= V_p(f:\Pi) + V_p(g:\Pi).
\end{aligned}$$

Taking the supremum over all such partitions on each side gives that
$\|f+g\|_p \le \|f\|_p + \|g\|_p$. □

Remark 2.2.2. *Note that for each $f \in \mathcal{C}_p$ and Lipschitz mapping $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, the composition $\varphi \circ f \in \mathcal{C}_p$. In particular, $\|\varphi \circ f\|_p \le K\|f\|_p$, where K denotes the Lipschitz constant of φ.*

One may also define the normalized space of p-variation paths:

$$\mathcal{C}_{0,p}([0,T],\mathbb{R}^d) := \{f \in \mathcal{C}_p([0,T],\mathbb{R}^d) : f(0) = 0\}.$$

As in the non-normalized case, we will refer to this simply as $\mathcal{C}_{0,p}$ when convenient.

Lemma 2.2.3. *$\|\cdot\|_p$ is a norm on $\mathcal{C}_{0,p}$.*

Proof. As a result of Propsition 2.2.1, one only needs to know that $\|f\|_p = 0$ for $f \in \mathcal{C}_{0,p}$ iff $f \equiv 0$. But one may readily verify that the only continuous functions for which $\|f\|_p = 0$ are those which are constant; from this, the claim follows immediately from the definition of the normalized variational space. □

In fact, the space $\mathcal{C}_{0,p}$ is complete under this norm, and so one has the following proposition which is presented without proof.

Proposition 2.2.4 (Theorem 5.25 of [FV10])**.** *The vector space $(\mathcal{C}_{0,p}, \|\cdot\|_p)$ is a non-separable Banach space.*

The non-separability of $\mathcal{C}_{0,p}$ is mentioned above as it will become a minor issue later on when attempting to decide on a path-space upon which the fractional Brownian motion should reside – this will be discussed further in Section 2.4. One may easily verify that a function $f \in \mathcal{C}_p(\mathbb{R}^d)$ (resp. $f \in \mathcal{C}_{0,p}(\mathbb{R}^d)$) if and only if each of the coordinate functions $f^i := f \cdot e_i$ is in $\mathcal{C}_p(\mathbb{R})$ (resp. $\mathcal{C}_{0,p}(\mathbb{R})$). It is also easy to check that for given α and p such that $\alpha < \frac{1}{p}$, any α-Hölder continuous function is in $\mathcal{C}_p$; thus, we may deduce the following:

Lemma 2.2.5. *The sample paths of B are almost surely of p variation for any $p > \frac{1}{H}$.*

Proof. This result follows immediately from the above remark and Lemma 2.1.1. □

The following lemmas allow one to see how the various variational spaces are related to one another and to the larger space of uniformly continuous functions in which they all reside.

Lemma 2.2.6. *The identity map is a contraction from $\mathcal{C}_{0,p}$ into $\mathcal{C}$.*

Proof. Note that for $f \in \mathcal{C}_{0,p}$

$$|f(t)|^p = |f(t) - f(0)|^p \leq |f(t) - f(0)|^p + |f(T) - f(t)|^p \leq \|f\|_p^p.$$

By raising each side of this inequality to the $\frac{1}{p}$-th power and take the supremum over $t \in [0, T]$, we see that

$$\|f\|_u \leq \|f\|_p$$

where $\|\cdot\|_u$ denotes the standard uniform norm. □

Lemma 2.2.7. *For each $1 \leq p < q$, $\mathcal{C}_{0,p} \subset \mathcal{C}_{0,q}$; in particular, if $f \in \mathcal{C}_{0,p}$, one has the bound*

$$\|f\|_q \leq (2\|f\|_u)^{1-\frac{p}{q}} \|f\|_p^{\frac{p}{q}},$$

where $\|\cdot\|_u$ denotes the uniform norm on $\mathcal{C}([0, T])$.

Proof. Pick $f \in \mathcal{C}_{0,p}$, and let $q > p$. Then it is immediately clear that for each $s, t \in [0, T]$,

$$|f(t) - f(s)| \leq 2\|f\|_u.$$

The claim then follows from noting that for each $\Pi \in \mathcal{P}[0, T]$,

$$\begin{aligned}(V_q(f : \Pi))^q &= \sum_{i=1}^{N} |\Delta_i f|^q = \sum_{i=1}^{N} |\Delta_i f|^p \, |\Delta_i f|^{q-p} \\ &\leq \sup_{\Pi} |\Delta_i f|^{q-p} \sum_{i=1}^{N} |\Delta_i f|^p \leq (2\|f\|_u)^{q-p} \, (V_p(f : \Pi))^p \,.\end{aligned}$$

As usual, we take the supremum over partitions to complete the proof. ☐

Given $f \in \mathcal{C}_p, g \in \mathcal{C}_q$, where p and q are such that $\frac{1}{p} + \frac{1}{q} > 1$, one may use the variational properties to develop a notion of integration of f against g. We record some of the basics regarding the existence of and estimates for such an integral below. To begin, we make the following definition for ease of notation: given f, g as above, and $\Pi := \{t_0, t_1, \ldots, t_N\} \in \mathcal{P}[s, t]$ for $0 \leq s < t \leq T$, let $S(f, g, \Pi)$ denote an expression of the form

$$S(f, g, \Pi) := \sum_{i=1}^{N} f(c_i) \Delta_i g$$

where $c_i \in [t_{i-1}, t_i]$.

Theorem 2.2.8 (Theorem 3.3.1 of [LQ02])**.** *Suppose p, q are positive numbers with $\frac{1}{p} + \frac{1}{q} > 1$, and let $f \in \mathcal{C}_p, g \in \mathcal{C}_q$. Then the expression*

$$\int_s^t f \; dg := \lim_{n \to \infty} S(f, g, \Pi_n)$$

exists for each collection $\{\Pi_n := \{t_i^{(n)}\}\} \subset \mathcal{P}[s, t]$ and $\{c_i^{(n)}\}$ with $c_i^{(n)} \in [t_{i-1}^{(n)}, t_i^{(n)}]$, such that the mesh size $|\Pi_n| := \max_{t_i^{(n)} \in \Pi_n} |t_i^{(n)} - t_{i-1}^{(n)}|$ tends to 0 as $n \to \infty$; this expression is independent of the choice of $\{\Pi_n\}$ and $\{c_i^{(n)}\}$. Furthermore, the mapping $t \mapsto \int_0^t f \; dg$ is contained in $\mathcal{C}_{0,p}$, and the mapping on $\mathcal{C}_{0,p} \oplus \mathcal{C}_{0,q}$ given by $(f, g) \mapsto \int_0^{\cdot} f \; dg$ is continuous in each term and its image is contained in $\mathcal{C}_{0,p}$.

The element $\int_s^t f\, dg$ is referred to as *the Young's integral of f against g*. This expression was originally formulated in [You36]. We have the following estimate on the value of this expression (see Formula 10.9 of [You36])

$$\left| \int_s^t f\, dg - [f(t)\,(g(t)-g(s))] \right| \tag{2.2.1}$$
$$\leq \left[1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right)\right] \|f\big|_{[s,t]}\|_p \|g\big|_{[s,t]}\|_q;$$

here ζ denotes the Riemann zeta function. We will use the following integration by parts formula repeatedly in the sequel.

Proposition 2.2.9. *Suppose $f \in C_p, g \in C_q$ with $\theta := \frac{1}{p}+\frac{1}{q} > 1$. Then for each $0 \leq s < t \leq T$, the following identity holds:*

$$\int_s^t f\, dg = f(t)g(t) - f(s)g(s) - \int_s^t g\, df.$$

Proof. First note that $f(t)g(t) - f(s)g(s) = \sum_{\Pi} \Delta_i(fg)$ for each $\Pi \in \mathcal{P}[s,t]$. Given some collection $\{\Pi_n\}\} \subset \mathcal{P}[s,t]$ with $|\Pi_n| \overset{n\to\infty}{\to} 0$, we may use this identity to write

$$\begin{aligned} f(t)g(t) - f(s)g(s) - \int_s^t g\, df - \int_s^t f\, dg \\ &= \lim_{n\to\infty} \sum_{\Pi_n} \Delta_i(fg) - g(t_{i-1})\Delta_i f - f(t_{i-1})\Delta_i g \\ &= \lim_{n\to\infty} \sum_{\Pi_n} (\Delta_i f)(\Delta_i g). \end{aligned}$$

To prove the claim, it suffices to show that the last sum tends to zero. Let $\varepsilon = \theta - 1$ and let p' be the Hölder conjugate of q; then it follows that

$$\frac{1}{p'} = 1 - \frac{1}{q} = 1 - \theta + \frac{1}{p} = \frac{1}{p} - \varepsilon.$$

We note that $p'\theta > p' > p$, and so Lemma 2.2.7 implies $f \in C_{p'\theta}$. Hölder's inequality gives us that

$$\begin{aligned}\left|\sum_{\Pi_n}(\Delta_i f)(\Delta_i g)\right| &\leq \sup_{\Pi_n}|\Delta_i f|^{\varepsilon}\sum_{\Pi_n}|\Delta_i f|^{\theta}|\Delta_i g|. \\ &\leq \sup_{\Pi_n}|\Delta_i f|^{\varepsilon}\left(\sum_{\Pi_n}|\Delta_i f|^{p'\theta}\right)^{\frac{1}{p'}}\left(\sum_{\Pi_n}|\Delta_i g|^{q}\right)^{\frac{1}{q}} \\ &\leq \sup_{\Pi_n}|\Delta_i f|^{\varepsilon}\left(V_{p'\theta}\left(f\right)\right)^{\theta}\left(V_q\left(g\right)\right).\end{aligned}$$

This expression will tend to zero as n tends to infinity. □

We will also make use of a version fundamental theorem of calculus, presented below without proof. The statement is a particular application of Theorem 5.3.1 in [LQ02].

Theorem 2.2.10. *Suppose $f \in \mathcal{C}_p$ for some $p > 2$ and $\varphi \in C^1(\mathbb{R}^d)$ with Lipschitz continuous derivative $\dot{\varphi}$. Then for each $0 \leq s < t \leq T$,*

$$[\varphi(f)]\,(t) - [\varphi(f)]\,(s) = \int_s^t \dot{\varphi}(f)\; df.$$

2.2.2 Two-Dimensional Case

In a similar spirit to the previous section, one may construct a Young's integration theory for functions of two variables. Given some $f \in \mathcal{C}([0,T]^2, \mathbb{R}^d)$, and partitions $\Pi_1 = \{s_i\}, \Pi_2 = \{t_j\} \in \mathcal{P}[0,T]$, we may define

$$\begin{aligned}\Delta_{ij} f &:= f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1}), \\ V_p(f : \Pi_1, \Pi_2) &:= \left(\sum_{i=1}^{\#(\Pi_1)}\sum_{j=1}^{\#(\Pi_2)}|\Delta_{ij} f|^p\right)^{\frac{1}{p}}, \\ \|f\|_p^{(2D)} &:= \sup_{\Pi_1,\Pi_2\in\mathcal{P}[0,T]} V_p(f : \Pi_1, \Pi_2).\end{aligned}$$

As in the (one-dimensional) case above, we shall define the spaces

$$\begin{aligned}&\mathcal{C}_p^{(2D)}([0,T]^2, \mathbb{R}^d) := \{f \in \mathcal{C}([0,T]^2, \mathbb{R}^d);\ \|f\|_p^{(2D)} < \infty\}; \\ &\mathcal{C}_{0,p}^{(2D)}([0,T]^2, \mathbb{R}^d) := \{f \in \mathcal{C}_p^{(2D)}([0,T]^2, \mathbb{R}^d) : f(0,\cdot) = f(\cdot,0) = 0\}.\end{aligned}$$

Just as before, references to domains and image spaces will be suppressed when mentioning these objects unless necessary. The following lemma connects the one- and two-dimensional normalized variational spaces.

Lemma 2.2.11.

(i) If $f \in \mathcal{C}_{0,p}^{(2D)}$, then for each $s \in [0,T]$, the function $f_s := (s,\cdot) \in \mathcal{C}_{0,p}$ and $\|f(s,\cdot)\|_p \le \|f\|_p^{(2D)}$.

(ii) If $f, g \in \mathcal{C}_{0,p}(\mathbb{R}^d)$, then the function $f \otimes g$ defined by

$$[(f \otimes g)(s,t)]^i := f^i(s) g^i(t) \qquad (i = 1, \dots, d)$$

is in $\mathcal{C}_{0,p}^{(2D)}(\mathbb{R}^d)$ and $\|f \otimes g\|_p^{(2D)} \le \|f\|_p \|g\|_p$.

Proof. (i) Fix $s \in [0,T]$. Then for each $\Pi \in \mathcal{P}[0,T]$,

$$\begin{aligned}
(V_p(f_s : \Pi))^p = \sum_{i=1}^{\#(\Pi)} |\Delta_i f(s,\cdot)|^p &= \sum_{i=1}^{\#(\Pi)} |\Delta_i f(s,\cdot) - \Delta_i f(0,\cdot)|^p \\
&\le \sum_{i=1}^{\#(\Pi)} \big(|\Delta_i f(s,\cdot) - \Delta_i f(0,\cdot)|^p \\
&\qquad + |\Delta_i f(T,\cdot) - \Delta_i f(s,\cdot)|^p\big) \\
&\le \left(\|f\|_p^{(2D)}\right)^p.
\end{aligned}$$

Taking supremums over all partitions of $[0,T]$ completes the proof.

(ii) Since having finite p-variation over $\mathbb{R}^d$ is equivalent to each coordinate function having finite p-variation, we may without loss of generality assume $d = 1$. If $\Pi_1, \Pi_2 \in \mathcal{P}[0,T]$, then

$$\begin{aligned}
(V_p(f \otimes g : \Pi_1, \Pi_2)) &= \sum_{i=1}^{\#(\Pi_1)} \sum_{j=1}^{\#(\Pi_2)} \|\Delta_{ij}(f \otimes g)\|^p \\
&= \sum_{i=1}^{\#(\Pi_1)} \sum_{j=1} \|\Delta_i f\|^p \|\Delta_j g\|^p \\
&= (V_p(f : \Pi_1))^p \, (V_p(g : \Pi_2))^p \,.
\end{aligned}$$

As before, by taking pth roots and supremums on each side, we acheive the desired result.

$\square$

We also record here the following embedding result; the proof, which is nearly identical to that of the corresponding one-dimensional result, is omitted.

Lemma 2.2.12. *For each $1 \leq p < q$, $\mathcal{C}_{0,p}^{(2D)} \subset \mathcal{C}_{0,q}^{(2D)}$; in particular, if $f \in \mathcal{C}_{0,p}^{(2D)}$, one has the bound*

$$\|f\|_q^{(2D)} \leq (4\|f\|_u)^{1-\frac{p}{q}} \left(\|f\|_p^{(2D)}\right)^{\frac{p}{q}}.$$

Analogously to the one-dimensional case, if we are given partitions $\Psi = \{s_0, s_1, \ldots, s_M\}$ and $\Pi = \{t_0, t_1, \ldots, t_N\} \in \mathcal{P}[0,t]$ with $t \leq T$ and $f \in \mathcal{C}_p^{(2D)}$ and $g \in \mathcal{C}_q^{(2D)}$ for p and q are such that $\frac{1}{p} + \frac{1}{q} > 1$, we may define

$$S(f, g, \Psi, \Pi) := \sum_{i=1}^{M} \sum_{j=1}^{N} f(c_i, d_j) \Delta_{ij} g$$

with $c_i \in [s_{i-1}, s_i], d_j \in [t_{j-1}, t_j]$.

Theorem 2.2.13 (Theorem 1.2 of [Tow02])**.** *Suppose p, q are positive numbers with $\frac{1}{p} + \frac{1}{q} > 1$, and let $f \in \mathcal{C}_p^{(2D)}, g \in \mathcal{C}_q^{(2D)}$.*

(i) The integral

$$\int_{[0,t]^2} f\, dg := \lim_{n\to\infty} S(f, g, \Psi_n, \Pi_n)$$

exists for each collection $\{\Psi_n := \{s_i^{(n)}\}\} \bigcup \{\Pi_n := \{t_j^{(n)}\}\} \subset \mathcal{P}[0,t]$ and values $\{c_i^{(n)}\}, \{d_j^{(n)}\}$ with $c_i^{(n)} \in [s_{i-1}^{(n)}, s_i^{(n)}], d_j^{(n)} \in [t_{j-1}^{(n)}, t_j^{(n)}]$ such that the maximum mesh size $|\Pi_n| \vee |\Psi_n|$ tends to 0 as $n \to \infty$; this expression is independent of the choice of $\{\Pi_n\}, \{\Psi_n\}, \{c_i^{(n)}\}$, or $\{d_j^{(n)}\}$.

(ii) For each f, g as above,

$$\left|\int_{[0,t]^2} f\, dg\right| \leq C\|g|_{[0,t]^2}\|_q^{(2D)} \times \left(\|f|_{[0,t]^2}\|_p^{(2D)} + \|f|_{[0,t]}(0,\cdot)\|_p + \|f|_{[0,t]}(\cdot,0)\|_p + |f(0,0)|\right),$$

where C is a positive constant depending only on p and q.

2.3 Dyadic Approximation

For each $m \in \mathbb{N}$, we will let D_m denote the dyadic partitioning of $[0,T]$; i.e.,

$$D_m := \{k2^{-m}T; k = 0, 1, \ldots, 2^m\}.$$

We define the m-th dyadic approximator $\pi_m : \mathcal{C}([0,T],\mathbb{R}^d) \longrightarrow \mathcal{C}([0,T],\mathbb{R}^d)$ as the unique projection operator such that, for any given $f \in \mathcal{C}([0,T],\mathbb{R}^d)$,

$$\begin{aligned} \pi_m f(t) &= f(t), \quad (t \in D_m) \\ \frac{d^2}{dt^2}\pi_m f(t) &= 0. \quad (t \notin D_m) \end{aligned}$$

In words, $\pi_m f$ is nothing more than the piecewise linear path agreeing with f on the set D_m. We will regularly use the shorthand notation $f_m := \pi_m f$ where convenient. It will be helpful to record here a pair of results relating the variation of paths with their linear approximations.

Theorem 2.3.1 (Propositions 5.20 and 5.60 of [FV10])**.**

(i) Suppose $x \in \mathcal{C}_p$, and let $x_m := \pi_m x$ be the dyadic approximation to x as defined above. Then one has that

$$\|x_m\|_p \leq 3^{p-1}\|x\|_p.$$

(ii) Suppose $x \in \mathcal{C}_p^{(2D)}$, and let $x_m := \pi_m x$ be the dyadic approximation to x as defined above. Then one has that

$$\|x_m\|_p^{(2D)} \leq 9^{p-1}\|x\|_p^{(2D)}.$$

Lemmas 2.2.7 and 2.2.12 also imply that for any $x \in \mathcal{C}_{0,p}$ (resp. $\mathcal{C}_{0,p}^{(2D)}$), the dyadic approximations x_m converge to x in $\mathcal{C}_{0,q}$ (resp. $\mathcal{C}_{0,q}^{(2D)}$) for any $q > p$.

We will also make use of dyadic approximations for the driving stochastic process. Specifically, we will define the m-th dyadic approximation of fractional Brownian motion $B_m := \pi_m(B)$ in the following manner:

$$(B_m)_t := B_{t_-} + (t - t_-)2^m[B_{t_+} - B_{t_-}], \quad (0 \leq t \leq T)$$

where t_- is the largest member of D_m such that $t_- \leq t$ and t_+ is the smallest member of D_m such that $t \leq t_+$. Each dyadic approximation is again a centered Gaussian process, with covariance

$$\begin{aligned}\mathbb{E}\left[(B_m)_s(B_m)_t\right] = R(s_-,t_-) &+ 2^m(t-t_-)\left[R(s_-,t_+) - R(s_-,t_-)\right] \\ &+ 2^m(s-s_-)\left[R(s_+,t_-) - R(s_-,t_-)\right] \\ &+ 2^{2m}(t-t_-)(s-s_-)\times \\ &\quad \left[R(s_+,t_+) - R(s_-,t_+) - (s_+,t_-) + R(s_-,t_-)\right].\end{aligned}$$

2.4 Gaussian Measure Spaces

Let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ denote a separable Banach space. We will say that a measure $\mathbb{P}$ on $\mathcal{W}$ is Gaussian if there exists a symmetric bilinear form $q : \mathcal{W}^* \times \mathcal{W}^* \longrightarrow \mathbb{R}$ such that for all $\varphi \in \mathcal{W}^*$,

$$\int_{\mathcal{W}} \exp\left(i\varphi(\omega)\right)\, d\mathbb{P}(\omega) = \exp\left(-\frac{1}{2}q(\varphi,\varphi)\right).$$

By setting φ as the zero functional of $\mathcal{W}$ above, one can see that $\mathbb{P}$ is a probability measure on $\mathcal{W}$.

Let $\mathcal{B}$ refer to the Borel σ-algebra on $\mathcal{W}$; we will call the triple $(\mathcal{W}, \mathcal{B}, \mathbb{P})$ a Gaussian space. One of the most useful estimates on a Gaussian space was proven by V. Fernique; the theorem that provides this estimate, which we provide below without proof, bears his name.

Theorem 2.4.1 (see, for example, Theorem 2.6 of [DPZ92])**.** *Suppose that $\mathbb{P}$ is a Gaussian measure on $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$. Then there exists $\alpha > 0$ such that*

$$\int_{\mathcal{W}} \exp\left(\alpha\|\omega\|_{\mathcal{W}}^2\right)\, d\mathbb{P}(\omega) < \infty.$$

Remark 2.4.2. *Note that a consequence of Fernique's Theorem is that the map $\omega \mapsto \|\omega\|_{\mathcal{W}}^p$ is integrable for all $p \geq 1$.*

We will denote by $\eta : L^2(\mathbb{P}) \to \mathcal{W}$ the continuous mapping with action given by

$$\eta f := \int_{\mathcal{W}} \omega f(\omega)\, d\mathbb{P}(\omega)$$

where the above expression is a Bochner integral, which is guaranteed to exist as

$$\int_{\mathcal{W}} \|\omega f(\omega)\| d\mathbb{P} \leq \|f\|_2 \left(\int_{\mathcal{W}} \|\omega\|^2 d\mathbb{P} \right)^{\frac{1}{2}} < \infty.$$

Define $\mathcal{H}$ as the image of η restricted to the space $\overline{\mathcal{W}^*}^{L^2(\mathbb{P})}$; this space may be equipped with inner product given by

$$\langle \eta f, \eta g \rangle_{\mathcal{H}} = \langle f, g \rangle_{L^2(\mathbb{P})}.$$

We will refer to $\mathcal{H}$ as the *Cameron-Martin space* associated to the Gaussian space $(\mathcal{W}, \mathcal{B}, \mathbb{P})$.

The best-known example of a Gaussian measure space is the classical d-dimensional Wiener space. In this case, our underlying Banach space is given by

$$\mathcal{W} := \{\omega \in \mathcal{C}([0,T], \mathbb{R}) : \omega(0) = 0\}$$

equipped with the uniform norm; then a Gaussian measure $\mathbb{P}$ may be constructed on $\mathcal{W}$ such that the coordinate process $\{B_t\}_{0 \leq t \leq T}$ defined by

$$B_t(\omega) = \omega(t) \qquad (\omega \in \mathcal{W})$$

is a Brownian motion. Details of the construction of this measure may be found in [Wie23], [Wie24], [Kuo75], and [Str93], among others.

In this case, the Cameron-Martin space is given by

$$\mathcal{H} := \left\{ h \in \mathcal{W} : h(s) = \int_0^s \phi(u)\, du; \phi \in L^2([0,T], \mathbb{R}^d) \right\}$$

with inner product given by

$$\langle h, k \rangle_{\mathcal{H}} := \int_0^T h'(s) k'(s)\, ds.$$

A second example of a Gaussian measure space, which is pertinent to our discussion, is described in detail in [DÜ99] and is given as follows: fix $0 < H < \frac{1}{2}$ and let $\mathcal{W}$ be as above. Then, just as in the classical case, one may define $\mathbb{P}$ as the unique Gaussian measure on $\mathcal{W}$ such that the coordinate process $\{B_t\}_{0 \leq t \leq T}$ defined by

$$B_t(\omega) = \omega(t) \qquad (\omega \in \mathcal{W})$$

is a fractional Brownian motion with Hurst parameter H and $\mathbb{P} = \text{Law}(B)$. By following Proposition 2.1.2 of [BHØZ08], we have that the Cameron-Martin space $\mathcal{H}$ consists of functions of the form

$$h(t) = \int_0^t K_H(t,s)\hat{h}(s)\,ds,$$

where $\hat{h} \in L^2[0,T]$ and

$$\begin{aligned} K_H(t,s) := b_H\Bigg[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} & (t-s)^{H-\frac{1}{2}} \\ & - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}}\,du\Bigg], \end{aligned}$$

where b_H is some suitable normalization constant. The inner product of this space is given by

$$\langle h, k\rangle_{\mathcal{H}} := \langle \hat{h}, \hat{k}\rangle_{L^2[0,T]}.$$

As a vector space, $\mathcal{H}$ is equal to the fractional integral space $I_{0+}^{H+\frac{1}{2}}(L^2[0,T])$; that is to say, each $h \in \mathcal{H}$ is given by

$$h(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} \varphi(s)\,ds$$

for some $\varphi \in L^2[0,T]$ (see Theorems 2.1 and 3.3 of [DÜ99]).

For each fixed $t \in [0,T]$, the function $R(t,\cdot) = \mathbb{E}[B_t B_\cdot] \in \mathcal{H}$ is the *reproducing kernel* for the space; that is to say, for any $h \in \mathcal{H}$, we have the following:

$$\langle h, R(t,\cdot)\rangle_{\mathcal{H}} = h(t).$$

We will repeatedly make use of the following basic lemma without reference.

Lemma 2.4.3. *Suppose $\mathcal{H}$ is a Hilbert space with reproducing kernel R. Then for any orthonormal set of basis vectors $\{k_n\}$; one has the identity*

$$\sum_{n=1}^{\infty} k_n(s)k_n(t) = R(s,t).$$

Proof. This follows from straightforward calculation, as

$$\begin{aligned}\sum_{n=1}^{\infty} k_n(s)k_n(t) &= \sum_{n=1}^{\infty} \langle k_n, R(s,\cdot)\rangle \langle k_n, R(t,\cdot)\rangle \\ &= \langle R(s,\cdot), R(t,\cdot)\rangle = R(s,t).\end{aligned}$$

□

To suit our purposes later on, it will be worthwhile to restrict our sample path space $\mathcal{W}^d$ to a variational space, as we will be using the variational properties to define (almost everywhere) pathwise integrals against our process B. While it might be tempting to replace $\mathcal{W}^d$ by $\mathcal{C}_{0,p}(\mathbb{R}^d)$, we recall that the latter space is non-separable; hence, this is not a suitable choice. We will instead use as our ambient path-space a slightly smaller variational space, outlined below.

Fix

$$p \in \left(\frac{1}{H}, \frac{1}{1-2H}\right), \tag{2.4.1}$$

and define (following Section 5.3.3 of [FV10]) the spaces

$$\begin{aligned}\mathcal{W}_p &:= \overline{C^\infty([0,T],\mathbb{R}) \cap \mathcal{W}}^{\|\cdot\|_p}; \\ \mathcal{W}_p^d &:= \bigoplus_{i=1}^{d} \mathcal{W}_p \cong \overline{C^\infty([0,T],\mathbb{R}^d) \cap \mathcal{W}^d}^{\|\cdot\|_p}.\end{aligned}$$

Proposition 2.4.4 (Corollary 5.33 and Proposition 5.36 of [FV10])**.**

(i) For $p > 1$, one has the following set inclusions:

$$\bigcup_{1 \le q < p} \mathcal{C}_{0,q}(\mathbb{R}^d) \subset \mathcal{W}_p^d \subset \mathcal{C}_{0,p}(\mathbb{R}^d).$$

(ii) For each $p > 1$, the space $(\mathcal{W}_p^d, \|\cdot\|_p)$ is a separable Banach space.

Corollary 2.4.5. *Let B be a d-dimensional fractional Brownian motion of Hurst parameter H. Then the sample paths $t \mapsto B_t$ are almost surely contained in $\mathcal{W}_p^d$ for $p := \frac{1}{H}$.*

Proof. This follows from part (i) of the above proposition, along with Lemma 2.2.5. □

This corollary implies that in our above construction of the Gaussian measure space associated to the fractional Brownian motion, our measure $\mathbb{P}$ is fully supported on $\mathcal{W}_p$. Hence, from here on out we will consider our process $\{B_t\}$ to be restricted to the probability space $(\mathcal{W}_p^d, \mathcal{B}_{\mathcal{W}_p^d}, \mathbb{P}|_{\mathcal{W}_p^d})$; the details of this restriction are included in the Appendix. Most importantly, the Cameron-Martin space $\mathcal{H}^d$ associated to the restriction of our measure is the same as in the original construction. The variational properties of $\mathcal{H}^d$ are of particular interest to us; in preparation, we state the following lemma.

Proposition 2.4.6. *Fix $0 < H < \frac{1}{2}$, and let $r := \frac{1}{2H}$. Then the covariance kernel $R(s,t)$ for fractional Brownian motion of Hurst parameter H as defined in Equation* (2.1.1) *has finite two-dimensional r-variation.*

The proof for the above proposition is given in Appendix A - see Proposition A.0.8 for details. Using this fact, we are in a position to show the following:

Proposition 2.4.7 (see [CFV09] or [FV10])**.** *Let $r := \frac{1}{2H}$. Then the Cameron-Martin space $\mathcal{H}^d$ may be embedded in the space $\mathcal{C}_r$; for each $h \in \mathcal{H}^d$,*

$$\|h\|_r \leq \left[\|R\|_r^{(2D)}\right]^{\frac{1}{2}} \|h\|_{\mathcal{H}^d}.$$

Proof. We may assume $d = 1$, since the same result holds for higher dimensions by working componentwise. Fix $h \in \mathcal{H}$. For each $\Pi = \{t_0, t_1, \ldots, t_N\} \in \mathcal{P}[0,T]$ and $B := \{b_i\}_{i=1}^N \subset \mathbb{R}$, one has the identity

$$\begin{aligned}\sum_{i=1}^N b_i \Delta_i h &= \sum_{i=1}^N b_i \left\langle h, \Delta_i R(t,\cdot)\right\rangle_{\mathcal{H}} \\ &= \left\langle h, \sum_{i=1}^N b_i \Delta_i R(t,\cdot)\right\rangle_{\mathcal{H}}.\end{aligned}$$

The Cauchy-Schwarz inequality implies that

$$\left|\sum_{i=1}^N b_i \Delta_i h\right| \leq \|h\|_{\mathcal{H}} \left\|\sum_{i=1}^N b_i \Delta_i R(t,\cdot)\right\|_{\mathcal{H}}.$$

The reproducing kernel property, along with Hölder's inequality, gives us that

$$\begin{aligned}\left\|\sum_{i=1}^{N} b_i \Delta_i R(t,\cdot)\right\|_{\mathcal{H}}^2 &= \sum_{i,j=1}^{N} b_i b_j \left\langle \Delta_i R(t,\cdot), \Delta_j R(t,\cdot)\right\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^{N} b_i b_j \Delta_{ij} R \\ &\leq \left(\sum_{i,j=1}^{N} |b_i b_j|^{r'}\right)^{\frac{1}{r'}} \left(\sum_{i,j=1}^{N} |\Delta_{ij} R|^{r}\right)^{\frac{1}{r}} \\ &\leq \|R\|_r^{(2D)} \|B\|_{\ell^{r'}}^2.\end{aligned}$$

where $r' = \frac{1}{1-2H}$ is the Hölder conjugate of r. Combining inequalities, we see that

$$\left|\sum_{i=1}^{N} b_i \Delta_i h\right| \leq \|h\|_{\mathcal{H}} (\|R\|_r^{(2D)})^{\frac{1}{2}} \|B\|_{\ell^{r'}}$$

By the converse of Hölder's inequality (Theorem 6.14 of [Fol99], for example), it follows that

$$V_r(h:\Pi) = \left(\sum_{i=1}^{N} |\Delta_i h|^r\right)^{\frac{1}{r}} \leq \|h\| (\|R\|_r^{(2D)})^{\frac{1}{2}}$$

Taking the supremum over all partitions of $[0,T]$ completes the proof. □

We conclude the section with some results which illustrate how relatively "nice" functions also reside within our Cameron-Martin space.

Lemma 2.4.8. *Suppose $0 < H < \frac{1}{2}$, and $\mathcal{H}^d$ is the Cameron-Martin space associated with fractional Brownian motion of Hurst parameter H as constructed above. If $f : [0,T] \to \mathbb{R}^d$ piecewise linear with $f(0) = 0$, then $f \in \mathcal{H}^d$.*

Proof. Without loss of generality, we may assume that $d = 1$. By [SKM93], in order to prove $f \in \mathcal{H}$ it suffices to show that $D_{0+}^{H+\frac{1}{2}} f \in L^2[0,T]$, where

$$D_{0+}^{H+\frac{1}{2}} f(x) := \frac{1}{\Gamma(\frac{1}{2}-H)} \int_0^x f(t)(x-t)^{-(H+\frac{1}{2})}\, dt.$$

Let $\{s_i\}_{i=0}^{N+1} \in \mathcal{P}[0,T]$ be the collection of breakpoints of f and $\{m_i\}_{i=0}^{N} \subset \mathbb{R}$ be the piecewise slopes of f; that is, $\{s_i\}$ and $\{m_i\}$ are the sets of numbers such that we may write

$$f(t)=\sum_{i=0}^{N}\left(f(s_i)+m_i\,(t-s_i)\right)1_{[s_i,s_{i+1})}(t).$$

Let $M:=\max_i |m_i|$. For $x\in[0,T]\backslash\{s_i\}$, let I denote the largest integer such that $s_i<x$ for all $i\leq I$. Then for each such x we find that

$$\begin{aligned}
D_{0+}^{H+\frac{1}{2}}f(x) &= \frac{1}{\Gamma(\frac{1}{2}-H)}\frac{d}{dx}\sum_{i=0}^{N}\int_0^x \left(f(s_i)+m_i\,(t-s_i)\right)1_{[s_i,s_{i+1})}(t)(x-t)^{-(H+\frac{1}{2})}\,dt\\
&= \frac{1}{\Gamma(\frac{1}{2}-H)}\frac{d}{dx}\Bigg(\sum_{i=0}^{I-1}\int_{s_i}^{s_{i+1}}\left(f(s_i)+m_i\,(t-s_i)\right)(x-t)^{-(H+\frac{1}{2})}\,dt\\
&\qquad\qquad + \int_{s_I}^{x}\left(f(s_I)+m_I\,(t-s_I)\right)(x-t)^{-(H+\frac{1}{2})}\,dt\Bigg).
\end{aligned}$$

For $i=0,\ldots,I-1$, we have that

$$\begin{aligned}
&\frac{d}{dx}\int_{s_i}^{s_{i+1}}\left(f(s_i)+m_i\,(t-s_i)\right)(x-t)^{-(H+\frac{1}{2})}\,dt\\
&\qquad= \frac{-1}{\frac{1}{2}-H}\frac{d}{dx}\Bigg[\left(f(s_i)+m_i(t-s_i)\right)(x-t)^{\frac{1}{2}-H}\Bigg|_{s_i}^{s_{i+1}}\\
&\qquad\qquad\qquad\qquad - m_i\int_{s_i}^{s_{i+1}}(x-t)^{\frac{1}{2}-H}\,dt\Bigg]\\
&\qquad= f(s_i)(x-s_i)^{-(H+\frac{1}{2})}-f(s_{i+1})(x-s_{i+1})^{-(H+\frac{1}{2})}\\
&\qquad\qquad\qquad\qquad + m_i\int_{s_i}^{s_{i+1}}(x-t)^{-(H+\frac{1}{2})}\,dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{d}{dx}\int_{s_I}^{x}\left(f(s_I)+m_I\left(t-s_I\right)\right)(x-t)^{-(H+\frac{1}{2})}\,dt\\
&\qquad=\frac{-1}{\frac{1}{2}-H}\frac{d}{dx}\left[\left(f(s_I)+m_I\left(t-s_I\right)\right)(x-t)^{\frac{1}{2}-H}\Big|_{s_I}^{x}-m_I\int_{s_I}^{x}(x-t)^{\frac{1}{2}-H}\,dt\right]\\
&\qquad=\frac{1}{\frac{1}{2}-H}\left(f(s_I)(x-s_I)^{\frac{1}{2}-H}+m_I\frac{d}{dx}\int_{s_I}^{x}(x-t)^{\frac{1}{2}-H}\,dt\right)\\
&\qquad=f(s_I)(x-s_I)^{-(H+\frac{1}{2})}+m_I\int_{s_I}^{x}(x-t)^{-(H+\frac{1}{2})}\,dt.
\end{aligned}$$

Hence, by telescoping sums and that fact that $f(s_0)=0$, we have that

$$D_{0+}^{H+\frac{1}{2}}f(x)=\frac{1}{\Gamma(\frac{1}{2}-H)}\left(\sum_{i=0}^{I-1}m_i\int_{s_i}^{s_{i+1}}(x-t)^{-(H+\frac{1}{2})}\,dt+m_I\int_{s_I}^{x}(x-t)^{-(H+\frac{1}{2})}\,dt\right)$$

and hence $|D_{0+}^{H+\frac{1}{2}}f(x)|\leq\frac{M}{\Gamma(\frac{1}{2}-H)}\int_0^x(x-t)^{-(H+\frac{1}{2})}\;dt=\frac{M}{\frac{1}{2}-H}x^{\frac{1}{2}-H}$. Therefore $D_{0+}^{H+\frac{1}{2}}f(x)\in L^\infty[0,T]\subset L^2[0,T]$. □

The next lemma is proven in a similar fashion, and so we will omit the details.

Lemma 2.4.9 (Lemma 31 of [FV06])**.** *If H and $\mathcal{H}^d$ are as above, then each $f\in C^1[0,T]$ with $f(0)=0$ is contained in $\mathcal{H}^d$.*

2.4.1 Calculus on Wiener spaces

Suppose $(\mathcal{W},\mathbb{P})$ is some Gaussian measure space with Cameron-Martin space $\mathcal{H}$. Let $\mathcal{S}$ refer to the space of *cylinder functionals* on $\mathcal{W}$; that is to say, functionals on $\mathcal{W}$ of the form

$$F(\omega)=f(\phi_1(\omega),\ldots,\phi_n(\omega)),$$

where $f\in C^\infty(\mathbb{R}^n)$ with all partial derivatives having at most polynomial growth, and $\{\phi_1,\ldots,\phi_n\}\subset\mathcal{W}^*$. On $\mathcal{S}$, we will let ∂_h denote the Frechet derivative in the direction of $h\in\mathcal{H}$:

$$\partial_h F:=\frac{d}{d\epsilon}\bigg|_{\epsilon=0}F(\omega+\epsilon h).$$

Let $D : \mathcal{S} \to \mathcal{S} \otimes \mathcal{H}^*$ be the operator defined by the action by

$$\begin{aligned} DF(\omega)k &:= \frac{d}{dt}|_{t=0} F\left(\omega + tk\right) \\ &= \sum_{i=1}^{n} \partial_i f(\phi_1(\omega), \ldots, \phi_n(\omega))\phi_i(k) \quad (k \in \mathcal{H})\,. \end{aligned}$$

For $1 \leq q < \infty$, we will let $\mathbb{D}^{1,q}$ denote the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,q} := (\mathbb{E}[|F|^q] + \mathbb{E}[\|DF\|^q_{\mathcal{H}^*}])$$

One can naturally define an iterated derivate operator D^k taking values in $\mathcal{H}^{\otimes k}$; from this we can define the seminorm

$$\|F\|_{k,q} := \left(\mathbb{E}[|F|^q] + \sum_{j=1}^{k} \mathbb{E}[\|D^j F\|^q_{\mathcal{H}^{\otimes k}}]\right)$$

and we will denote by $\mathbb{D}^{k,q}$ the closure of $\mathcal{S}$ with respect to $\|\cdot\|_{k,q}$. Also, let

$$\mathbb{D}^\infty := \bigcap_{k\in\mathbb{N}} \bigcap_{q\geq 1} \mathbb{D}^{k,q}.$$

Given some $F \in \mathbb{D}^{1,q}$, we may define the *Malliavin covariance matrix* γ by

$$\gamma := DF(DF)^*.$$

The proofs of our main results will hinge upon the following theorem.

Theorem 2.4.10 (Theorem III.5.1 of [Mal97], for example)**.** *Suppose F is a random vector satisfying the following conditions:*

1. *$F \in \mathbb{D}^{1,p}$ for all $p \geq 1$;*

2. *The Malliavin covariance matrix $\gamma = DF(DF)^*$ is almost surely invertible.*

Then F admits a density with respect to Lebesgue measure. If, in addition, one has that

1. *$F \in \mathbb{D}^\infty$;*

2. *$(\det\gamma)^{-1} \in L^{\infty-}$;*

then this density is C^∞.

Chapter 3

The Heisenberg Case

This chapter will be devoted to the proof of Theorem 1.2.1.

3.1 Moments of the Area Process and Its Approximations

We begin by recording some basic results for the area-like process A_T and its approximations $(A_m)_T$. Recall that A_m is defined as

$$\begin{aligned}(A_m)_T &:= \frac{1}{2}\left[\int_0^T (B_m^1)_t \, d(B_m^2)_t - \int_0^T (B_m^2)_t \, d(B_m^1)_t\right] \\ &= \sum_{t_i \in D_m} \left(\frac{B_{t_i}^1 + B_{t_{i-1}}^1}{2}\right) \Delta_i B^2 - \left(\frac{B_{t_i}^2 + B_{t_{i-1}}^2}{2}\right) \Delta_i B^1.\end{aligned}$$

It is immediate that for each $m \in \mathbb{N}$, $(A_m)_T$ is a centered Gaussian random variable.

Fix $m \in \mathbb{N}$. To simplify notation, we will define $t_i := \frac{i}{2^m}T$; to that same end, we will let

$$
\begin{aligned}
\gamma(i,j) &:= \mathbb{E}\left[(B^1_{t_i}+B^1_{t_{i-1}})(B^2_{t_j}+B^2_{t_{j-1}})\right]\mathbb{E}\left[\Delta_i B^1 \Delta_j B^2\right] \\
&= \left(R(t_i,t_j)+R(t_{i-1},t_j)+R(t_i,t_{j-1})+R(t_{i-1},t_{j-1})\right) \\
&\qquad \times \left(R(t_i,t_j)-R(t_{i-1},t_j)-R(t_i,t_{j-1})+R(t_{i-1},t_{j-1})\right) \\
&= \left(R(t_i,t_j)+R(t_{i-1},t_j)+R(t_i,t_{j-1})+R(t_{i-1},t_{j-1})\right)\Delta_{ij}R. \\
\sigma(i,j) &:= \mathbb{E}\left[(B^1_{t_i}+B^1_{t_{i-1}})\Delta_i B^1\right]\mathbb{E}\left[(B^2_{t_j}+B^2_{t_{j-1}})\Delta_j B^2\right], \\
&= \left(R(t_i,t_i)-R(t_{i-1},t_{i-1})\right)\left(R(t_j,t_j)-R(t_{j-1},t_{j-1})\right) \\
&= (t_i^{2H}-t_{i-1}^{2H})(t_j^{2H}-t_{j-1}^{2H}).
\end{aligned}
$$

Using Wick's theorem and the fact that B^1 and B^2 are i.i.d., we find that

$$
\begin{aligned}
&\mathbb{E}\left[|(A_m)_T|^2\right] \\
&\quad = \sum_{i,j=1}^{2^n} \mathbb{E}\Bigg[\left(\frac{B^1_{t_i} + B^1_{t_{i-1}}}{2} \Delta_i B^2 - \frac{B^2_{t_i} + B^2_{t_{i-1}}}{2} \Delta_i B^1 \right) \\
&\qquad\qquad \times \left(\frac{B^1_{t_j} + B^1_{t_{j-1}}}{2} \Delta_j B^2 - \frac{B^2_{t_j} + B^2_{t_{j-1}}}{2} \Delta_j B^1 \right) \Bigg] \\
&\quad = \frac{1}{4} \sum_{i,j=1}^{2^n} \mathbb{E}\Big[(B^1_{t_i} + B^1_{t_{i-1}}) \Delta_i B^2 (B^1_{t_j} + B^1_{t_{j-1}}) \Delta_j B^2 \\
&\qquad\qquad - (B^2_{t_i} + B^2_{t_{i-1}}) \Delta_i B^1 (B^1_{t_j} + B^1_{t_{j-1}}) \Delta_j B^2 \\
&\qquad\qquad - (B^1_{t_i} + B^1_{t_{i-1}}) \Delta_i B^2 (B^2_{t_j} + B^2_{t_{j-1}}) \Delta_j B^1 \\
&\qquad\qquad + (B^2_{t_i} + B^2_{t_{i-1}}) \Delta_i B^1 (B^2_{t_j} + B^2_{t_{j-1}}) \Delta_j B^1 \Big] \\
&\quad = \frac{1}{4} \sum_{i,j=1}^{2^n} \Big(\mathbb{E}[(B^1_{t_i} + B^1_{t_{i-1}})(B^1_{t_j} + B^1_{t_{j-1}})] \mathbb{E}[\Delta_i B^2 \Delta_j B^2] \\
&\qquad\qquad - \mathbb{E}[(B^2_{t_i} + B^2_{t_{i-1}}) \Delta_j B^2] \mathbb{E}[\Delta_i B^1 (B^1_{t_j} + B^1_{t_{j-1}})] \\
&\qquad\qquad - \mathbb{E}[(B^1_{t_i} + B^1_{t_{i-1}}) \Delta_j B^1] \mathbb{E}[\Delta_i B^2 (B^2_{t_j} + B^2_{t_{j-1}})] \\
&\qquad\qquad + \mathbb{E}[(B^2_{t_i} + B^2_{t_{i-1}})(B^2_{t_j} + B^2_{t_{j-1}})] \mathbb{E}[\Delta_i B^1 \Delta_j B^1] \Big) \\
&\quad = \frac{1}{2} \sum_{i,j=1}^{2^n} \left(\gamma(i,j) - \sigma(i,j) \right).
\end{aligned}
$$

One may check via telescoping sums that

$$\sum_{i,j=1}^{2^n} \sigma(i,j) = T^{4H},$$

and hence once has the formula

$$\mathbb{E}\left[|(A_m)_T|^2\right] = \frac{1}{2} \left(\sum_{i,j=1}^{2^m} \gamma(i,j) \right) - \frac{T^{4H}}{2}.$$

From this, we may conclude the following:

Proposition 3.1.1. *For each $m \in \mathbb{N}$, one has the bound*

$$\mathbb{E}\left[|(A_m)_T|^2\right] \leq 2\left(\|R\|_u \|R\|_r^{(2D)}\right) - \frac{T^{4H}}{2}.$$

Since R is of finite two-dimensional r-variation with $r = \frac{1}{2H}$, it follows from Theorem 2.2.13 that $\sum_{i,j=1}^{2^n} \gamma(i,j)$ is bounded so long as $H > \frac{1}{4}$.

Proposition 3.1.2. *Let A_T be the area-process associated to fractional Brownian motion with $\frac{1}{4} < H < \frac{1}{2}$; then*

$$\mathbb{E}[A_T^2] = 2\int_{[0,T]^2} R(s,t)\, dR(s,t) - \frac{T^{4H}}{2}.$$

Proof. Since$(A_m)_T \to A_T$ in L^2, the statement above is equivalent to the convergence

$$\lim_{m\to\infty} \sum_{i,j=1}^{2^m} \gamma(i,j) = 4\int_{[0,T]^2} R(s,t)\, dR(s,t).$$

From the definition of the two-dimensional Young's integral, it will suffice to show that

$$\lim_{n\to\infty} \left| \sum_{i,j=1}^{2^m} \gamma(i,j) - 4R(t_{i-1}, t_{j-1})\Delta_{ij} R \right| = 0$$

To that end, we will make use of Hölder's inequality. Let $r' := \frac{r}{r-1}$ be the conjugate exponent of r, and define

$$\delta_m := \max_{|t-t'|\leq 2^{-m}, |s-s'|\leq 2^{-m}} |R(s,t) - R(s',t')|.$$

Note that δ_m converges to zero as m tends to infinity. It follows that

$$\begin{aligned}
&\left|\sum_{i,j=1}^{2^m}\gamma(i,j) - 4R(t_{i-1},t_{j-1})\Delta_{ij}R\right| \\
&\quad = \left|\sum_{i,j=1}^{2^m}\left(R(t_i,t_j) + R(t_{i-1},t_j) + R(t_i,t_{j-1}) - 3R(t_{i-1},t_{j-1})\right)\Delta_{ij}R\right| \\
&\quad \leq \sum_{i,j=1}^{2^m}\left|R(t_i,t_j) + R(t_{i-1},t_j) + R(t_i,t_{j-1}) - 3R(t_{i-1},t_{j-1})\right|\left|\Delta_{ij}R\right| \\
&\quad \leq \left(\sum_{i,j=1}^{2^m}\left|R(t_i,t_j) + R(t_{i-1},t_j) + R(t_i,t_{j-1}) - 3R(t_{i-1},t_{j-1})\right|^{r'}\right)^{\frac{1}{r'}} \\
&\qquad\qquad\qquad\qquad \times \left(\sum_{i,j=1}^{2^m}\left|\Delta_{ij}R\right|^r\right)^{\frac{1}{r}} \\
&\quad \leq \|R\|^{(2D)})_r\left(\sum_{i,j=1}^{2^m}\left|R(t_i,t_j) + R(t_{i-1},t_j) + R(t_i,t_{j-1}) - 3R(t_{i-1},t_{j-1})\right|^{r'}\right)^{\frac{1}{r'}} \\
&\quad \leq C\delta_m\|R\|_r^{(2D)}\Bigg(\sum_{i,j=1}^{2^m}\left|R(t_i,t_j) - R(t_{i-1},t_{j-1})\right|^r + \left|R(t_{i-1},t_j) - R(t_{i-1},t_{j-1})\right|^r \\
&\qquad\qquad + \left|R(t_i,t_{j-1}) - R(t_{i-1},t_{j-1})\right|^r\Bigg)^{\frac{1}{r'}} \\
&\quad \leq C\delta_m\left(\|R\|_r^{(2D)}\right)^{1+\frac{r}{r'}} \longrightarrow 0.
\end{aligned}$$

□

Remark 3.1.3. *This result gives one some understanding as to why the construction of the area process fails for fractional Brownian motion for $H < \frac{1}{4}$; one can see from the above calculations that the second moments of the approximating processes become unbounded under such a regime.*

Let us record a corollary, which is an immediate consequence of the positivity of the expectation operator.

Corollary 3.1.4. *For each fixed $\frac{1}{4} < H < \frac{1}{2}$, one has the bound*

$$\int_{[0,T]^2} R(u,v)\, dR(u,v) \geq \frac{T^{4H}}{4}.$$

3.2 Operator Realization

In order to prove the conditions required for a regular density, we will recast the process Y of Theorem 1.2.1 in terms of a linear operator on the path-space $\mathcal{W}_p^2$.

Suppose $\{e_1, e_2\}$ is the Euclidean basis on $\mathbb{R}^2$. Given some $\omega := \omega^1 e_1 + \omega^2 e_2 \in \mathcal{W}_p^2$, we will let $\tilde{\omega}$ be the element of $\mathcal{W}_p^2$ defined by

$$\tilde{\omega} := \omega^2 e_1 - \omega^1 e_2 = J\omega,$$

where $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is clockwise rotation by $\pi/2$.

We begin by constructing an operator on the Cameron-Martin space $\mathcal{H}^2$ which can be seen as a polarization of the area process. Recall that $\mathcal{H}^2$ is contained in $\mathcal{C}_r$ for $r := \frac{1}{2H}$. Let $q : \mathcal{H}^2 \times \mathcal{H}^2 \to \mathbb{R}$ denote the symmetric quadratic form given by

$$q(h,k) = \frac{1}{2}\left(\int_0^T h(t) \cdot d\tilde{k}(t) + \int_0^T k(t) \cdot d\tilde{h}(t)\right) \tag{3.2.1}$$

The integrals given in the definition of q are to be considered as Young's integrals; by our previous assumption that $\frac{1}{3} < H < \frac{1}{2}$, we have that $\frac{2}{r} = 4H > 1$, and so q is well-defined. We may also use integration by parts to write

$$q(h,k) = \int_0^T h(t) \cdot d\tilde{k}(t) + \frac{1}{2}\left(k(T) \cdot \tilde{h}(T)\right); \tag{3.2.2}$$

we will frequently and freely change between these equivalent expressions without further remark.

Lemma 3.2.1. *The operator q defined above is continuous in each variable.*

Proof. This is an immediate result of the estimate in Equation (2.2.1) along with Proposition 2.4.7, as

$$|q(h,k)| \leq C\|h\|_r\|k\|_r \leq C\|h\|_{\mathcal{H}}\|k\|_{\mathcal{H}}. \tag{3.2.3}$$

□

As a result of Lemma 2.4.8, we may now rewrite our approximate solutions Y_m in the following form:

$$Y_m = \left((B_m)_T, \frac{1}{2} q(B_m, B_m) \right).$$

As a consequence of the Riesz Representation Theorem, we have the existence of a linear operator $Q : \mathcal{H}^2 \to \mathcal{H}^2$ such that $\langle Qh, k \rangle_{\mathcal{H}^2} = q(h, k)$.

Before finding an explicit form for Q, we require the following key proposition which is a presentation of mappings from $[0, T]$ to Young's integrals against the reproducing kernel of $\mathcal{H}$.

Proposition 3.2.2. *Fix $\alpha \in \mathcal{W}_p$; for each partition $\Pi = \{t_i\}_{i=0}^N \in \mathcal{P}[0, T]$, define the vector $S_\Pi \in \mathcal{H}$ in the following manner:*

$$S_\Pi(\cdot) := \sum_{i=1}^N \alpha(c_i) \left[R(t_i, \cdot) - R(t_{i-1}, \cdot) \right],$$

where $c_i \in (t_{i-1}, t_i)$. Then $\mathcal{H} - \lim_{k \to \infty} S_{\Pi_k}$ exists, where $\{\Pi_k\}_{k=1}^\infty \subset \mathcal{P}[0, T]$ with $|\Pi_k|$ converging to zero as $k \longrightarrow \infty$; furthermore, this limit is independent of the family of partitions. We will denote this limit by

$$\int_0^T \alpha(t) R(dt, \cdot).$$

This limit satisfies the following properties:

1. *$\left\| \int_0^T \alpha(t) R(dt, \cdot) \right\|_{\mathcal{H}}^2 = \int_{[0,T]^2} \alpha \otimes \alpha \; dR$; hence, there exists a constant $C > 0$ such that*
$$\left\| \int_0^T \alpha(t) R(dt, \cdot) \right\|_{\mathcal{H}}^2 \leq C \|\alpha\|_p^2 \|R\|_r^{(2D)}.$$

2. *For each $h \in \mathcal{H}$, $\left\langle \int_0^T \alpha(t) R(dt, \cdot), h \right\rangle_{\mathcal{H}} = \int_0^T \alpha(t) dh(t)$.*

3. *$\left(\int_0^T \alpha(t) R(dt, \cdot) \right)(s) = \int_0^T \alpha(t) R(dt, s)$.*

Proof. First note that $\frac{1}{p} + \frac{1}{r} > (1 - 2H) + 2H = 1$, which implies that

1. the Young's integral of α against $R(\cdot, s)$ for any $s \in [0, T]$ is well-defined, since $R(\cdot, s) \in \mathcal{H}$;

2. the 2D-Young's integral of $\alpha \otimes \alpha$ against R is well-defined.

Then for each k,

$$\begin{aligned}\|S_{\Pi_k}\|_{\mathcal{H}}^2 &= \sum_{i=1}^{\#(\Pi_k)} \sum_{j=1}^{\#(\Pi_k)} \alpha(c_i)\alpha(c_j)\, \langle \Delta_i R(t_i,\cdot), \Delta_j R(t_j,\cdot)\rangle_{\mathcal{H}} \\ &= \sum_{i=1}^{\#(\Pi_k)} \sum_{j=1}^{\#(\Pi_k)} \alpha(c_i)\alpha(c_j)\Delta_{ij} R \\ &\leq C\|\alpha\|_p^2 \|R\|_r^{(2D)},\end{aligned} \tag{3.2.4}$$

where the inequality follows from Theorem 2.1 of [Tow02] and with C being a constant depending only on p and r.

Given any two partitions $\Pi_n = \{s_i\}, \Pi_m = \{t_k\}$ in the family, for $c_i \in [s_{i-1}, s_i]$ and $d_k \in [t_{k-1}, t_k]$,

$$\begin{aligned}\|S_{\Pi_n} - S_{\Pi_m}\|_{\mathcal{H}}^2 &= \|S_{\Pi_n}\|_{\mathcal{H}}^2 + \|S_{\Pi_m}\|_{\mathcal{H}}^2 - 2\langle S_{\Pi_n}, S_{\Pi_m}\rangle_{\mathcal{H}} \\ &= \sum_{i=1}^{\#(\Pi_n)} \sum_{j=1}^{\#(\Pi_n)} \alpha(c_i)\alpha(c_j)\, [\Delta_{ij} R(s_i, s_j)] \\ &\quad + \sum_{k=1}^{\#(\Pi_m)} \sum_{l=1}^{\#(\Pi_m)} \alpha(d_k)\alpha(d_l)\, [\Delta_{kl} R(t_k, t_l)] \\ &\quad - 2 \sum_{i=1}^{\#(\Pi_n)} \sum_{k=1}^{\#(\Pi_m)} \alpha(c_i)\alpha(d_k)\, [\Delta_{ik} R(s_i, t_k)] \\ &\xrightarrow{n,m\to\infty} \int_{[0,T]^2} (\alpha\otimes\alpha)(s,t)\; dR(s,t) \\ &\quad + \int_{[0,T]^2} (\alpha\otimes\alpha)(s,t)\; dR(s,t) \\ &\quad - 2\int_{[0,T]^2} (\alpha\otimes\alpha)(s,t)\; dR(s,t) = 0.\end{aligned}$$

Hence, the completeness of $\mathcal{H}$ implies the existence of $\lim_{n\to\infty} S_{\Pi_n} = \int_0^T \alpha(t) R(dt,\cdot)$. Since the 2D-Young's integral is independent of choice of partitions, one may also see from the calculation above that the limit of S_{Π_k} is also independent of choice of partition, as claimed. Letting k tend to infinity in (3.2.4) proves property (1).

For an arbitrary $h \in \mathcal{H}$, we note that

$$\begin{aligned}
\left\langle \int_0^T \alpha(t)\ R(dt,\cdot), h \right\rangle_{\mathcal{H}} &= \lim_{|\Pi|\to 0} \langle S_\Pi, h\rangle_{\mathcal{H}} \\
&= \lim_{|\Pi|\to 0} \sum_{i=1}^{\#(\Pi)} \alpha(c_i) \big\langle R(t_{i+1},\cdot) - R(t_i,\cdot), h \big\rangle_{\mathcal{H}} \\
&= \lim_{|\Pi|\to 0} \sum_{i=1}^{\#(\Pi)} \alpha(c_i) \left[h(t_{i+1}) - h(t_i)\right] \\
&= \int_0^T \alpha(t)\ dh(t),
\end{aligned}$$

and so (2) holds. In particular, by setting $h = R(s,\cdot)$, (3) is a consequence of (2). $\square$

Proposition 3.2.3. *Let $Q : \mathcal{H}^2 \to \mathcal{H}^2$ be the bounded operator defined by*

$$q(h,k) = \langle Qh, k\rangle_{\mathcal{H}^2},$$

where q is as in Equations (3.2.1) *or* (3.2.2)*. Then the action of Q on elements of $\mathcal{H}^2$ is given by*

$$Qh := \frac{1}{2} R(T,\cdot)\, \tilde{h}(T) - \int_0^T \tilde{h}(t)\, R(dt,\cdot)\,. \tag{3.2.5}$$

Proof. Pick an arbitrary $k \in \mathcal{H}^2$. The inner product of k against each of the terms on the right hand side of (3.2.5) is given as

$$\begin{aligned}
\left\langle R(T,\cdot)\, \tilde{h}(T), k \right\rangle_{\mathcal{H}^2} &= \tilde{h}^1(T) \left\langle R(T,\cdot), k^1 \right\rangle_{\mathcal{H}} + \tilde{h}^2(T) \left\langle R(T,\cdot), k^2 \right\rangle_{\mathcal{H}} \\
&= \tilde{h}^1(T) k^1(T) + \tilde{h}^2(T) k^2(T) = \tilde{h}(T) \cdot k(T),
\end{aligned}$$

and, as a result of Proposition 3.2.2,

$$\begin{aligned}\left\langle \int_0^T \tilde{h}\,(t)\, R\,(dt,\cdot)\,, k \right\rangle_{\mathcal{H}^2} &= \left\langle \int_0^T \tilde{h}^1\,(t)\, R\,(dt,\cdot)\,, k^1 \right\rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^T \tilde{h}^2\,(t)\, R\,(dt,\cdot)\,, k^2 \right\rangle_{\mathcal{H}} \\ &= \int_0^T \tilde{h}^1\,(t)\, dk^1(t) + \int_0^T \tilde{h}^2\,(t)\, dk^2(t) \\ &= \int_0^T \tilde{h}\,(t)\cdot dk(t) = -\int_0^T h\,(t)\cdot d\tilde{k}(t).\end{aligned}$$

By combining these terms and comparing to (**??**), we see that the claim is proven.

□

Proposition 3.2.4. *Let $Q : \mathcal{H}^2 \longrightarrow \mathcal{H}^2$ be the operator defined above.*

1. *Q may be extended to an operator from $\mathcal{W}_p^2$ into $\mathcal{H}^2$, which will also be denoted by Q; for any $\omega \in \mathcal{W}_p^2$,*

$$Q\omega := \frac{1}{2}R(T,\cdot)\tilde{\omega}(T) - \int_0^T \tilde{\omega}(t)\; R(dt,\cdot).$$

2. *Q is a bounded operator on $\mathcal{W}_p^2$.*

Proof. 1. That Q is well-defined as an operator on $\mathcal{W}_p^2$ follows from Proposition 3.2.2; it is then of immediate consequence that $Q\omega \in \mathcal{H}^2$ for any $\omega = (\omega^1, \omega^2) \in \mathcal{W}_p^2$.

2. For a fixed $\omega \in \mathcal{W}_p^2$,

$$
\begin{aligned}
\|Q\omega\|^2_{\mathcal{H}^2} &= \sum_{i=1}^2 \left\| \frac{1}{2} R(T,\cdot)\tilde{\omega}^i(T) - \int_0^T \tilde{\omega}^i(t)\; R(dt,\cdot) \right\|^2_{\mathcal{H}} \\
&= \sum_{i=1}^2 \frac{1}{4} \left\| R(T,\cdot)\tilde{\omega}^i(T) \right\|^2_{\mathcal{H}} + \left\| \int_0^T \tilde{\omega}^i(t)\; R(dt,\cdot) \right\|^2_{\mathcal{H}} \\
&\qquad - \left\langle R(T,\cdot)\tilde{\omega}^i(T), \int_0^T \tilde{\omega}^i(t)\; R(dt,\cdot) \right\rangle_{\mathcal{H}} \\
&= \sum_{i=1}^2 \frac{T^{2H}|\tilde{\omega}^i(T)|^2}{4} + \left\| \int_0^T \tilde{\omega}^i(t)\; R(dt,\cdot) \right\|^2_{\mathcal{H}} \\
&\qquad - \tilde{\omega}^i(T) \int_0^T \tilde{\omega}^i(t)\; R(dt,T) \\
&\leq \sum_{i=1}^2 \frac{T^{2H}|\tilde{\omega}^i(T)|^2}{4} + \left| \int_{[0,T]^2} \left(\tilde{\omega}^i \otimes \tilde{\omega}^i\right)(s,t)\; dR(s,t) \right| \\
&\qquad + \left| \int_0^T \tilde{\omega}^i(T)\tilde{\omega}^i(t)\; R(dt,T) \right| .
\end{aligned}
$$

The first term is the sum is bounded above by $\frac{T^{2H}}{4}\|\tilde{\omega}^i\|_p^2$ by Lemma 2.2.6. The second term is bounded above by a positive multiple of $\|\tilde{\omega}^i\|_p^2\|R\|_r^{(2D)}$ through the application of Theorem 2.2.13 and Lemma 2.2.11. Finally, the third term may be bounded above by some multiple of $\|\tilde{\omega}^i\|_p^2\|R(T,\cdot)\|_r$ as a result of Lemma 2.2.6 and Equation 2.2.1. Putting these bounds together, we may conclude that

$$
\|Q\omega\|^2_{\mathcal{H}^2} \leq C\left(\|\tilde{\omega}^1\|_p^2 + \|\tilde{\omega}^2\|_p^2\right) \leq \hat{C}\|\omega\|_p^2
$$

for suitable constants C and $\hat{C}$. This expression is finite by Proposition 2.4.7.

□

Let us denote by QB the random variable taking values in $\mathcal{H}^2$:

$$
QB := \frac{1}{2} R(T,\cdot)\tilde{B}_T - \int_0^T \tilde{B}_t\; R(dt,\cdot),
$$

where $\tilde{B} := (B^2, -B^1)$.

3.3 Malliavin Derivative

We are now in a position to calculate the derivative of the process Y_T. Let us begin by recording the following helpful convergence theorem, which will be of great use in the section.

Theorem 3.3.1 (Theorem 15.72 of [Jan97])**.** *Let $1 \leq p \leq \infty$, $k \geq 1$. Suppose for some sequence $\{X_n\}_{n=1}^{\infty} \subset \mathbb{D}^{k,p}$, there exists $X \in L^p$ and $Y_j \in L^p(\mathcal{H}^{\otimes j})$ for $j = 1, \ldots, k$ such that $X_n \to X$ in L^p and $D^j X_n \to Y_j$ in $L^p(\mathcal{H}^{\otimes j})$ as $n \to \infty$. Then $X \in \mathbb{D}^{k,p}$, $D^j X = Y_j$, and $X_n \to X$ in $\mathbb{D}^{k,p}$.*

For $i \in \{1, 2\}$ let us denote by R_t^i the linear operator on $\mathcal{H}^2$ with action given by

$$R_t^i h = \langle R(t,\cdot)e_i, h\rangle e_i = h^i(t)e_i,$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. The fact that $DB = R_T^1 + R_T^2$ is immediately clear from the definition of the Malliavin derivative.

Proposition 3.3.2. *The process Y_T has a derivative, DY_T, taking values in the space of linear operators from $\mathcal{H}^2$ into $\mathbb{R}^3$, with action given by*

$$DY_T h = \left(R_T^1 h, R_T^2 h, \langle QB, h\rangle_{\mathcal{H}^2}\right) \ \ a.s.$$

Proof. We have that Y_T is continuously $\mathcal{H}^2$-differentiable by Proposition 3 of [CFV07], and Corollaries 16 and 20 of [CQ02] imply that $\mathbb{E}\left|(Y_m)_T - Y_T\right|^2 \to 0$ as $m \to \infty$. We claim that $DA_T h = \langle QB, h\rangle$; in order to prove this, it suffices by Theorem 3.3.1 to show that

$$\mathbb{E}\left\|\langle QB,\cdot\rangle - D\left(\frac{1}{2}q(B_m, B_m)\right)\right\|^2_{(\mathcal{H}^2)^*} \overset{m\to\infty}{\longrightarrow} 0. \tag{3.3.1}$$

Recall that the process B_m was defined as the dyadic linear approximator to our fractional Brownian motion B; similarly, we will denote by $R_m(u,v)$ the m-th dyadic approximation of the kernel $R(\cdot, v) \in \mathcal{H}$ in the first variable; i.e.,

$$R_m(u,v) := \pi_m\big(R(\cdot,v)\big)(u).$$

Recall that $\tilde{B} = JB$. Since $B(T) \cdot \tilde{B}(T) = 0$, it follows that

$$\begin{aligned} q(B_m, B_m) &= \int_0^T (B_m)_t \cdot d(\tilde{B}_m)_t \\ &= \frac{1}{2} \sum_{k=0}^{2^m} \left(B_{t_i} + B_{t_{i-1}} \right) \cdot \left(\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}} \right). \end{aligned}$$

Using the definition of the Malliavin derivative, one has that

$$\begin{aligned} Dq(B_m, B_m)h &= \frac{1}{2} \sum_{k=0}^{2^m} \Big[\left(h(t_i) + h(t_{i-1}) \right) \cdot \left(\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}} \right) \\ &\qquad + \left(B_{t_i} + B_{t_{i-1}} \right) \cdot \left(\tilde{h}(t_i) - \tilde{h}(t_{i-1}) \right) \Big] \\ &= \int_0^T h_m(t) \cdot d(\tilde{B}_m)_t + \int_0^T (B_m)_t \cdot d\tilde{h}_m(t) \\ &= h(T) \cdot \tilde{B}_T - 2 \int_0^T (\tilde{B}_m)_t \cdot dh_m(t). \\ &= \left\langle R(T, \cdot)\tilde{B}_T - 2 \int_0^T (\tilde{B}_m)_t \; R_m(dt, \cdot), h \right\rangle_{\mathcal{H}^2} \end{aligned}$$

and so $Dq(B_m, B_m) = R(T, \cdot)\tilde{B}_T - 2 \int_0^T (\tilde{B}_m)_t \; R_m(dt, \cdot)$. Since $\|\langle h, \cdot \rangle\|_{\mathcal{H}^*} = \|h\|_{\mathcal{H}}$ for any Hilbert space $\mathcal{H}$, we can use the above calculations to rewrite the left side of (3.3.1) as

$$\begin{aligned} &\mathbb{E} \left\| QB - \left(\frac{1}{2} R(T, \cdot)\tilde{B}_T - \int_0^T (\tilde{B}_m)_t \; R_m(dt, \cdot) \right) \right\|_{\mathcal{H}^2}^2 \\ &\qquad = \mathbb{E} \left\| \int_0^T \tilde{B}_t \; R(dt, \cdot) - \int_0^T (\tilde{B}_m)_t \; R_m(dt, \cdot) \right\|_{\mathcal{H}^2}^2. \end{aligned}$$

Hence, to prove the claim is it required for us to show that

$$\mathbb{E} \left\| \int_0^t B^i(s) R(ds, \cdot) - \int_0^t B^i_m(s) R_m(ds, \cdot) \right\|_{\mathcal{H}}^2 \stackrel{m \to \infty}{\longrightarrow} 0.$$

Let us begin by noting that

$$\begin{aligned}\left\| \int_0^T B^i(s)\ R(ds,\cdot) - \int_0^T B^i_m(s)\ R_m(ds,\cdot)\right\|^2_{\mathcal{H}} \\ \leq \Bigg(\left\| \int_0^T B^i(s)\ (R-R_m)(ds,\cdot)\right\|^2_{\mathcal{H}} \\ + \left\| \int_0^T (B^i - B^i_m)(s) R_m(ds,\cdot)\right\|^2_{\mathcal{H}} \Bigg).\end{aligned}$$

In order to prove convergence of the above expression, it will be helpful to introduce the following notation:

$$\begin{aligned}\Psi_m(u,v) &:= \langle (R-R_m)(u,\cdot),(R-R_m)(v,\cdot)\rangle_{\mathcal{H}} \\ &= R(u,v) - R_m(u,v) - R_m(v,u) + \langle R_m(u,\cdot), R_m(v,\cdot)\rangle_{\mathcal{H}} \\ &= R(u,v) - R_m(u,v) - R_m(v,u) + \pi_m\left[R_m(u,\cdot)\right](v).\end{aligned}$$

Note that Ψ_m converges uniformly to zero as m tends to infinity. As a linear combination of R and R_m, Ψ_m has finite two-dimensional r-variation for $r = \frac{1}{2H}$. Let r' be a number such that $r' > r$ and $\frac{1}{r'} + \frac{1}{r} > 1$; then it follows that Ψ_m has finite two-dimensional r'-variation; furthermore, Lemma 2.2.12 implies that

$$\lim_{m\to\infty} \|\Psi_m\|^{(2D)}_{r'} = 0.$$

Using the continuity of the inner product, we find that

$$\begin{aligned}&\left\| \int_0^t B^i(s)\ (R-R_m)(ds,\cdot)\right\|^2_{\mathcal{H}} \\ &\quad= \lim_{|\Pi|\to 0} \sum_{u_j,v_k\in\Pi} B^i(c_j)B^i(d_k)\langle \Delta_i (R-R_m)(u_j,\cdot), \Delta_j(R-R_m)(v_k,\cdot)\rangle_{\mathcal{H}} \\ &\quad= \lim_{|\Pi|\to 0} \sum_{u_j,v_k\in\Pi} \big(B^i(c_j)B^i(d_k)\big)\,\Delta_{ij}\Psi_m(u_j,v_k) = \int_{[0,T]^2} B^i\otimes B^i\ d\Psi_m \\ &\quad\leq C\|\Psi_m\|^{(2D)}_{r'}\|B^i\|^2_p,\end{aligned}$$

with the inequality resulting from Lemma 2.2.11 and Theorem 2.2.13. Hence,

$$\mathbb{E}\left|\int_{[0,T]^2} B^i \otimes B^i \, d\Psi_m\right| \leq C\|\Psi_m\|_{r'}^{(2D)}\mathbb{E}\|B^i\|_p^2;$$

Fernique's Theorem guarantees that this expression is finite; from our above remarks, we have that its value tends to zero as $m \to \infty$. We may then conclude that

$$\mathbb{E}\left\|\int_0^T B^i(s)\,(R - R_m(ds,\cdot))\right\|_{\mathcal{H}}^2 \to 0.$$

We can approach the convergence of the second term in a similar manner. Choose p' such that $p' > p$ and $\frac{1}{p'} + \frac{1}{r} > 1$; then the sample paths of $B^i - B^i_m$ has finite r-variation and Lemma 2.2.7 tells us that $\|B^i - B^i_m\|_{p'}$ tends to zero as m tends to infinity. Therefore, it follows that

$$\begin{aligned}\mathbb{E}\left\|\int_0^T (B^i - B^i_m)(s)R_m(ds,\cdot)\right\|_{\mathcal{H}}^2 &= \mathbb{E}\int_{[0,T]^2} (B^i - B^i_m)\otimes(B^i - B^i_m)\, dR_m \\ &\leq \mathbb{E}\left[\|B^i - B^i_m\|_{p'}^2\right]\|R\|_r^{(2D)} \to 0\end{aligned}$$

$\square$

Remark 3.3.3. *In fact, we have shown something slightly stronger in the above proof. By changing the exponent on the left-hand side of* (3.3.1), *we may conclude that* $\|D(Y_m)_T - DY_T\|_{(\mathcal{H}^2)^*}$, *converges to zero in all* $L^j, j \geq 1$. *By applying the triangle inequality, we also find that* $\|DY_T\|_{(\mathcal{H}^2)^*} \in L^{\infty-}$.

Proposition 3.3.4. *The random variable* Y_T *is in* $\mathbb{D}^\infty$.

Proof. Corollaries 16 and 20 of [CQ02] implies that $\mathbb{E}[|A_T|^2] < \infty$, and that $A_T = L^2 - \lim_{m\to\infty}(A_m)_T$. Hence, A_T is in the second-order Itô chaos; it follows from hypercontractivity (pp. 61-63 of [Nua06], for example), that $\mathbb{E}[|A_T|^j] < \infty$ for all $1 \leq j < \infty$. Combining this with the above remark, we find that

$$\begin{aligned}\|Y_T\|_{1,j}^j &= \mathbb{E}[|Y_T|^j] + \mathbb{E}\left[\|DY_T\|_{(\mathcal{H}^2)^*}^j\right] \\ &\leq \mathbb{E}[|B_T|^j] + \mathbb{E}[|A_T|^j] + \mathbb{E}\left[\|DY_T\|_{(\mathcal{H}^2)^*}^j\right] < \infty.\end{aligned}$$

In fact, this is sufficient to prove the claim, as one may perform calculations similar to those in Proposition 3.3.2 to show that

$$D^2 Y_T = (0, 0, QR(T, \cdot)),$$

which is clearly deterministic and thus has finite moments of all orders. From this, we also know that $D^k Y_T \equiv 0$ for all $k \geq 3$, and so it must be the case that $\|Y_T\|_{k,j} < \infty$ for all k and j. $\square$

3.4 Integrability of the Malliavian Covariance Determinant

We begin by recording some more general results, which will be useful in proving integrability of $(\det \gamma)^{-j}$.

Lemma 3.4.1. *Suppose that X is a non-negative random variable such that, for each $j \geq 1$, there exists a constant $C_j > 0$ for which*

$$\mathbb{E}\left[e^{-sX}\right] \leq C_j s^{-j} \quad \forall\, s \geq 1.$$

Then $X^{-1} \in L^{\infty-}$.

Proof. Fix $j \geq 1$. We note that for any $k \geq 0$,

$$\int_0^\infty s^{j-1} e^{-ks}\, ds = k^{-j}\Gamma(j),$$

where Γ denotes the standard Gamma function. By letting $k = X$, we find that

$$\mathbb{E}[X^{-j}] = \frac{1}{\Gamma(j)}\mathbb{E}\left[\int_0^\infty s^{j-1} e^{-sX}\, ds\right] = \frac{1}{\Gamma(j)}\int_0^\infty s^{j-1}\mathbb{E}\left[e^{-sX}\right]\, ds.$$

It is sufficient for completion of the proof to note that, under the assumption given, the right-hand expression is finite:

$$\int_0^1 s^{j-1} E\left[e^{-sX}\right]\, ds \leq \int_0^1 s^{j-1} ds = \frac{1}{j};$$
$$\int_1^\infty s^{j-1} E\left[e^{-sX}\right]\, ds \leq \int_1^\infty s^{j-1}(C_{j+1}s^{-(j+1)})\, ds = C_{j+1}.$$

$\square$

Theorem 3.4.2 (see Melcher [Mel04, pp.26-27])**.** *Let* $(\mathcal{W}, \mathcal{B}, \mathbb{P})$ *be a Gaussian measure space with associated Cameron-Martin space* $\mathcal{H}$*, and suppose* $\Phi : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ *is a bounded non-negative quadratic form. Then the operator* $\hat{\Phi} : \mathcal{H} \to \mathcal{H}$ *given by*

$$\Phi(h, k) = \langle \hat{\Phi} h, k \rangle_{\mathcal{H}}$$

is trace-class. In addition, if $\hat{\Phi}$ *is not a finite rank operator, then*

$$\Phi^{-1} \in L^{\infty-}(\mathcal{W}, \mathbb{P}).$$

Proof. By Theorem 5.3.32 of [Str93], we have that there exists a set of independent, identically distributed standard normal random variables $\{\xi_n\}_{n=1}^{\infty}$ such that the series $B^N := \sum_{n=1}^{N} \xi_n h_n$ converges in $\mathcal{W}$ to B $\mathbb{P}$-a.s. and in all $L^j, j \geq 1$ as $N \to \infty$, and

$$\text{Law}\left(\sum_{n=1}^{\infty} \xi_n h_n\right) = \mathbb{P}.$$

In particular, the fact that $\mathbb{E}\|B^N - B\|_{\mathcal{W}}^2 \to 0$ implies that $\Phi(B^N, B^N) \to \Phi(B, B)$ in L^1. Thus, Fernique's Theorem allows us to conclude that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left\langle \hat{\Phi} h_n, h_n \right\rangle_{\mathcal{H}} &= \lim_{N\to\infty} \sum_{n=1}^{N} \Phi(h_n, h_n) \\
&= \lim_{N\to\infty} \mathbb{E}\left[\Phi(B^N, B^N)\right] \\
&= \mathbb{E}[\Phi(B)] \leq C\mathbb{E}[\|B\|_{\mathcal{W}}^2] < \infty.
\end{aligned}$$

Thus, $\hat{\Phi}$ is trace-class.

Suppose that $\hat{\Phi}$ is not finite rank. Since $\hat{\Phi}$ is compact, there exists an orthonormal basis $\{h_n\}_{n=1}^{\infty} \subset \mathcal{H}$ for which $\hat{\Phi} h_n = \lambda_n h_n$; our assumption guarantees that $\#\{n : \lambda_n > 0\} = \infty$. Using this, it is easy to check that

$$\Phi(B^N, B^N) = \left\langle \hat{\Phi} B^N, B^N \right\rangle_{\mathcal{H}} = \sum_{n=1}^{N} \lambda_n \xi_n^2$$

and so

$$\Phi(B, B) = L^1 - \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \xi_n^2.$$

We will let $K_N := \#\{1 \leq n \leq N : \lambda_n > 0\}$; it is clear that $\{K_N\}$ is an non-decreasing sequence with $K_N \overset{N\to\infty}{\longrightarrow} \infty$. Therefore, for each fixed N and positive s,

$$\begin{aligned}\mathbb{E}\left[\exp\left(-s\Phi\left(B,B\right)\right)\right] &= \mathbb{E}\left[\exp\left(-s\lim_{N\to\infty}\sum_{n=1}^{N}\lambda_n\xi_n^2\right)\right]\\ &\leq \mathbb{E}\left[\exp\left(-s\sum_{n=1}^{N}\lambda_n\xi_n^2\right)\right]\\ &= \prod_{n=1}^{N}\left(\frac{1}{2\lambda_n s+1}\right)^{\frac{1}{2}} \leq C_N s^{-\frac{K_N}{2}}.\end{aligned}$$

Applying Lemma 3.4.1 finishes the proof. $\square$

In order to apply Theorem 3.4.2, we will explicitly calculate a formula for the determinant of the Malliavin covariance matrix associated to Y.

Lemma 3.4.3. *Given any $a \neq 0$, $C \in M_{m,n}(\mathbb{R})$, and $D \in M_n(\mathbb{R})$, one has that*

$$\det\begin{bmatrix} aI_m & C \\ C^{tr} & D\end{bmatrix} = a^m\left(\det(D - a^{-1}C^{tr}C)\right),$$

where C^{tr} is the transpose of C.

Proof. This claim follows immediately when one writes

$$\begin{bmatrix} aI_m & C \\ C^{tr} & D\end{bmatrix} = \begin{bmatrix} aI_m & 0 \\ C^{tr} & I_n\end{bmatrix}\begin{bmatrix} I_m & a^{-1}C \\ 0 & D - a^{-1}C^{tr}C\end{bmatrix}.$$

$\square$

Proposition 3.4.4. *Define the map $\gamma : \mathcal{W}_p^2 \to M^3(\mathbb{R})$ in the following manner:*

$$\gamma(\omega) := \begin{bmatrix} T^{2H}I_2 & \left(Q\omega\right)\left(T\right) \\ \left[\left(Q\omega(T)\right)\right]^{tr} & \|Q\omega\|^2_{\mathcal{H}^2}\end{bmatrix}$$

Also, define the quadratic form Φ on $\mathcal{W}_p^2$ as follows:

$$\Phi(\omega) = T^{4H}\|Q\omega\|^2_{\mathcal{H}^2} - T^{2H}|Q\omega(T)|^2.$$

Then

1. $DY_T(DY_T)^* = \gamma$ *a.s.*

2. $\Phi = \det\gamma$.

Proof. 1. We begin by calculating the adjoint operator $(DY)^*$.

Recall that $R_t^i h = h^i(t)e_i$. For a fixed $\lambda = (\lambda^1, \lambda^2) \in \mathbb{R}^2$, $h \in \mathcal{H}^2$, and $i = 1, 2$,

$$\begin{aligned}\left\langle (R_T^i)^*\lambda, h\right\rangle_{\mathcal{H}^2} &= \lambda \cdot h^i(T)e_i \\ &= \lambda^i h^i(T) = \left\langle \lambda^i R(T,\cdot)e_i, h\right\rangle_{\mathcal{H}^2}.\end{aligned}$$

Thus, one has that

$$R_T^i(R_T^j)^*\lambda = R_T^i\lambda^j R(T,\cdot)e_j = \delta_{ij}T^{2H}\lambda^j.$$

Suppose $\mathbf{x} \in \mathbb{R}^3$.

$$\begin{aligned}\langle (DY_T)^*\mathbf{x}, k\rangle &= \mathbf{x}\cdot DY_T k \\ &= \mathbf{x}\cdot (k(T), \langle QB, k\rangle) \\ &= (x^1, x^2)\cdot k(T) + x^3(\langle QB, k\rangle) \\ &= \langle (R_T^1)^*(x^1,x^2) + (R_T^2)^*(x^1,x^2) + x^3 QB, k\rangle \\ &= \langle x^1 R(T,\cdot)e_1 + x^2 R(T,\cdot)e_2 + x^3 QB, k\rangle.\end{aligned}$$

Using this, we may now verify the claim:

$$\begin{aligned}DY_T(DY_T)^*\mathbf{x} &= DY_T\left(x^1R(T,\cdot)e_1 + x^2R(T,\cdot)e_2 + x^3QB\right) \\ &= DY_T(x^1R(T,\cdot)e_1) + DY_T(x^2R(T,\cdot)e_2) + DY_T(x^3QB) \\ &= (T^{2H}x^1, 0, \langle QB, x^1R(T,\cdot)e_1\rangle) \\ &\quad + (0, T^{2H}x^2, \langle QB, x^2R(T,\cdot)e_2\rangle) \\ &\quad + (x^3QB^1(T), x^3QB^2(T), \langle QB, x^3QB\rangle) \\ &= \begin{bmatrix} T^{2H}I_2 & QB(T) \\ (QB(T))^{tr} & \|QB\|^2_{\mathcal{H}^2}\end{bmatrix}\mathbf{x}.\end{aligned}$$

2. This follows from Lemma 3.4.3.

□

Lemma 3.4.5. *The quadratic form Φ is positive semidefinite and has a trivial nullspace.*

Proof. The Cauchy-Schwarz inequality allows us to see that

$$|(Q\omega)^i(T)|^2 = |\langle R(T,\cdot),(Q\omega)^i\rangle_{\mathcal{H}}|^2 \leq \|R(T,\cdot)\|_{\mathcal{H}}^2 \|(Q\omega)^i\|_{\mathcal{H}}^2 = T^{2H}\|(Q\omega)^i\|_{\mathcal{H}}^2.$$

Hence, Φ is non-negative and $\Phi = 0$ if and only if $Q\omega = \mathbf{c}R(T,\cdot)$ for some constant vector $\mathbf{c} \in \mathbb{R}^2$. Suppose, then, that ω satisfies this condition; then by the definition of Q, it would follow that

$$0 = \mathbf{c}R(T,\cdot) - \frac{\tilde{\omega}(T)R(T,\cdot)}{2} + \int_0^T \tilde{\omega}(t)\ R(dt,\cdot) = \int_0^T \left(\tilde{\omega}(t) - \frac{\tilde{\omega}(T)}{2} + \mathbf{c}\right)\ R(dt,\cdot)$$

which implies by Proposition 3.2.2 that for all $h \in \mathcal{H}^2$,

$$0 = \int_0^T \left(\tilde{\omega}(t) - \frac{\tilde{\omega}(T)}{2} + \mathbf{c}\right) \cdot dh(t).$$

As a result of Lemma 2.4.9, one has that $\mathcal{C}_c^\infty(0,T) \subset \mathcal{H}$; then for each $\varphi \in \mathcal{C}_c^\infty(0,T)$, integration by parts allows us to conclude that

$$0 = \int_0^T \varphi(t) \cdot d\left(\tilde{\omega}(t) - \frac{\tilde{\omega}(T)}{2} + \mathbf{c}\right) = \int_0^T \varphi(t) \cdot d\tilde{\omega}(t),$$

which implies that $\tilde{\omega}(t)$ is a constant function. Thus $\omega(t) = \omega(0) = 0$ for all $t \in [0,T]$. □

Corollary 3.4.6. $(\Phi)^{-1} \in L^{\infty-}(\mathcal{W}_p^2, \mathbb{P})$.

Proof. Combining Lemma 3.4.5 and Theorem 3.4.2 gives us the desired result. □

Portions of Chapters 3 are adapted from material submitted for publication as Driscoll, Patrick, "Smoothness of Density for the Area Process of Fractional Brownian Motion." The dissertation author was the sole author of this paper.

Chapter 4

General Case

This chapter is devoted to the proof of Theorem 1.2.2. Many of the objects of study and proof techniques are directly analogous to those used in the previous chapter.

4.1 Step-2 Nilpotent Fields

As before, we fix $\frac{1}{3} < H < \frac{1}{2}$; let $B := \{B^1, \ldots, B^n\}$ denote n-dimensional fractional Brownian motion with Hurst parameter H.

Let $k \in \{1, \ldots, \frac{n(n-1)}{2}\}$, and suppose that $\{\alpha_1, \ldots, \alpha_k\}$ is a collection of maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ with the following properties:

(I) Each α_l is a skew-symmetric bilinear form;

(II) The set $\{\alpha_l\}$ is a linearly independent set; i.e., the bilinear form $\sum_l c_l \alpha_l$ is the zero map if and only if $c_l = 0$ for all l.

Define $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ by $\alpha := (\alpha_1, \ldots, \alpha_k)$; this map induces a function from $\mathcal{W}^n \times \mathcal{W}^n$ into $\mathcal{W}^k$, which we will also refer to as α; its action is given by

$$[\alpha(\omega, \tau)](t) := \alpha(\omega(t), \tau(t)).$$

Just as in the case of the Heisenberg group, we may use the skew-symmetric form α to define a Lie group $(G, \circ)$, where $G := \mathbb{R}^n \times \mathbb{R}^k$ and $\circ : G \times G \to G$ is the multiplication operation defined by

$$(\mathbf{v}, \mathbf{x}) \circ (\mathbf{w}, \mathbf{y}) = \left(\mathbf{v} + \mathbf{w}, \mathbf{x} + \mathbf{y} + \frac{1}{2}\alpha(\mathbf{v}, \mathbf{w})\right).$$

Additionally, for any $\xi = (\mathbf{w}, \mathbf{y}) \in \mathfrak{g} \simeq T_e G$, one may readily calculate the the left-invariant vector field $\tilde{\xi}$;

$$\tilde{\xi}_{(\mathbf{v},\mathbf{x})} = \left(\mathbf{w}, \mathbf{y} + \frac{1}{2}\alpha\left(\mathbf{v}, \mathbf{w}\right)\right).$$

The Jacobian vector fields $\{X_i := \widetilde{(e_i, 0)}\}_{i=1}^n$ for $\mathfrak{g}$ are given by

$$(X_i)_{(\mathbf{v},\mathbf{x})} = \left(e_i, \frac{1}{2}\alpha\left(\mathbf{v}, e_i\right)\right).$$

Similar to the Heisenberg case, we have that

$$[X_i, X_j] = (0, \alpha(e_i, e_j)).$$

The following theorem ensures us that any step-2 stratified group on $\mathbb{R}^{n+k}$ may be generated in such a way.

Theorem 4.1.1 (Theorem 3.2.1 of [BLU07])**.** *Each homogeneous Carnot group of step two on $\mathbb{R}^{n+k}$ is characterized by an operator α satifisying Properties I and II above.*

4.2 Stochastic Differential Equation Solutions

Suppose $B = (B^1, \ldots, B^n)$ is an n-dimensional fractional Brownian motion with Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$. Given a step-2 homogeneous Carnot group G with Jacobian basis $\{X_i\}_{i=1}^n$ as in the previous section, one may make sense of the differential equation

$$dY = \sum_i X_i(Y) dB^i \tag{4.2.1}$$

as in Chapter 2; that is, we consider the solution Y as the limiting process of solutions of (4.2.1) driven by dyadic approximation processes $B_m = \pi_m B$. Suppose

α is the skew-symmetric operator on $\mathbb{R}^n \times \mathbb{R}^n$ as defined in Section 1; then the stochastic differential equation in the dyadic approximation case is of the form

$$\begin{aligned} dY_m = \sum_i X_i(Y_m) dB_m^i &= \sum_i \left(e_i, \frac{1}{2}\alpha(y, e_i) \right) dB_m^i \\ &= \left(dB_m^1, \ldots, dB_m^n, \frac{1}{2} \sum_i \alpha(y, e_i) dB_m^i \right). \end{aligned}$$

It is easy to check that the solution is given as $Y_m := (y_m, \hat{y}_m)$, where

$$\begin{aligned} (y_m)_T &= (B_m)_T, \\ (\hat{y}_m)_T &= \frac{1}{2} \int_0^T \alpha((B_m)_t, d(B_m)_t) := \frac{1}{2} \sum_i \int_0^T \alpha((B_m)_t, e_i) d(B_m^i)_t, \end{aligned}$$

with the integrals above interpreted as Riemann-Stieltjies integrals since the piecewise linearity of B_m implies that $\alpha(B_m, e_i)$ is piecewise linear as well for each i.

Theorem 2 of [CQ02] and Theorem 4.1.1 of [Lyo98] imply that the limiting process $Y := \lim_{m\to\infty} Y_m$ exists a.s. We will suggestively write this process heuristically as

$$Y_T = \left(B_T, \frac{1}{2} \int_0^T \alpha(B_t, dB_t) \right).$$

We record here a pair of simple lemmas which allow for some control of the process Y.

Lemma 4.2.1. *Suppose α is a continuous bilinear form on $\mathbb{R}^n$. Then for each fixed $v \in \mathbb{R}^n$, the mapping*

$$f \mapsto \alpha(f, v)$$

is a map from $\mathcal{C}_p([0,T], \mathbb{R}^n)$ into $\mathcal{C}_p([0,T], \mathbb{R})$ for all $p \geq 1$; more explicitly, one has the bound

$$\|\alpha(f, v)\|_p \leq \|\alpha\|_{L(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})} |v| \|f\|_p.$$

Proof. Fix $f \in \mathcal{C}_p([0,T],\mathbb{R}^n)$; and let $\Pi = \{t_i\}_{i=0}^N \in \mathcal{P}[0,T]$. Then one has that

$$\begin{aligned}\sum_{i=1}^N |\alpha(f(t_{i+1}),v) - \alpha(f(t_i),v)|^p &= \sum_{i=1}^N |\alpha(f(t_{i+1}) - f(t_i),v)|^p \\ &\leq \sum_{i=1}^N \|\alpha\|^p_{(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})} \|v\|^p_U \left|f(t_{i+1}) - f(t_i)\right|^p.\end{aligned}$$

Taking the supremum over all such partitions of $[0,T]$ will then complete the proof. $\square$

Lemma 4.2.2. *Let α be a continuous bilinear map on $\mathbb{R}^n$, and suppose $v \in \mathbb{R}^n$. If p,q be constants such that $\frac{1}{p} + \frac{1}{q} > 1$, then for any $f \in \mathcal{C}_p([0,T],\mathbb{R}^n)$, $g \in \mathcal{C}_q([0,T],\mathbb{R})$, the Young's integral*

$$\int_s^t \alpha(f(\tau),v)dg(\tau)$$

is well defined for all $0 \leq s < t \leq T$, and satisfies the bounds

$$\left|\int_s^t \alpha(f(\tau),v)dg(\tau) - \alpha(f(s),v)\left(g(t)-g(s)\right)\right| \leq C\|f|_{[s,t]}\|_p \|g|_{[s,t]}\|_q,$$

where C is a constant depending on $p,q,\alpha,$ and v.

Proof. This immediately follows from Lemma 4.2.1, along with Theorem 2.2.8 and the bound given in (2.2.1). $\square$

4.3 Operator Realization

Recall that we have restricted the value of the Hurst parameter to $\frac{1}{3} < H < \frac{1}{2}$.

For $j = 1,\ldots,k$, define a quadratic form q_j on $\mathcal{H}^n \times \mathcal{H}^n$ as follows:

$$\begin{aligned}q_j(h,k) &:= \frac{1}{2}\left[\int_0^T \alpha_j(h(s),dk(s)) + \int_0^T \alpha_j(k(s),dh(s))\right] \qquad (4.3.1)\\ &= \int_0^T \alpha_j(h(s),dk(s)) - \frac{\alpha_j(h(T),k(T))}{2}.\end{aligned}$$

Note that the above integrals are to be intepreted in the manner of Young, and are well defined as by Lemma 4.2.2, along with the fact that the Cameron-Martin space may be embedded into $\mathcal{C}_r$ with $r := \frac{1}{2H}$. Since piecewise linear continuous functions are contained in $\mathcal{H}$, we may write our approximating process Y_m as:

$$(Y_m)_T = \left((B_m)_T, \frac{1}{2}q_1(B_m, B_m), \ldots, \frac{1}{2}q_k(B_m, B_m)\right).$$

Define the linear mapping $a : \mathcal{W}_p \to \mathcal{H}$ by

$$a\omega := \frac{1}{2}\omega(T)R(T, \cdot) - \int_0^T \omega(t)R(dt, \cdot).$$

The above integral is to be interpreted in the manner of Young, and the mapping above is well-defined as a result of Proposition 3.2.2.

Lemma 4.3.1. *The operator a is bounded on $\mathcal{W}_p$.*

Proof. This proof is almost identical to the second part of the proof for Proposition 3.2.4. For any $\omega \in \mathcal{W}_p$, one may readily verify using Proposition 3.2.2 that

$$\|a\omega\|_{\mathcal{H}}^2 = \frac{1}{4}T^{2H}|\omega(T)|^2 - \omega(T)\int_0^T \omega(t)R(dt, T) + \int_{[0,T]^2} \omega(s)\omega(t)dR(s, t).$$

We may now bound the absolute value of each term on the right-hand side; the first term by using the p-variational embedding of $\mathcal{C}$ as given in Lemma 2.2.6, the second term by Lemma 2.2.6 and the Young's integral bounds in Theorem 2.2.8, and the third term using the two-dimensional Young's integral bounds given in Theorem 2.2.13. $\square$

Given a skew-symmetric bilinear form ξ on $\mathbb{R}^n$, we define J_ξ as the linear map on $\mathbb{R}^n$ with action given by

$$J_\xi x = \sum_i \xi(e_i, x)e_i.$$

We will regularly refer to $J_l := J_{\alpha_l}$ for the operators $\{\alpha_l\}$ defined in Section 4.1.

Using this notation, one has the identity

$$q_l(B_m, B_m) = \int_0^T (B_m)_t \, dJ_l(B_m)_t.$$

One may take the tensor product of these two operators to form an operator on $\mathcal{H} \otimes \mathbb{R}^n \cong \mathcal{H}^n$:

$$
\begin{aligned}
(a \otimes J_\xi)h &= \frac{1}{2} R(T, \cdot) \otimes J_\xi h(T) - \int_0^T R(dt, \cdot) \otimes J_\xi h(t) \\
&= \frac{1}{2}\left[\int_0^T R(t\cdot) \otimes dJ_\xi h(t) - \int_0^T R(dt, \cdot) \otimes J_\xi h(t)\right].
\end{aligned}
$$

Example 2. *If $n = 2$, all skew-symmetric forms are scalar multiples of ξ, where $\xi(x, y) = (y, -x)$. Then in this case, one has that*

$$
(a \otimes J_\xi)h = \frac{1}{2} R(T\cdot) \otimes \xi h(T) - \int_0^T R(dt, \cdot) \otimes \xi h(t)
$$

which is equivalent to the operator Q as defined in Proposition 3.2.3.

Lemma 4.3.2. *Suppose α is a continuous skew-symmetric bilinear form on $\mathbb{R}^n$. Then for any $h \in \mathcal{H}^n$ and for each partition $\Pi = \{t_j\}_{j=0}^N \in \mathcal{P}[0, T]$, define the vector $S_\Pi \in \mathcal{H}$ in the following manner:*

$$
S_\Pi(\cdot) := \sum_{j=1}^N \alpha(h(c_j), [R(t_j, \cdot) - R(t_{j-1}, \cdot)]\, e_i)
$$

where $c_j \in (t_{j-1}, t_j)$. Then $\mathcal{H} - \lim_{k\to\infty} S_{\Pi_k}$ exists, where $\{\Pi_k\}_{k=1}^\infty \subset \mathcal{P}[0, T]$ with $|\Pi_k|$ converging to zero as $k \longrightarrow \infty$; furthermore, this limit is independent of the family of partitions. We will denote this limit by

$$
\int_0^T \alpha(h(t), R(dt, \cdot)e_i).
$$

This limit satisfies the following properties:

1. *$\left\|\int_0^T \alpha(h(t), R(dt, \cdot)e_i)\right\|_{\mathcal{H}}^2 = \int_{[0,T]^2} \alpha(h(s), e_i)\alpha(h(t), e_i)\; dR(s, t)$; hence, there exists a constant $C > 0$ such that*

$$
\left\|\int_0^T \alpha(h(t), R(dt, \cdot)e_i)\right\|_{\mathcal{H}}^2 \leq C\|\alpha(h, e_i)\|_p^2 \|R\|_r^{(2D)}.
$$

2. *For each $k \in \mathcal{H}$, $\left\langle \int_0^T \alpha(h(t), R(dt, \cdot)e_i), k\right\rangle_{\mathcal{H}} = \int_0^T \alpha(h(t), e_i)\; dk(t)$.*

3. $\left(\int_0^T \alpha(h(t), R(dt,\cdot)e_i)\right)(s) = \int_0^T \alpha(h(t), R(dt,s)e_i)$.

Proof. This is an application of Proposition 3.2.2 along with Lemma 4.2.1. □

Proposition 4.3.3. *For each symmetric form q_l, as defined in Equation* (4.3.1) *with $l = 1, \ldots, k$, one has the identity*

$$q_l(h, \tilde{h}) = \left\langle (a \otimes J_l)h, \tilde{h} \right\rangle_{\mathcal{H}\otimes\mathbb{R}^n}.$$

Proof. This result follows from computing the right-hand inner product, using part 2 of Lemma 4.3.2. □

Proposition 4.3.4. *For each skew-symmetric bilinear form ξ on $\mathbb{R}^n$, the map $a \otimes J_\xi$ may be extended to a bounded mapping from $\mathcal{W}_p \otimes \mathbb{R}^n$ to $\mathcal{H} \otimes \mathbb{R}^n$; we will also denote this extension by $a \otimes J_\xi$. Its action is given by*

$$(a \otimes J_\xi)\omega = \frac{1}{2}\left[\int_0^T R(t,\cdot) \otimes dJ_\xi\omega(t) - \int_0^T R(dt,\cdot) \otimes J_\xi\omega(t)\right].$$

Proof. This proof follows in the same manner as the analogous result detailed in Proposition 3.2.4; we will only record here an explicit upper bound on the operator norm of $a \otimes J_\xi$. To that end, given an $\omega \in \mathcal{W}_p \otimes \mathbb{R}^n$, we have

$$\begin{aligned}
\|(a \otimes J_\xi)\omega\|^2_{\mathcal{H}\otimes\mathbb{R}^n} = \sum_{i=1}^n \Bigg(&\left\|\frac{\xi(\omega(T), e_i)R(t,\cdot)}{2}\right\|^2 + \left\|\int_0^T \xi(e_i, omega(t))R(dt,\cdot)\right\|^2 \\
&- \left\langle \xi(\omega(T), e_i)R(T,\cdot), \int_0^T \xi(e_i,\omega(t))R(dt,\cdot)\right\rangle \Bigg) \\
= \sum_{i=1}^n \Bigg(&(\xi(\omega(T), e_i))^2 \frac{T^{2H}}{4} + \int_{[0,T]^2} \xi(e_i,\omega(s))\xi(e_i,\omega(t))dR(s,t) \\
&+ \xi(\omega(T), e_i)\int_0^T \xi(e_i,\omega(t))R(dt,T)\Bigg) \\
\leq \sum_{i=1}^n \Bigg(&\|\xi\|^2 \Bigg(|\omega(T)|^2 \frac{T^{2H}}{4} + \|\omega\|_p^2 \|R\|_r^{(2D)} \\
&+ |\omega(T)|\|\omega\|_p\|R\|_r \Bigg)\Bigg) \\
\leq n\|\xi\|^2 &\left(\frac{T^{2H}}{4} + \|R\|_r^{(2D)} + \|R\|_r\right)\|\omega\|_p^2.
\end{aligned}$$

□

We may thus define random processes $(a \otimes J_l)B$ for each $l = 1, \ldots, k$ by the formula

$$(a \otimes J_l)B = \frac{1}{2}\left[\int_0^T R(t,\cdot) \otimes dJ_l B_t - \int_0^T R(dt,\cdot) \otimes J_l B_t)\right]. \qquad (a.s.)$$

4.4 Malliavin Derivative

Let $R_T^j : \mathcal{H} \otimes \mathbb{R}^n \to \mathbb{R}$ be the evaluation operator on the j-th coordinate; i.e.,

$$R_T^j h := h^j(T) = \langle R(T,\cdot) \otimes e_j, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n}.$$

Proposition 4.4.1. *The process Y_T has derivative DY_T taking values in $\mathcal{L}(\mathcal{H} \otimes \mathbb{R}^n, \mathbb{R}^{n+k})$, with action given by*

$$DY_T h = \left(R_T^1 h, \ldots, R_T^n h, \langle (a \otimes J_1)B, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n}, \ldots, \langle (a \otimes J_k)B, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n}\right) \quad (a.s.)$$

Proof. We begin by computing the derivative of $q_l(B_m, B_m)$ for $l = 1, \ldots, k$ in the direction of some $h \in \mathcal{H} \otimes \mathbb{R}^n$. Again, we let $T_j = \frac{j}{2^m}T$.

$$\begin{aligned}
Dq_l(B_m, B_m)h &= D\left[\sum_{j=1}^{2^m} \frac{B_{T_j} + B_{T_{j-1}}}{2}\left(J_l B_{T_j} - J_l B_{T_{j-1}}\right)\right] h \\
&= \frac{1}{2}\sum_{j=1}^{2^m} \left(h(T_j) + h(T_{j-1})\right)\left(J_l B_{T_j} - J_l B_{T_{j-1}}\right) \\
&\qquad\qquad + \left(B_{T_j} + B_{T_{j-1}}\right)\left(J_l h(T_j) - J_l h(T_{j-1})\right) \\
&= \frac{1}{2}\sum_{j=1}^{2^m} \left(h(T_j) + h(T_{j-1})\right)\left(J_l B_{T_j} - J_l B_{T_{j-1}}\right) \\
&\qquad\qquad - \left(J_l B_{T_j} + J_l B_{T_{j-1}}\right)\left(h(T_j) - h(T_{j-1})\right) \\
&= \frac{1}{2}\left[\int_0^T h_m(t) d(J_l B_m)_t - \int_0^T (J_l B_m)_t dh_m(t)\right] \\
&= \langle (a \otimes J_l)B_m, h\rangle_{\mathcal{H}\otimes\mathbb{R}^n}.
\end{aligned}$$

Thus, in order to prove the claim, it suffices to show that for each $l =$

$1, \ldots, k$,

$$\mathbb{E} \left\| \langle (a \otimes J_l) B, \cdot \rangle - \langle (a \otimes J_l) B_m, h \rangle \right\|^2_{(\mathcal{H} \otimes \mathbb{R}^n)^*}$$
$$= \mathbb{E} \left\| \int_0^T R(dt, \cdot) \otimes (J_l B)_t - \int_0^T R_m(dt, \cdot) \otimes (J_l B_m)_t \right\|^2_{\mathcal{H} \otimes \mathbb{R}^n}$$

tends to zero as $m \to \infty$. Yet one can dominate each term by

$$C_l \mathbb{E} \left\| \int_0^T R(dt, \cdot) \otimes B_t - \int_0^T R_m(dt, \cdot) \otimes (B_m)_t \right\|^2_{\mathcal{H} \otimes \mathbb{R}^n}$$

for some suitable constant C_l depending only on α_l. This term vanishes in the limit by the same argument as is used in the proof of Proposition 3.3.2. $\square$

4.5 Integrability of the Malliavin Covariance Determinant

Recall that the Malliavin covariance matrix is defined as the operator $DY_T \, (DY_T)^*$. We begin the chapter by defining two matrix-valued operations on our path-space that will be the higher-dimensional analogues of the expressions $Q\omega(T)$ and $\|Q\omega\|^2$ which appeared in the Malliavin covariance matrix for the Heisenberg case.

We will indicate by Ψ the Gram matrix on our operators $\{a \otimes J_l\}$; more precisely, Ψ will be the function from $\mathcal{W}_p \otimes \mathbb{R}^n$ to $M_k(\mathbb{R})$ defined by

$$[\Psi(\omega)]_{ij} = \langle (a \otimes J_i)\omega, (a \otimes J_j)\omega \rangle_{\mathcal{H} \otimes \mathbb{R}^n}.$$

We will let Θ indicate the linear mapping from $\mathcal{W}_p \otimes \mathbb{R}^n$ into the space $M_{n,k}(\mathbb{R})$ of $n \times k$ matrices with real-valued entries given by

$$[\Theta\omega]_{ij} := [(a \otimes J_j)\omega]\,(T) \cdot e_i.$$

Each of these matrices may be extended to matrix-valued random variables $\Psi(B)$ and ΘB in the usual manner.

Proposition 4.5.1. *Suppose $\gamma : \mathcal{W}_p \otimes \mathbb{R}^n \to M^{n+k}(\mathbb{R})$ is given by*

$$\gamma(\omega) := \begin{bmatrix} T^{2H} I_n & \Theta\omega \\ (\Theta\omega)^{tr} & \Psi(\omega) \end{bmatrix}.$$

then $DY_T \left(DY_T\right)^* = \gamma(B)$ almost surely.

Proof. Given $\mathbf{x} \in \mathbb{R}^{n+k}$ and $h \in \mathcal{H} \otimes \mathbb{R}^n$,

$$\begin{aligned} \langle (DY_T)^* \mathbf{x}, h \rangle &= \mathbf{x} \cdot \left(R_T^1 h, \ldots, R_T^n h, \langle (a \otimes J_1) B, h \rangle_{\mathcal{H}\otimes\mathbb{R}^n}, \ldots, \langle (a \otimes J_k) B, h \rangle_{\mathcal{H}\otimes\mathbb{R}^n} \right) \\ &= \left\langle \sum_{i=1}^n x^i R(T, \cdot) \otimes e_i + \sum_{j=1}^k x^{n+j} (a \otimes J_j) B, h \right\rangle_{\mathcal{H}\otimes\mathbb{R}^n}. \end{aligned}$$

Direct calculations reveal the following identities:

$$\begin{gathered} R_T^l (R(T, \cdot) \otimes e_i) = \delta_{il} T^{2H}; \\ \langle (a \otimes J_l) B, R(T, \cdot) \otimes e_i \rangle_{\mathcal{H}\otimes\mathbb{R}^n} = R_T^l \left((a \otimes J_l) B \right) = \left[(a \otimes J_l) B \right]_T \cdot e_i. \end{gathered}$$

By linearity, it follows that

$$\begin{aligned} DY_T (DY_T)^* \mathbf{x} &= \sum_{i=1}^n x^i DY_T \left[R(T, \cdot) \otimes e_i \right] + \sum_{j=1}^k x^{n+j} DY_T \left[(a \otimes J_j) B \right] \\ &= \sum_{i=1}^n x^i \left(T^{2H} e_i, \left[(a \otimes J_1) B \right]_T \cdot e_i, \ldots, \left[(a \otimes J_k) B \right]_T \cdot e_i \right) \\ &\quad + \sum_{j=1}^k x^{n+j} \Big(\left[(a \otimes J_j) B \right]_T \cdot e_1, \ldots, \left[(a \otimes J_j) B \right]_T \cdot e_n, \\ &\qquad \langle (a \otimes J_1) B, (a \otimes J_j) B \rangle, \ldots, \langle (a \otimes J_k) B, (a \otimes J_j) B \rangle \Big). \end{aligned}$$

One may readily verify that this is almost surely equivalent to $[\gamma(B)]\mathbf{x}$. $\square$

Recall that proving smoothness of the density of Y with respect to Lebesgue measure requires showing that

$$\left(\det \gamma(B)\right)^{-1} \in L^{\infty-}. \tag{4.5.1}$$

As an application of Lemma 3.4.3, we have that

$$\det \gamma(B) = T^{2Hn} \det \left(\Psi(B) - T^{-2H} (\Theta B)^{tr} \Theta B \right).$$

For each $\mathbf{y} \in S^{k-1}$, define the linear operator $\Phi_{\mathbf{y}}$ by

$$\Phi_{\mathbf{y}}(\omega) := \left(\Psi(\omega) - T^{-2H} (\Theta \omega)^{tr} \Theta \omega \right) \mathbf{y} \cdot \mathbf{y} \tag{4.5.2}$$

This operator is continuous both in ω and $\mathbf{y}$. Once again, we may almost surely identify this operator with a random variable $\Phi_{\mathbf{y}}(B)$. We note that

$$(\det \gamma(B))^{-1} \leq T^{-2Hn} \left(\min_{\mathbf{y} \in S^{k-1}} \Phi_{\mathbf{y}}(B) \right)^{-(n+k)};$$

hence our desired integrability condition will be implied by showing that

$$\left(\min_{\mathbf{y} \in S^{k-1}} \Phi_{\mathbf{y}}(B) \right)^{-1} \in L^{\infty-}.$$

For each $\mathbf{y} \in S^{k-1}$ we will let $\mathbf{y} \cdot \alpha := \sum_{i=1}^{k} y_i \alpha_i$. It is easy to check that

$$(\Psi(B))\mathbf{y} \cdot \mathbf{y} = \|(a \otimes J_{\mathbf{y}\cdot\alpha})B\|^2_{\mathcal{H} \otimes \mathbb{R}^n}, \quad \left((\Theta B)^{tr} \Theta B \right) \mathbf{y} \cdot \mathbf{y} = |([a \otimes J_{\mathbf{y}\cdot\alpha})B](T)|^2.$$

Thus, each $\Phi_{\mathbf{y}}$ is non-negative as a result of the Cauchy-Schwarz inequality. In addition, $\Phi_{\mathbf{y}}$ is a bounded operator as by Lemma 4.3.2; Theorem 3.4.2 implies that the operator $\hat{\Phi}_{\mathbf{y}} : \mathcal{H}^n \to \mathcal{H}^n$ defined by

$$\left\langle \hat{\Phi}_{\mathbf{y}} h, h \right\rangle_{\mathcal{H}^n} = \Phi_{\mathbf{y}} h$$

is trace-class.

Proposition 4.5.2. *For each fixed $\mathbf{y} \in S^{k-1}$, the map $\Phi_{\mathbf{y}}$ is contained in $L^{\infty-}(\mathbb{P})$.*

Proof. Again recalling Theorem 3.4.2, it suffices to show that $\hat{\Phi}$ is not a finite rank operator. To that aim, we begin by noting that $\hat{\Phi}_{\mathbf{y}} h = 0$ implies that $\Phi_{\mathbf{y}}(h) = 0$. By Cauchy-Schwarz, this is true if and only if

$$(a \otimes J_{\mathbf{y}\cdot\alpha})\, h = \mathbf{c} R(T, \cdot)$$

for some $\mathbf{c} \in \mathbb{R}^n$. By definition, this is equivalent to the statement that

$$\int_0^T R(dt, \cdot) \otimes (J_{\mathbf{y}\cdot\alpha} h(T) - J_{\mathbf{y}\cdot\alpha} h(t) - \mathbf{c}) = 0.$$

By taking the inner product of each side against an arbitrary $\varphi \in \mathcal{C}_c^\infty \otimes \mathbb{R}^n \subset \mathcal{H} \otimes \mathbb{R}^n$ and applying integration by parts, we obtain the identity

$$0 = \int_0^T \varphi(t) \otimes dJ_{\mathbf{y}\cdot\alpha} h(t) \qquad (\forall \varphi \in \mathcal{C}_c^\infty \otimes \mathbb{R}^n)$$

which implies $J_{\mathbf{y}\cdot\alpha} h \equiv 0$; that is, $h(t) \in \mathrm{Null}\,(J_{\mathbf{y}\cdot\alpha})$ for all $t \in [0, T]$. By the assumption on our skew-symmetric operators, $\mathbf{y} \cdot \alpha$ is not the zero map. Thus, we may pick some $\mathbf{v} \neq 0$ for which $J_{\mathbf{y}\cdot\alpha}\mathbf{v}$ is non-zero; it follows that for each fixed non-zero $k \in \mathcal{H}$, the element $k \otimes J_{\mathbf{y}\cdot\alpha}\mathbf{v}(t) \neq 0$ for all $t \in (0, T]$ such that $k(t) \neq 0$. Thus the set

$$\{k \otimes \mathbf{v} : 0 \neq k \in \mathcal{H}\}$$

is contained in the complement of the kernel of $\hat{\Phi}$; it is clear that the cardinality of this set is infinite. □

Proposition 4.5.3. *For all $1 \leq p < \infty$, the expectations $\mathbb{E}[\Phi_{\mathbf{y}}^{-p}]$ are uniformly bounded in $\mathbf{y}$; i.e.,*

$$\sup_{\mathbf{y} \in S^{k-1}} \mathbb{E}[\Phi_{\mathbf{y}}^{-p}] < \infty.$$

Proof. We begin by noting that the operator $\hat{\Phi}_{\mathbf{y}}$ may be written as

$$\begin{aligned}\hat{\Phi}_{\mathbf{y}} &= (a \otimes J_{\mathbf{y}\cdot\alpha})^*(a \otimes J_{\mathbf{y}\cdot\alpha}) - T^{-2H}(a \otimes J_{\mathbf{y}\cdot\alpha})^* R_T^* R_T (a \otimes J_{\mathbf{y}\cdot\alpha}) \\ &= (a^*a - T^{-2H} a^* R_T^* R_T a) \otimes J_{\mathbf{y}\cdot\alpha}^* J_{\mathbf{y}\cdot\alpha},\end{aligned}$$

where R_T denotes the evaluation operator at time T. We note that the quadratic form A on $\mathcal{W}_p$ given by

$$A(\omega, \tau) = \langle a\omega, a\tau \rangle_{\mathcal{H}} - (a\omega(T) \cdot a\tau(T))$$

is non-negative and bounded by Lemma 4.3.1, and hence Theorem 3.4.2 implies that $(a^*a - T^{-2H} a^* R_T^* R_T a)$ is trace-class and a fortiori compact. Note that for each $\mathbf{y}$, the (non-negative) eigenvalues $\{\lambda_n^{\mathbf{y}}\}$ of the operator $\hat{\Phi}_{\mathbf{y}}$ are given by the products of eigenvalues of $(a^*a - T^{-2H}a^*R_T^*R_Ta)$ and $J_{\mathbf{y}\cdot\alpha}^*J_{\mathbf{y}\cdot\alpha}$. Recall from the proof of Theorem 3.4.2 that one has the equation

$$\Phi_{\mathbf{y}}(B) = L^1 - \lim_{N\to\infty}\sum_{n=1}^{N}\lambda_n^{\mathbf{y}}\xi_n^2$$

where $\{\xi_n\}$ are a set of independent standard normal random variables.

Let $\{\sigma_n\}$ denote the eigenvalues of the operator $(a^*a - T^{-2H}a^*R_T^*R_Ta)$ and $\rho_{\mathbf{y}}$ the spectral radius of $J_{\mathbf{y}\cdot\alpha}^*J_{\mathbf{y}\cdot\alpha}$. Define the set $\mathcal{E}_{\mathbf{y}}$ as the collection of non-zero eigenvalues of $\hat{\Phi}_{\mathbf{y}}$ of the form $\rho_{\mathbf{y}}\sigma_n$. Since a has a trivial kernel, and $J_{\mathbf{y}\cdot\alpha}$ is not the zero map, we have that $\#(\mathcal{E}_{\mathbf{y}}) = \infty$. Without loss of generality, we may order our eigenvalues such that members of $\mathcal{E}_{\mathbf{y}}$ are listed "first"; i.e., $\lambda_n^{\mathbf{y}} \in \mathcal{E}_{\mathbf{y}}$ for any $n \in \mathbb{N}$.

For each $1 \le p < \infty$, let N be the first integer for which $N > 2p$. Then

$$\begin{aligned}
\mathbb{E}[(\Phi_{\mathbf{y}}(B))^{-p}] &\le \mathbb{E}\left[\left(\sum_{n=1}^{N}\lambda_n^{\mathbf{y}}\xi_n^2\right)^{-p}\right] \\
&\le \left(\min_{n=1,\dots,N}\lambda_n^{\mathbf{y}}\right)^{-p}\mathbb{E}\left[\left(\sum_{n=1}^{N}\xi_n^2\right)^{-p}\right] \\
&= \left(\min_{n=1,\dots,N}\lambda_n^{\mathbf{y}}\right)^{-p}\int_0^\infty r^{-2p}r^{N-1}e^{\frac{-r^2}{2}}\,dr
\end{aligned}$$

which is certainly a finite expression. In particular, a uniform bound on $\mathbb{E}[(\Phi_{\mathbf{y}}(B))^{-p}]$ will be proven if we can find a constant M for which

$$\left(\min_{n=1,\dots,N}\lambda_n^{\mathbf{y}}\right)^{-p} \le M$$

for all $\mathbf{y} \in S^{k-1}$. We note that

$$\min_{n=1,\dots,N}\lambda_n^{\mathbf{y}} = \rho_{\mathbf{y}}\min_{n=1,\dots,N}\sigma_n \ge C\rho_{\mathbf{y}}$$

where the constant C is dependent only on the value of N. Thus it is only left for us to prove that

$$\max_{\mathbf{y}\in S^{k-1}}\rho_{\mathbf{y}}^{-p} \le M;$$

yet this is equivalent to the statement that

$$\min_{\mathbf{y}\in S^{k-1}} \rho_{\mathbf{y}} > 0,$$

which is true by the compactness of the unit sphere and the non-degeneracy condition imposed upon α. □

Lemma 4.5.4 (Lemma 6.6 of [Bel06])**.** *Suppose X is a non-negative random variable such that*

$$\mathbb{P}(X < \varepsilon) = O(\varepsilon^{\infty-}) \qquad (\varepsilon \longrightarrow 0).$$

Then $X^{-1} \in L^{\infty-}(\mathbb{P})$.

Proof. Fix some $p \geq 1$. Pick some $q > p$; then by assumption, there exists some constants $K = K_q, M = M_q$ such that

$$\mathbb{P}\left(X < \varepsilon\right) \leq K\varepsilon^q,$$

provided $\varepsilon < \frac{1}{M}$. Using this, we see that

$$\begin{aligned}
\mathbb{E}[X^{-p}] &= \int_0^\infty \tau^{p-1}\mathbb{P}(X^{-1} > \tau)\, d\tau \\
&= \int_0^\infty \tau^{p-1}\mathbb{P}(X < \tau^{-1})\, d\tau \\
&= \int_0^M \tau^{p-1}\mathbb{P}(X < \tau^{-1})\, d\tau + \int_M^\infty \tau^{p-1}\mathbb{P}(X < \tau^{-1})\, d\tau \\
&\leq \int_0^M \tau^{p-1}\, d\tau + K\int_M^\infty \tau^{p-1}\tau^{-q}\, d\tau \\
&\leq \frac{M^p}{p} + \frac{M^{p-q}}{q-p} < \infty.
\end{aligned}$$

□

We are now in a position to prove our desired result.

Theorem 4.5.5. *Let $\Phi_{\mathbf{y}}$ be defined as in Equation* (4.5.2)*. Then*

$$\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B)\right)^{-1} \in L^{\infty-}(\mathcal{W}_p^n, \mathbb{P}).$$

Proof. By Lemma 4.5.4, it suffices to check that for all q,

$$\mathbb{P}\left\{\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B)\right) < \varepsilon\right\} \leq C_q\varepsilon^q$$

for some suitable constant C_q dependent only on q. Fix $\varepsilon > 0$. We pick a natural number $N(\varepsilon)$ and vectors $\{\mathbf{y}_i\}_{i=1}^{N(\varepsilon)}$ such that

$$\bigcup_{i=1}^{N(\varepsilon)} B(\mathbf{y}_i;\varepsilon^2)$$

form an open cover of S^{k-1}. Note that the value of $N(\varepsilon)$ is bounded above by $2^k\varepsilon^{-2k}$; one may see this by slicing the cube $[-1,1]^k$ into disjoint cubes of size length ε^2.

Define the following sets:

$$\mathcal{A}_i := \left\{\inf_{\mathbf{z}\in B(\mathbf{y}_i;\varepsilon^2)} \Phi_{\mathbf{z}}B < \varepsilon : \|B\|_p^2 \leq \frac{1}{\varepsilon}\right\};$$
$$\mathcal{B}_i := \left\{\inf_{\mathbf{z}\in B(\mathbf{y}_i;\varepsilon^2)} \Phi_{\mathbf{z}}B < \varepsilon : \|B\|_p^2 > \frac{1}{\varepsilon}\right\}.$$

Then one has that

$$\mathbb{P}\left\{\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B)\right) < \varepsilon\right\} \leq \sum_{i=1}^{N}\Big(\mathbb{P}(\mathcal{A}_i) + \mathbb{P}(\mathcal{B}_i)\Big).$$

Suppose $\mathbf{z} \in B(\mathbf{y}_i;\varepsilon^2)$. Then on $\mathcal{A}_i$, one has the inequality

$$|\Phi_{\mathbf{y}_i}(B)| \leq |\Phi_{\mathbf{y}_i}(B) - \Phi_{\mathbf{z}}(B)| + |\Phi_{\mathbf{z}}(B)| < C\|B\|_p^2\varepsilon^2 + \varepsilon = (1+C)\varepsilon$$

for a suitable constant C. Therefore $\mathcal{A}_i \subset \{\Phi_{\mathbf{y}_i}(B) < (1+C)\varepsilon\}$. Letting $M_q := \sup_{\mathbf{y}\in S^{k-1}} \mathbb{E}[(\Phi_{\mathbf{y}}(B))^{-q}]$ (a finite quantity by Proposition 4.5.3) and using Markov's inequality, we obtain the bound

$$\begin{aligned}\mathbb{P}(\mathcal{A}_i) &\leq \mathbb{P}\left\{\Phi_{\mathbf{y}_i}(B) < 2\varepsilon\right\} = \mathbb{P}\left\{(\Phi_{\mathbf{y}_i}(B))^{-q} > (2\varepsilon)^{-q}\right\} \\ &\leq ((1+C)\varepsilon)^q\mathbb{E}[(\Phi_{\mathbf{y}_i}(B))^{-q}] \leq (2^qM_q)\varepsilon^q.\end{aligned}$$

Another application of Markov's inequality gives us that

$$\begin{aligned}\mathbb{P}(\mathcal{B}_i) &\leq \mathbb{P}\left\{\|B\|_p^2 > \frac{1}{\varepsilon}\right\} = \mathbb{P}\left\{\|B\|_p^{2q} > \frac{1}{\varepsilon^q}\right\} \\ &\leq \varepsilon^q \mathbb{E}[\|B\|_p^{2q}],\end{aligned}$$

which is finite as a consequence of Fernique's Theorem.

We note that each inequality is independent of i, and so, for suitable constant K_q, we obtain the bound

$$\begin{aligned}\mathbb{P}\left\{\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B)\right) < \varepsilon\right\} &\leq N(\varepsilon)((1+C)^q M_q + \mathbb{E}[\|B\|_p^{2q}])\varepsilon^q \\ &\leq K_q \varepsilon^{q-2k}.\end{aligned}$$

As such a bound holds for all $q \geq 1$, we may conclude that

$$\mathbb{P}\left\{\left(\min_{\mathbf{y}\in S^{k-1}} \Phi_{\mathbf{y}}(B)\right) < \varepsilon\right\} = O(\varepsilon^{\infty-}),$$

as desired. $\square$

Appendix A

r-Variation of the Covariance Function

Fix $0 < H < 1/2$ and $T > 0$. We will, as usual, denote by $R : [0,T]^2 \longrightarrow \mathbb{R}$ the covariance function for fractional Brownian motion with Hurst parameter H on $[0,T]$; that is,

$$\mathbb{E}[B_s^H B_t^H] = R(s,t) := \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right].$$

We may associate a **finitely addivite signed measure** μ_R on the algebra generated by rectangles of the form
$\{(a,b] \times (c,d] \subset (0,T]^2\}$ with R in the following way:

$$\begin{aligned}\mu_R((a,b] \times (c,d]) &= R(b,d) - R(a,d) - R(b,c) + R(a,c) \\ &= \frac{1}{2}\left(|d-a|^{2H} + |c-b|^{2H} - |d-b|^{2H} - |c-a|^{2H}\right).\end{aligned}$$

It is easy to check the covariance of the process increments $B_b^H - B_a^H$ and $B_d^H - B_c^H$ is given by the μ_R-measure of the rectangle $(a,b] \times (c,d]$, or

$$\mathbb{E}[(B_d^H - B_c^H)(B_b^H - B_a^H)] = \mu_R((a,b] \times (c,d]).$$

So, by Lemma 2.1.2, one has that $\mu_R\left((a,b] \times (c,d]\right) < 0$ for all $0 \le a < b < c < d \le T$. Throughout the sequel, we record the following lemma, which we will use repeatedly with the coefficient $\alpha = 2H$.

Lemma A.0.6. *Given some $0 < \alpha < 1$, one has that for all $x, y \geq 0$,*

$$\begin{aligned}(x+y)^\alpha - x^\alpha &\leq y^\alpha;\\ x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}} &\leq (x+y)^{\frac{1}{\alpha}}.\end{aligned}$$

Proof. These inequalities are a result of the subadditivity of $x \mapsto x^\alpha$ and the superadditivity of $x \mapsto x^{\frac{1}{\alpha}}$; in the first case, one has that

$$\frac{1}{2}(x+y)^\alpha < \frac{1}{2^\alpha}(x+y)^\alpha = \left(\frac{x+y}{2}\right)^\alpha \leq \frac{1}{2}x^\alpha + \frac{1}{2}y^\alpha.$$

The argument for the second inequality is similar. □

Lemma A.0.7. *Let μ_R be the measure defined above. Then for any intervals $(a,b], (c,d] \subset [0,T]$, one has the bound*

$$|\mu_R((a,b] \times (c,d])| \leq (b-a)^{2H} \wedge (d-c)^{2H}.$$

Proof. We will need to consider three possible cases:

(1) One interval is nested within the other,

(2) the intervals partially overlap, or

(3) the intervals are disjoint.

Case 1. *$a \leq c < d \leq b$.*

In this scenario, the claimed upper bound is clearly $(d-c)^{2H}$. Using this, we have that

$$\begin{aligned}|\mu_R((a,b] \times (c,d])| &= \frac{1}{2}\left|(d-a)^{2H} + (b-c)^{2H} - (b-d)^{2H} - (c-a)^{2H}\right|\\ &\leq \frac{1}{2}\left((d-a)^{2H} - (c-a)^{2H}\right)\\ &\qquad\qquad + \frac{1}{2}\left((b-c)^{2H} - (b-d)^{2H}\right)\\ &\leq (d-c)^{2H}.\end{aligned}$$

Case 2. *$a < c \leq b < d$.*

In this case, we know that

$$
\begin{aligned}
|\mu_R((a,b] \times (c,d])| &= \frac{1}{2}\left|(d-a)^{2H} + (b-c)^{2H} - (d-b)^{2H} - (c-a)^{2H}\right| \\
&\leq \frac{1}{2}\left((d-a)^{2H} - (d-b)^{2H}\right) \\
&\qquad\qquad + \frac{1}{2}\left|(b-c)^{2H} - (c-a)^{2H}\right| \\
&\leq \frac{1}{2}(b-a)^{2H} + \frac{1}{2}\left((b-c)^{2H} \vee (c-a)^{2H}\right) \\
&\leq (b-a)^{2H}.
\end{aligned}
$$

In a similar manner,

$$
\begin{aligned}
|\mu_R((a,b] \times (c,d])| &= \frac{1}{2}\left|(d-a)^{2H} + (b-c)^{2H} - (d-b)^{2H} - (c-a)^{2H}\right| \\
&\leq \frac{1}{2}\left((d-a)^{2H} - (c-a)^{2H}\right) \\
&\qquad\qquad + \frac{1}{2}\left|(b-c)^{2H} - (d-b)^{2H}\right| \\
&\leq \frac{1}{2}(d-c)^{2H} + \frac{1}{2}\left((b-c)^{2H} \vee (d-b)^{2H}\right) \\
&\leq (d-c)^{2H}.
\end{aligned}
$$

Case 3. $a < b \leq c < d$.

Here, we will use the concavity inequality twice to generate the desired bound. Firstly, we calculate that

$$
\begin{aligned}
|\mu_R((a,b] \times (c,d])| &= \frac{1}{2}\left|(d-a)^{2H} + (c-b)^{2H} - (d-b)^{2H} - (c-a)^{2H}\right| \\
&\leq \frac{1}{2}\left((d-a)^{2H} - (c-a)^{2H}\right) \\
&\qquad\qquad + \frac{1}{2}\left((d-b)^{2H} - (c-b)^{2H}\right) \\
&\leq (d-c)^{2H}.
\end{aligned}
$$

In much the same manner, we find that

$$
\begin{aligned}
|\mu_R((a,b] \times (c,d])| &= \frac{1}{2}\left|(d-a)^{2H} + (c-b)^{2H} - (d-b)^{2H} - (c-a)^{2H}\right| \\
&\leq \frac{1}{2}\left((d-a)^{2H} - (d-b)^{2H}\right) \\
&\qquad\qquad + \frac{1}{2}\left((c-a)^{2H} - (c-b)^{2H}\right) \\
&\leq (b-a)^{2H}.
\end{aligned}
$$

□

Proposition A.0.8. *Let* $r := \frac{1}{2H} > 1$. *Then the function* R *has finite two-dimensional* r*-variation over* $[0, T]^2$*; more specifically,*

$$\|R\|_r^{(2D)} \leq (5T)^{2H}.$$

Proof. Let

$$\Pi := \{s_0 := 0 < s_1 < \ldots < s_M := T\},$$
$$\Psi := \{t_0 := 0 < t_1 < \ldots < t_N := T\}$$

be two partitions of $[0, T]$. Fix a $j \in \{1, \ldots, T\}$. We will let A be the unique integer such that $s_{A-1} \leq t_{j-1} < s_A$, and $L \geq A$ will denote the unique integer for which $s_{L-1} < t_j \leq s_L$.

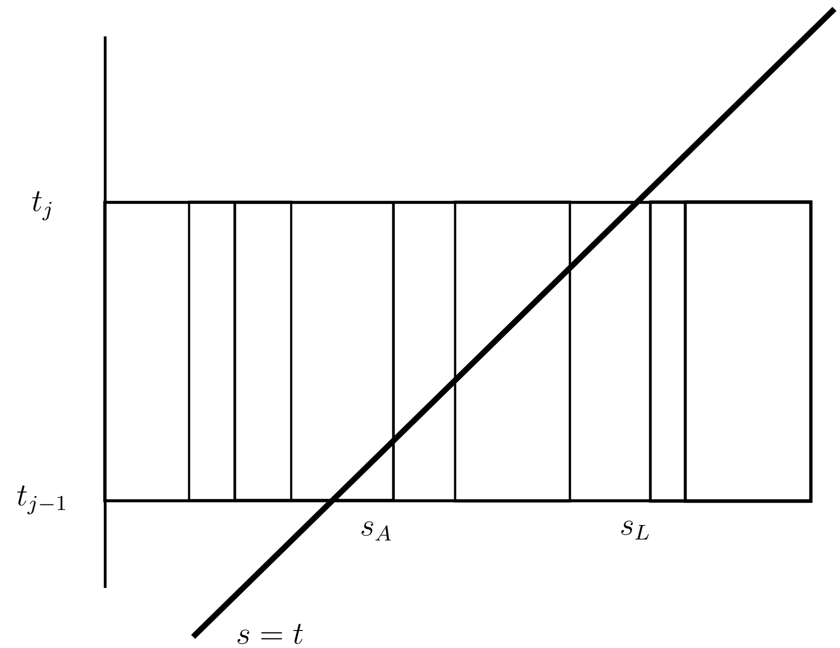

Figure A.1: An example partition in s for a fixed strip $t_{j-1} < t \leq t_j$.

As usual, we define

$$\Delta_{ij} R := \mu_R \left((s_{i-1}, s_i] \times (t_{j-1}, t_j] \right).$$

Then

$$\begin{aligned} \sum_{i=1}^{M} |\Delta_{ij} R|^r \leq \sum_{i=1}^{A-1} |\Delta_{ij} R|^r + |\Delta_{Aj} R|^r \\ + \sum_{i=A+1}^{L-1} |\Delta_{ij} R|^r + |\Delta_{Lj} R|^r + \sum_{i=L+1}^{M} |\Delta_{ij} R|^r . \end{aligned} \tag{A.0.1}$$

It follows from Lemma A.0.7 that

$$|\Delta_{Aj} R|^r \leq (t_j - t_{j-1}), \qquad |\Delta_{Lj} R|^r \leq (t_j - t_{j-1}). \tag{A.0.2}$$

Lemma A.0.7 also implies that $|\Delta_{ij}R|^r \leq (s_i - s_{i-1})$; hence, we may use telescoping to bound the third term:

$$\sum_{i=A+1}^{L-1} |\Delta_{ij}R|^r \leq \sum_{i=A+1}^{L-1} (s_i - s_{i-1}) = (s_{L-1} - s_A) \leq (t_j - t_{j-1}). \tag{A.0.3}$$

Let us now focus on the first and last terms of Equation (A.0.1). Note that on each of these sums, Lemma 2.1.2 implies that $\Delta_{ij}R < 0$. We may use this fact along with Lemma A.0.7 to see that

$$\begin{aligned}
\sum_{i=1}^{A-1} |\Delta_{ij}R|^r &+ \sum_{i=L+1}^{M} |\Delta_{ij}R|^r \\
&\leq \left(\sum_{i=1}^{A-1} |\Delta_{ij}R|\right)^r + \left(\sum_{i=L+1}^{M} |\Delta_{ij}R|\right)^r \\
&= \left|\sum_{i=1}^{A-1} \Delta_{ij}R\right|^r + \left|\sum_{i=L+1}^{M} \Delta_{ij}R\right|^r \\
&= |\mu_R((0, s_{A-1}] \times (t_{j-1}, t_j])|^r \\
&\qquad + |\mu_R((s_L, T] \times (t_{j-1}, t_j])|^r \\
&\leq 2(t_j - t_{j-1}).
\end{aligned} \tag{A.0.4}$$

Combining Equations (A.0.1)–(A.0.4) allows us to conclude that

$$\sum_{i=1}^{M} |\Delta_{ij}R|^r \leq 5(t_j - t_{j-1}).$$

Hence,

$$\sum_{j=1}^{N} \sum_{i=1}^{M} |\Delta_{ij}R|^r \leq \sum_{j=1}^{N} 5(t_j - t_{j-1}) = 5T.$$

This completes the proof, since the two-dimensional r-variation of R is given as

$$\|R\|_r^{(2D)} = \left(\sup_{\Pi, \Psi \in \mathcal{P}[0,T]} \sum_{\Pi} \sum_{\Psi} |\Delta_{ij}R|^r\right)^{\frac{1}{r}} \leq (5T)^{2H}.$$

$\square$

Appendix B

Restriction of Gaussian Measures

At first blush, it may seem natural to have our process B have the classical Wiener space $\mathcal{W}^2 := \mathcal{C}([0,T],\mathbb{R}^2)$ as its sample space. However, doing so is not ideal, since many of the operators we will be considering are only defined on smaller spaces, such as the p-variation spaces.

We begin with a general result regarding σ-algebras.

Lemma B.0.9. *Let X be any real separable Banach space and $\mathcal{L}$ be any Then $\|\cdot\|_X$ is $\sigma(\mathcal{L})$ – measurable if and only if $\mathcal{B}_X = \sigma(\mathcal{L})$.*

Proof. It it easy to see that, in any case, $\sigma(\mathcal{L}) \subset \mathcal{B}_X$. Also, since $\|\cdot\|_X$ is continuous it is always Borel measurable; therefore, if $\mathcal{B}_X = \sigma(\mathcal{L})$ then $\|\cdot\|_X$ is clearly $\sigma(\mathcal{L})$ – measurable.

Suppose that $\|\cdot\|_X$ is $\sigma(\mathcal{L})$ – measurable; then for each $x_0 \in \sigma(\mathcal{L})$, $\|\cdot - x_0\|_X$ is also $\sigma(\mathcal{L})$ – measurable, and $x \to x - x_0$ is $\sigma(\mathcal{L})/\sigma(\mathcal{L})$ – measurable. From this observation, it follows that $\sigma(\mathcal{L})$ contains all balls in X. Since X is separable, every open subset of X may be written as a countable union of open balls. It follows, then, that $\sigma(\mathcal{L})$ contains all open subsets of X and therefore that $\mathcal{B}_X \subset \sigma(\mathcal{L})$. $\square$

Theorem B.0.10. *Suppose $(X, \mathcal{B} = \mathcal{B}_X, \mu)$ is a Gaussian probability space, and $\tilde{X}$ is a linear subspace of X. Also let $\|\cdot\|_{\tilde{X}}$ is a norm on $\tilde{X}$ such that*

1. *The space $(\tilde{X}, \|\cdot\|_{\tilde{X}})$ is a separable Banach space,*

2. *The embedding of $\tilde{X}$ into X is continuous,*

3. $\tilde{X} \in \mathcal{B}$ *and* $\mu(\tilde{X}) = 1$,

4. $\tilde{\mathcal{B}} := \mathcal{B}_{\tilde{X}} = \{A \cap \tilde{X} : A \in \mathcal{B}\}$.

Then $\tilde{\mu} := \mu|_{\tilde{X}}$ *is a Gaussian measure and* $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ *is a Gaussian probability space. Furthermore,* (X, μ) *and* $(\widetilde{X}, \tilde{\mu})$ *share the same Cameron-Martin space* $\mathcal{H}$.

Proof. Let $R_{\pi/4} : X \times X \to X \times X$ is the rotation map defined by

$$R_{\pi/4}(x, y) = \left(\frac{\sqrt{2}}{2}(x - y), \frac{\sqrt{2}}{2}(x + y)\right);$$

then by the rotational invariance of Gaussian measures (see, for example, Theorem 3.1.1 of [Bry95]), proving the statement that $\tilde{\mu}$ is Gaussian is equivalent to proving that

$$\int_{\tilde{X} \times \tilde{X}} f(x, y)\, d\tilde{\mu}(x) d\tilde{\mu}(y) = \int_{\widetilde{X} \times \widetilde{X}} f \circ R_{\pi/4}(x, y)\, d\tilde{\mu}(x) d\tilde{\mu}(y)$$

for any bounded $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$-measurable function f. Let f be such a function; since $\widetilde{X}$ is of full μ measure, we may extend f to an $\mathcal{B} \times \mathcal{B}$-measurable function (which we shall also refer to as f) such that $\int_{\widetilde{X} \times \widetilde{X}} f d\mu d\mu = \int_{X \times X} f d\mu d\mu$ (this extension may be done by setting a function equal to f on $\widetilde{X} \times \widetilde{X}$ and equal to zero on the complement, for example). Then it follows that

$$\begin{aligned}
\int_{\tilde{X} \times \tilde{X}} f(x, y)\, d\tilde{\mu}(x) d\tilde{\mu}(y) &= \int_{\tilde{X} \times \tilde{X}} f(x, y)\, d\mu(x) d\mu(y) \\
&= \int_{X \times X} f(x, y)\, d\mu(x) d\mu(y) \\
&= \int_{X \times X} f \circ R_{\pi/4}(x, y)\, d\mu(x) d\mu(y) \\
&= \int_{\widetilde{X} \times \widetilde{X}} f \circ R_{\pi/4}(x, y)\, d\mu(x) d\mu(y) \\
&= \int_{\widetilde{X} \times \widetilde{X}} f \circ R_{\pi/4}(x, y)\, d\tilde{\mu}(x) d\tilde{\mu}(y).
\end{aligned}$$

This proves the first assertion.

To see the equivalence of Cameron-Martin spaces, we recall that

$J : L^2(X, \mu) \to X$, defined by

$$\eta f := \int_X x f(x)\, d\mu(x),$$

maps onto $\mathcal{H}$. Again, by virtue of μ being fully supported on $\widetilde{X}$, we may extend any element of $L^2(\widetilde{X}, \tilde{\mu})$ to an element of $L^2(X, \mu)$; thus it is easy to see that $\eta(L^2(\widetilde{X}, \tilde{\mu})) = \eta(L^2(X, \mu)) = \mathcal{H}$, as desired. □

Remark B.0.11. *An alternate proof of the equivalence of Cameron-Martin spaces may be found in Proposition 2.8 of [DPZ92].*

Let us now focus on restricting the law of fractional Brownian motion with Hurst parameter $1/3 < H < 1/2$ to a variational space. The standard Gaussian space on which fBm is realized is $(\mathcal{W}, \mathcal{B}, \mathbb{P})$, where $\mathcal{W} = \{\omega \in \mathcal{C}([0, T], \mathbb{R}) : \omega(0) = 0\}$ and $\mathbb{P} = \text{Law}(B^H)$. Pick $0 < \epsilon << 1$ and fix $p := 1/H + \epsilon$. Let $\phi_t, 0 \leq t \leq T$ denote the evaluation map on $\mathcal{W}$; i.e., $\phi_t(x) = x(t)$ for any $x \in \mathcal{W}$. Since

$$\| \cdot \|_{\mathcal{W}} = \sup_{0 \leq t \leq T} \phi_t,$$

it follows that $\| \cdot \|_{\mathcal{W}}$ is a $\sigma(\{\phi_t : 0 \leq t \leq T\})$-measurable function, and by Lemma B.0.9, it then follows that $\sigma(\{\phi_t : 0 \leq t \leq T\}) = \mathcal{B}_{\mathcal{W}}$. Recall that we have defined the p-variation norm on $\mathcal{W}$ by

$$\|x\|_p = \sup_{\Pi \in \mathcal{P}[0,T]} \left(\sum_{i=1}^{(\#\Pi)} |\Delta_i x|^p \right)^{\frac{1}{p}}.$$

Recall that we have defined the space

$$\mathcal{W}_p = \overline{\{x \in \mathcal{C}_\infty([0, T], \mathbb{R}) : x(0) = 0\}}^{\|\cdot\|_p}.$$

By Corollary 5.35 and Proposition 5.36 of [FV10], this space is a separable Banach space under the p-variation norm and contains all q-variation paths starting at zero for any $1 \leq q < p$. Note that for $x \in \mathcal{W}_p$, Hölder's inequality implies that for any $t \in [0, T]$,

$$\begin{aligned}
|x(t)| &= |x(t) - x(0)| \\
&\leq |x(t) - x(0)| + |x(T) - x(0)| \\
&\leq 2^{\frac{p-1}{p}} \left(|x(t) - x(0)|^p + |x(T) - x(0)|^p\right)^{\frac{1}{p}} \\
&\leq 2^{\frac{p-1}{p}} \|x\|_p,
\end{aligned}$$

from which it follows that $\|x\|_{\mathcal{W}} \leq \|x\|_p$, and so the embedding of $\mathcal{W}_0^p$ into $\mathcal{W}$ is continuous. Observe that we may rewrite the p-variation norm as

$$\| \cdot \|_p = \sup_{\Pi \in \mathcal{P}[0,T]} \left(\sum_{i=1}^{(\#\Pi)} |\phi_{t_i} - \phi_{t_{i-1}}|^p \right)^{\frac{1}{p}}.$$

Thus, $\| \cdot \|_p$ is $\sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \leq t \leq T\})$-measurable, which implies that $\sigma(\mathcal{L}) = \mathcal{B}_{\mathcal{W}_p}$. Furthermore, by Theorem 5.31 of [FV10], we know that the space $\mathcal{W}_p$ is equivalent to

$$\left\{ x \in \mathcal{C}_p : \lim_{\delta \to 0} \sup_{\Pi \in \mathcal{P}[0,T] : |\Pi| < \delta} \sum_{i=1}^{\#(\Pi)} |x(t_i) - x(t_{i-1})|^p = 0 \right\}$$

If we now define

$$\alpha_p(x) := \lim_{n \to \infty} \sup_{\Pi \in \mathcal{P}[0,T] \cap \mathbb{N} : |\Pi| < \frac{1}{n}} \sum_{i=1}^{\#(\Pi)} |x(t_i) - x(t_{i-1})|^p,$$

then it follows that α_p is a $\sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \leq t \leq T\})$-measurable function, and that

$$\mathcal{W}_p = \mathcal{W}_p \cap \{\alpha_p = 0\} \in \mathcal{B}_{\mathcal{W}}.$$

Additionally, we may now use Lemma B.0.9 to conclude that

$$\begin{aligned}
\mathcal{B}_{\mathcal{W}_p} &= \sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \leq t \leq T\}) \\
&= \{A \cap \mathcal{W}_p : A \in \sigma(\{\phi_t|_{\mathcal{W}} : 0 \leq t \leq T\})\} \\
&= \{A \cap \mathcal{W}_p : A \in \mathcal{B}_{\mathcal{W}}\}.
\end{aligned}$$

Finally, we note that since the paths $t \mapsto B_t^H$ are a.s. Hölder continuous of order $\beta := H\left(1+\frac{\epsilon H}{2}\right)^{-1} < H$, each such path has finite q-variation for $q = \frac{1}{\beta} = \frac{1}{H} + \frac{\epsilon}{2}$. So by Corollary 5.35 of [FV06], $\mathbb{P}(\mathcal{W}_p) \geq \mathbb{P}(\mathcal{W}_q) = 1$. Thus, we may appeal to Theorem B.0.10 to conclude that $(\mathcal{W}_p, \mathcal{B}_{\mathcal{W}_p}, \mathbb{P}|_{\mathcal{W}_p})$ is also a Gaussian probability space, and that the associated Cameron-Martin space $\mathcal{H}$ coincides with the usual Cameron-Martin space corresponding to $\mathbb{P}$ on $\mathcal{W}$.

Appendix C

Trace of $\hat{\Phi}$

This section will be devoted to finding a quantitative bound for the trace of the operator

$$\hat{\Phi} : \mathcal{H}^2 \to \mathcal{H}^2$$

given by the formula $\left\langle \hat{\Phi} h, k \right\rangle_{\mathcal{H}^2} = \Phi(h,k)$, where Φ is the Malliavin covariant determinant as presented in Proposition 3.4.4.

Lemma C.0.12. *Suppose $\{k_n\}$ is an orthonormal set of basis vectors for the Cameron-Martin space $\mathcal{H}$ associated to fractional Brownian motion with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$. Then one has the following identities:*

$$\sum_{n=1}^{\infty} \int_0^T k_n(s) k_n(t) R(dt,u) = \int_0^T R(s,t)\ R(dt,u);$$
$$\sum_{n=1}^{\infty} \int_{[0,T]^2} k_n(s) k_n(t)\ dR(s,t) = \int_{[0,T]^2} R(s,t)\ dR(s,t);$$

the expressions on each side are intepreted as one- and two-dimensional Young's integrals, respectively.

Proof. We start by defining Q_N as the projection off of the first N basis vectors of $\mathcal{H}$; i.e.,

$$Q_N h := \sum_{n=N+1}^{\infty} \langle k_n, h \rangle\, k_n.$$

The first identity is relatively straightforward to prove. Standard one-dimensional Young's integral bounds along with the embedding of $\mathcal{H}$ into $\mathcal{C}_r$ for

$r := \frac{1}{2H}$ gives us the following estimate (where the value of the constant C may change from line to line as necessary):

$$\begin{aligned} \left| \int_0^T \left(R(s,t) - \sum_{n=1}^N k_n(s)k_n(t) \right) R(dt,u) \right| &\le C\|R(u,\cdot)\|_r \\ &\qquad \times \|(R(s,\cdot) - \sum_{n=1}^N k_n(s)k_n(\cdot)\|_r \\ &\le C\|R(u,\cdot)\|_r \|Q_N R(s,\cdot)\|_{\mathcal{H}}, \end{aligned}$$

with the inequality resulting from Proposition 2.4.7. Since $\|Q_N R(s,\cdot)\|_{\mathcal{H}}$ tends to zero as $N \to \infty$, we have obtained the desired result.

As for the second identity, we begin by noting that the mapping from $[0,T]$ $s \mapsto R(s,\cdot) \in \mathcal{H}$ is continuous; thus the set $\{R(s,\cdot) : 0 \le s \le T\}$ is compact in $\mathcal{H}$. For each s, $Q_N R(s,\cdot)$ converges to zero as N tends to infinity. Fix $\delta > 0$, and pick $\{s_i\}_{i=1}^M$ such that the the set of open balls in $\mathcal{H}$ of radius $\frac{\delta}{2}$ is an open covering for $\{R(s,\cdot) : 0 \le s \le T\}$. Let N be the first natural number for which $\max_{i=1,\ldots,M} \|Q_N R(s_i,\cdot)\| < \frac{\delta}{2}$ and for a given $s \in [0,T]$, denote by $I \in \{1,\ldots,M\}$ the number for which $R(s,\cdot) \in B(R(s_I,\cdot);\frac{\delta}{2})$. Then one has that

$$\|Q_N R(s,\cdot)\| \le \|Q_N R(s_I,\cdot) - Q_N R(s,\cdot)\| + \|Q_N R(s_I,\cdot)\| < \delta;$$

and so $Q_N R(s,\cdot)$ converges to zero uniformly in s.

Recall that the function $R - \left(\sum_{n=1}^{N-1} k_n \otimes k_n\right)$ is contained in $\mathcal{C}_q^{(2D)}$ for each $q \le r = \frac{1}{2H}$; hence, setting $q < r$ such that $\frac{1}{q} + \frac{1}{r} > 1$, one has the Towghi bound

$$\left| \int_{[0,T]^2} R - \left(\sum_{n=1}^{N-1} k_n \otimes k_n \right) \, dR \right| \le C\|R\|_r^{(2D)} \left\| R - \left(\sum_{i=1}^N k_n \otimes k_n \right) \right\|_q^{(2D)}$$

for suitable constant C. By Lemma 2.2.12; the right-hand expression is bounded above by a positive multiple of $\|R - \left(\sum_{n=1}^{N} k_n \otimes k_n\right)\|_u$. Now we note that

$$\begin{aligned}
\left\| R - \left(\sum_{n=1}^{N} k_n \otimes k_n \right) \right\|_u &= \sup_{(s,t)\in[0,T]^2} \left| R(s,t) - \left(\sum_{n=1}^{N} k_n(s) k_n(t) \right) \right| \\
&= \sup_{(s,t)\in[0,T]^2} |\langle R(s,\cdot), Q_N R(t,\cdot)\rangle| \\
&\leq \sup_{(s,t)\in[0,T]^2} \|R(s,\cdot)\|_{\mathcal{H}} \, \|Q_N R(t,\cdot)\|_{\mathcal{H}} \\
&\leq T^H \sup_{0\leq t\leq T} \|Q_N R(t,\cdot)\|_{\mathcal{H}},
\end{aligned}$$

which, as we observed, tends to zero as we let $N \to \infty$. □

Proposition C.0.13. *The operator $\hat{\Phi}$ described above is trace-class, and*

$$tr(\hat{\Phi}) \leq 2\left[T^{4H} \|R\|_r^{(2D)} + T^{2H} \|R\|_r^2\right] \|R\|_r^{(2D)}.$$

Proof. Recall from the proof of Proposition 3.2.4 that for $\omega \in \mathcal{W}_p^2$,

$$\begin{aligned}
\|Q\omega\|_{\mathcal{H}^2}^2 = \sum_{i=1}^{2} \frac{1}{4} T^{2H} |\tilde{\omega}^i(T)|^2 + \left\| \int_0^T \tilde{\omega}^i(t) \; R(dt,\cdot) \right\|_{\mathcal{H}}^2 \\
- \tilde{\omega}^i(T) \int_0^T \tilde{\omega}^i(t) \; R(dt,T).
\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}
|Q\omega(T)|^2 &= \sum_{i=1}^{2} \left| \frac{1}{2} T^{2H} \tilde{\omega}^i(T) - \int_0^T \tilde{\omega}^i(t) \; R(dt,T) \right|^2 \\
&= \sum_{i=1}^{2} \frac{1}{4} T^{4H} |\tilde{\omega}^i(T)|^2 + \left| \int_0^T \tilde{\omega}^i(t) \; R(dt,T) \right|^2 \\
&\quad - T^{2H} \tilde{\omega}^i(T) \int_0^T \tilde{\omega}^i(T) \; R(dt,T).
\end{aligned}$$

Combining terms, we obtain the following expression for Φ:

$$\Phi(\omega) = \sum_{i=1}^{2} \left(T^{4H} \left\| \int_0^T \tilde{\omega}^i(t) \; R(dt,\cdot) \right\|_{\mathcal{H}}^2 - T^{2H} \left| \int_0^T \tilde{\omega}^i(t) \; R(dt,T) \right|^2 \right).$$

Let $\{k_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$; then, as above, we have that $\{k_n e_1\} \bigcup \{k_n e_2\}$ is an orthonormal basis of $\mathcal{H}^2$, and it follows that

$$\begin{aligned}
\operatorname{tr}(\hat{\Phi}) &= \sum_{n=1}^{\infty} \left(\Phi(k_n e_1) + \Phi(k_n e_2)\right) \\
&= 2\sum_{n=1}^{\infty} \left[T^{4H} \left\| \int_0^T k_n(t)\ R(dt, \cdot) \right\|_{\mathcal{H}}^2 - T^{2H} \left| \int_0^T k_n(t)\ R(dt, T) \right|^2 \right].
\end{aligned}$$

To simplify the first term, one observes that for a family of partitions $\{\Pi_m\} \subset \mathcal{P}[0,T]$ with $\lim_{m\to\infty} |\Pi_m| = 0$,

$$\begin{aligned}
\left\| \int_0^T k_n(t)\ R(dt, \cdot) \right\|_{\mathcal{H}}^2 &= \lim_{m\to\infty} \left\| \sum_{i=1}^{\#(\Pi_m)} k_n(c_i)[R(\cdot, t_i) - R(\cdot, t_{i-1})] \right\|_{\mathcal{H}}^2 \\
&= \lim_{m\to\infty} \sum_{i,j=1}^{\#(\Pi_m)} k_n(c_i) k_n(c_j) \left\langle R(\cdot, t_i) - R(\cdot, t_{i-1}), R(\cdot, t_j) - R(\cdot, t_{j-1}) \right\rangle_{\mathcal{H}} \\
&= \lim_{m\to\infty} \sum_{i,j=1}^{\#(\Pi_m)} k_n(c_i) k_n(c_j) \Big[R(t_i, t_j) - R(t_i, t_{j-1}) \\
&\qquad\qquad - R(t_{i-1}, t_j) + R(t_{i-1}, t_{j-1}) \Big] \\
&= \int_{[0,T]^2} k_n(s) k_n(t) R(ds, dt)
\end{aligned}$$

Hence, it follows from Lemma C.0.12 that

$$\sum_{n=1}^{\infty} \left\| \int_0^T k_n(t)\ R(dt, \cdot) \right\|_{\mathcal{H}}^2 = \int_{[0,T]^2} R(s,t)\ dR(s,t).$$

The simplification of the second term is slightly more involved. Note that integration by parts implies

$$\begin{aligned}
\left|\int_0^T k_n(t)\ R(dt,T)\right|^2 &= \left|T^{2H}k_n(T) - \int_0^T R(t,T)\ dk_n(t)\right|^2 \\
&= T^{4H}\left|k_n(T)\right|^2 + \left|\int_0^T R(t,T)\ dk_n(t)\right|^2 \\
&\qquad - 2T^{2H}k_n(T)\int_0^T R(t,T)\ dk_n(t) \\
&= T^{4H}\left|k_n(T)\right|^2 + \left|\int_0^T R(t,T)\ dk_n(t)\right|^2 \\
&\qquad - 2T^{2H}k_n(T)\left(T^{2H}k_n(T) - \int_0^T k_n(t)\ R(dt,T)\right) \\
&= 2T^{2H}\int_0^T k_n(T)k_n(t)\ R(dt,T) - T^{4H}|k_n(T)|^2 \\
&\qquad + \left|\int_0^T R(t,T)\ dk_n(tu)\right|^2.
\end{aligned}$$

The Fundamental Theorem of Calculus for Young's integrals (Theorem 5.4.1 of [LQ02]), along with Lemma C.0.12, implies that

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_0^T k_n(T)k_n(t)R(dt,T) &= \int_0^T R(t,T)R(dt,T) \\
&= \frac{1}{2}[(R(T,T))^2 - (R(T,0))^2] = \frac{T^{4H}}{2}.
\end{aligned}$$

Combining this with the fact that $\sum\limits_{n=1}^{\infty}|k_n(T)|^2 = R(T,T) = T^{2H}$, we have that

$$\sum_{n=1}^{\infty}\left|\int_0^T k_n(t)\ R(dt,T)\right|^2 = \sum_{n=1}^{\infty}\left|\int_0^T R(t,T)\ dk_n(t)\right|^2$$

Let $P_N : \mathcal{H} \to \mathcal{H}$ denote the projection map onto the first N terms of our orthonormal basis; that is, $P_N h := \sum\limits_{n=0}^{N}\langle h, k_n\rangle k_n$. Then for any $h \in \mathcal{H}$, we may once again apply Proposition 3.2.2 to find that

$$\begin{aligned}
&\left\| P_N \int_0^T h(t)\ R(dt,\cdot) \right\|_{\mathcal{H}}^2 \\
&\qquad = \left\langle P_N \int_0^T h(s)\ R(ds,\cdot), P_N \int_0^T h(t)\ R(dt,\cdot) \right\rangle_{\mathcal{H}} \\
&\qquad = \sum_{m,n=0}^{N} \left(\int_0^T h(s) dk_n(s) \right) \left(\int_0^T h(t) dk_m(t) \right) \langle k_n, k_m \rangle_{\mathcal{H}} \\
&\qquad = \sum_{n=0}^{N} \left(\int_0^T h(t) dk_n(t) \right)^2 .
\end{aligned}$$

Using this fact along with the continuity of the norm , we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \int_0^T R(t,T)\ dk_n(t) \right|^2 &= \lim_{N\to\infty} \sum_{n=1}^{N} \left| \int_0^T R(t,T)\ dk_n(t) \right|^2 \\
&= \lim_{N\to\infty} \left\| P_N \int_0^T R(t,T)\ R(dt,T) \right\|_{\mathcal{H}}^2 \\
&= \left\| \int_0^T R(t,T)\ R(dt,T) \right\|_{\mathcal{H}}^2 \\
&= \int_{[0,T]^2} R(s,T) R(t,T)\ dR(s,t)
\end{aligned}$$

Therefore, we can rewrite the trace of our operator as

$$\begin{aligned}
\mathrm{tr}(\hat{\Phi}) &= 2 \int_{[0,T]^2} \left[T^{4H} R(s,t) - T^{2H} R(s,T) R(t,T) \right]\ R(ds,dt) \\
&\leq 2 \left[T^{4H} \|R\|_r^{(2D)} + T^{2H} \|R\|_r^2 \right] \|R\|_r^{(2D)}
\end{aligned}$$

Where the inequality results from the standard bounds on two-dimensional Young's integrals as given in Theorem 2.2.13. □

Portions of the Appendix are adapted from material submitted for publication as Driscoll, Patrick, "Smoothness of Density for the Area Process of Fractional Brownian Motion." The dissertation author was the sole author of this paper.

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